Automating the calculation of jet functions

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Motivation

- **Definition:**

[Farhi,77]

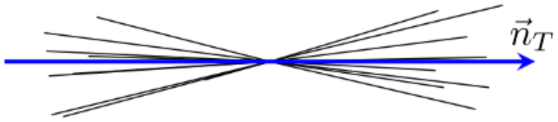
$$T = \max_{\vec{n}} \frac{\sum_i |\vec{n}\cdot\vec{p}_i|}{Q}$$

$T = 1 - \tau$

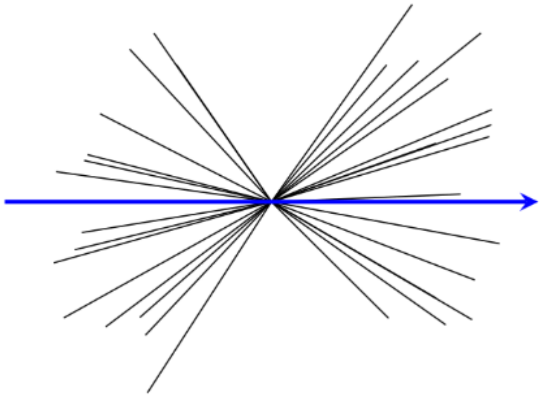
Thrust axis

Center-of-mass energy

- **Describe how pencil-like an event is.**



$T \simeq 1$



$T < 1$

Motivation

- **Definition:**

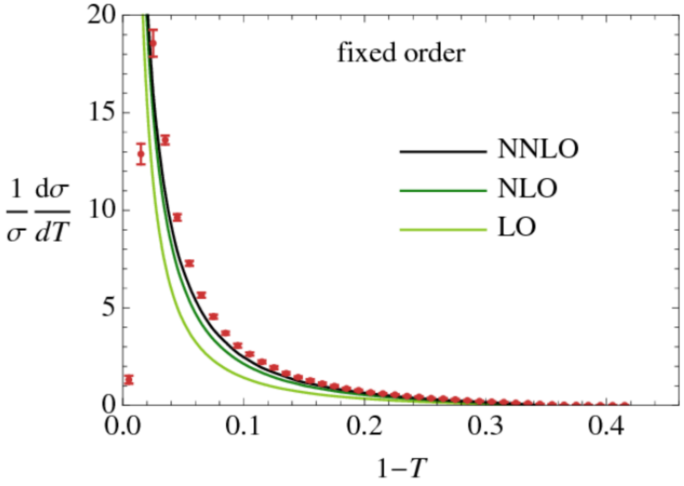
$$T = \max_{\vec{n}} \frac{\sum_i |\vec{n} \cdot \vec{p}_i|}{Q}$$

[Farhi,77]

- **Fixed-order calculation**

$$\int_0^{1-T} \mathrm{d}(1-T') \frac{1}{\sigma_0} \frac{\mathrm{d}\sigma}{\mathrm{d}T'} = 1 + \frac{2\alpha_s}{3\pi} \left(-2\ln^2(1-T) - 3\ln(1-T) + \ldots\right)$$

$T \simeq 1$: Fixed order calculation breaks down

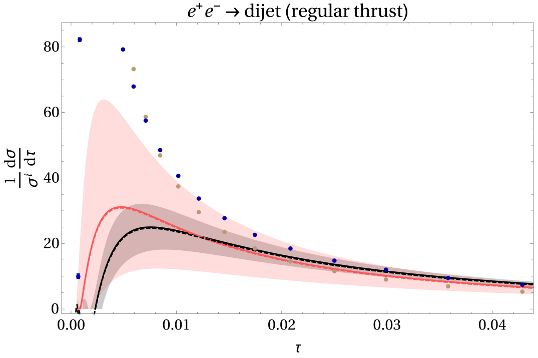


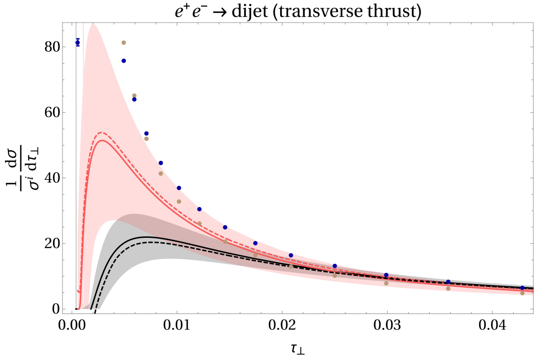
[Becher,Schwartz ,08]

- **⇒ Resummation**

Motivation

- **Resummation is useful to correctly describe observables at colliders**





[Becher, Tormo, Piclum, 16]

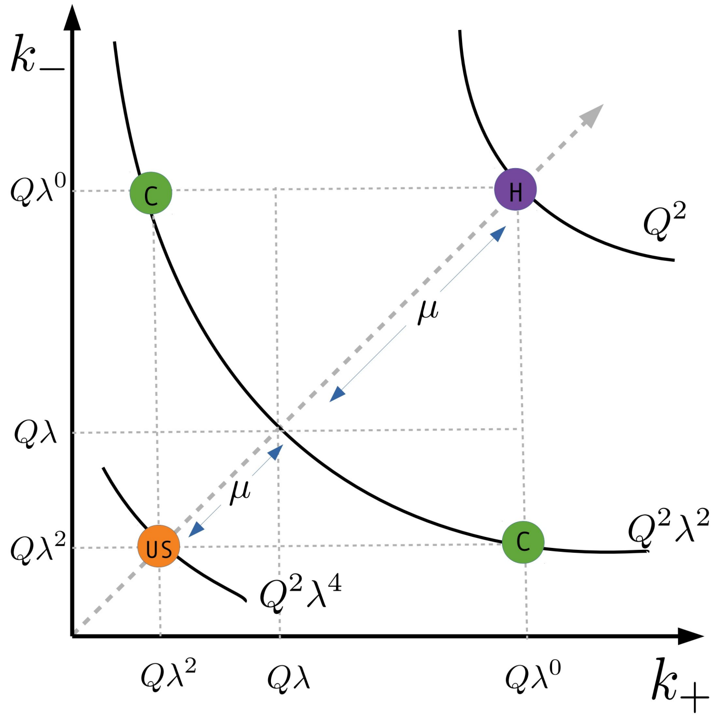
- **SCET has emerged as an important tool to study IR sector of QCD and resum large logarithms in a systematic framework**

- **The backbone relies on the underlying factorisation theorems**

Soft-Collinear Effective Theory (SCET)

- **Effective theory:**
  - Soft and collinear modes
  - Integrating out hard modes
- **At leading power soft and collinear modes decouple**

- **Typical scaling**
  - Hard Region: $k_H^\mu \sim (1, 1, 1)Q$
  - Collinear Region: $k_c^\mu \sim (1, \lambda^2, \lambda)Q$
  - Ultra-soft Region: $k_{us}^\mu \sim (\lambda^2, \lambda^2, \lambda^2)Q$
- **⇒ Complete Factorisation**



Factorisation

- **Generic factorisation theorem in SCET**

$$\mathrm{d}\sigma \simeq H(\mu_F) \cdot \prod_i B_i(\mu_F) \otimes \prod_j J_j(\mu_F) \otimes S(\mu_F)$$

Hard interaction

Collinear radiation(initial)

Collinear radiation(final)

Soft radiation

- **Each function can be computed perturbatively**
- **Resummation is performed by calculating them at their characteristic scales and evolving them to a common scale.**

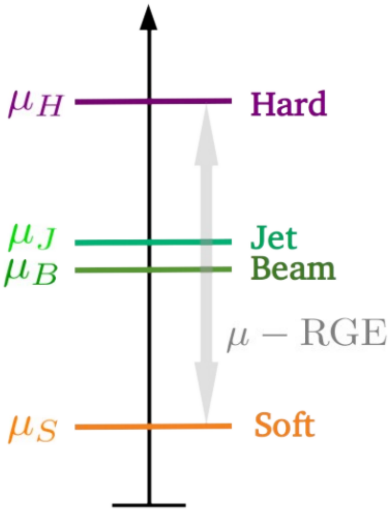
Resummation

- **Resummation through RGE**

- **Hard function RGE:**

$$\frac{\mathrm{d}H(Q,\mu)}{\mathrm{d}\ln\mu} = \gamma_{\text{Hard}}(Q,\mu)H(Q,\mu)$$

[Becher,Neubert,10]



  - Hard anomalous dimension

$$\gamma_{\text{Hard}}(Q,\mu) = \Gamma_{\text{Cusp}}(\alpha_s)\ln\frac{Q}{\mu} + \gamma_H(\alpha_s)$$

$$H(Q,\mu) = H(Q,\mu_H)U(\mu_H,\mu) \qquad U(\mu_H,\mu) = \exp\left[\int_{\mu_H}^{\mu} \mathrm{d}\ln\mu'\,\gamma_{\text{Hard}}(Q,\mu')\right]$$

Boundary term
(free of large logs)

Evolution kernel
(resums large logs)

Ingredients for Resummation

- **We need to have all anomalous dimensions and matching coefficients**

$$\underbrace{\Gamma_{\text{Cusp}},\, \gamma^H,\, c_H,}_{\text{Observable-independent}} \quad \underbrace{\gamma^S,\, c_S,\, \gamma^B,\, c_B,\, \gamma^J,\, c_J}_{\text{Observable-dependent}}$$

- **Observable-independent quantities are known**

- **Soft, Beam and Jet quantities are computed on a case-by-case basis**

- **Need two-loop matching coefficients to achieve NNLL' accuracy**

	$\Gamma_{\text{Cusp}}, \beta$	$\gamma^{H,S,B,J}$	$c_{H,S,B,J}$
NLL	2-loop	1-loop	1
NLL'	2-loop	1-loop	α_s
NNLL	3-loop	2-loop	α_s
NNLL'	3-loop	2-loop	α_s^2

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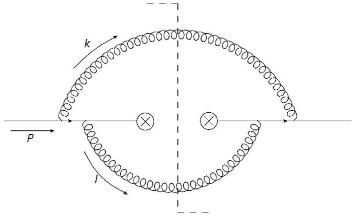
Automation of Soft/Jet/Beam functions

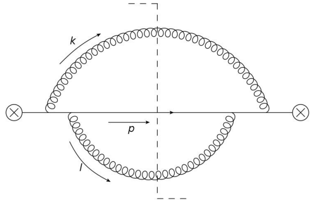
- **Set up a general framework to automatically calculate Jet, Beam, and Soft functions for a general class of observables**

- **Soft functions** → **2-particle final state** [Bell,Rahn,Talbert,18,20]
  - Complicated measurement function

- **Beam functions** → **2-particle final state** [Bell,KB,Das,Wald (in progress)]
  - Non-trivial matching onto PDFs

- **Jet functions** → **3-particle final state**
  - Complicated divergence structures





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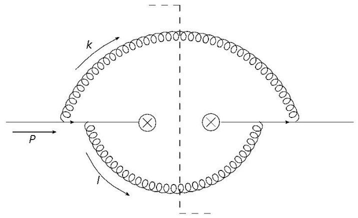

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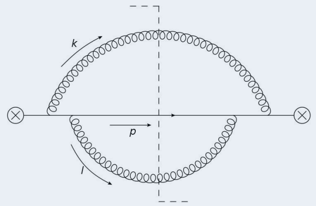


Jet functions

- **Definitions:**
  - Quark jet function $J_q(\tau, \mu)$

$$\left[\frac{\not{n}}{2}\right] J_q(\tau, \mu) = \frac{1}{\pi} \sum_{i \in X} (2\pi)^d \delta\left(Q - \sum\nolimits_i k_i^-\right) \delta^{d-2}\left(\sum\nolimits_i k_i^\perp\right) \mathcal{M}(\tau, \{k_i\}) \langle 0| \chi |X\rangle \langle X| \bar{\chi} |0\rangle$$

  - Gluon jet function $J_g(\tau, \mu)$

$$-g_\perp^{\mu\nu} \frac{\pi}{Q} \delta^{AB} g_s^2 J_g(\tau, \mu) = \sum_{i \in X} (2\pi)^d \delta\left(Q - \sum\nolimits_i k_i^-\right) \delta^{d-2}\left(\sum\nolimits_i k_i^\perp\right) \mathcal{M}(\tau, \{k_i\}) \langle 0| \mathcal{A}_\perp^{\mu,A} |X\rangle \langle X| \mathcal{A}_\perp^{\nu,B} |0\rangle$$

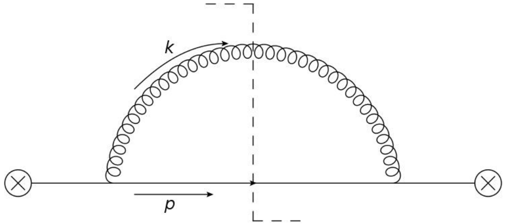
Phase space constraints

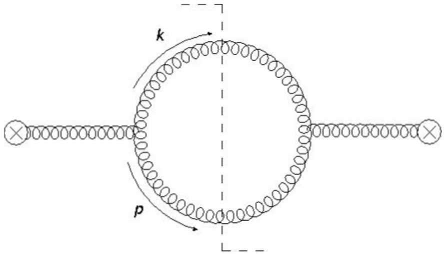
Collinear ME

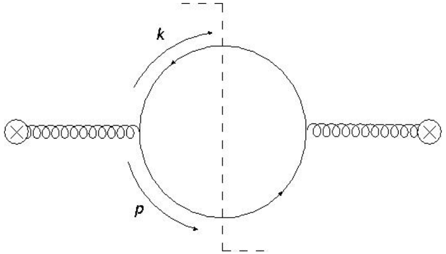
  - Generic measurement function $\mathcal{M}(\tau, \{k_i\})$

NLO

- **Automation exists at NLO** [KB's master thesis,18] [Basdew-Sharma et al,20]
- **Matrix element : LO splitting function** $P^{(0)}_{q^*\to gq}(z_k)$ $P^{(0)}_{g^*\to gg}(z_k)$ $P^{(0)}_{g^*\to q\bar{q}}(z_k)$ [Altarelli,Parisi,77]







$$P^{(0)}_{q^*\to gq}(z_k) \propto \frac{1}{k_-}C_F \qquad P^{(0)}_{g^*\to gg}(z_k) \propto \frac{1}{k_-p_-}C_A \qquad P^{(0)}_{g^*\to q\bar{q}}(z_k) \propto T_F n_f$$

- **Phase space parametrisation**

$$z_k = \frac{k_-}{Q}, \quad k_T = \sqrt{k_+k_-}, \quad t_k = \frac{1-\cos(\theta_k)}{2}$$

Measurement

- **Generic parametrisation of the measurement function in Laplace space**

$$\mathcal{M}_1(\tau, z_k, k_T, t_k) = \exp\left(-\tau k_T \left(\frac{k_T}{z_k \bar{z}_k Q}\right)^n f(z_k, t_k)\right)$$

Non-zero in the singular limits of ME

- **Example :**
  - Thrust : $n = 1$ $\quad f(z_k, t_k) = 1$
  - Transverse Thrust : $n = 1$ $\quad f(z_k, t_k) = 16 \frac{t_k \bar{t}_k}{\sin \theta_B}$
  - Angularities : $n = 1 - A$ $\quad f(z_k, t_k) = (1 - z)^{1-A} + z^{1-A}$

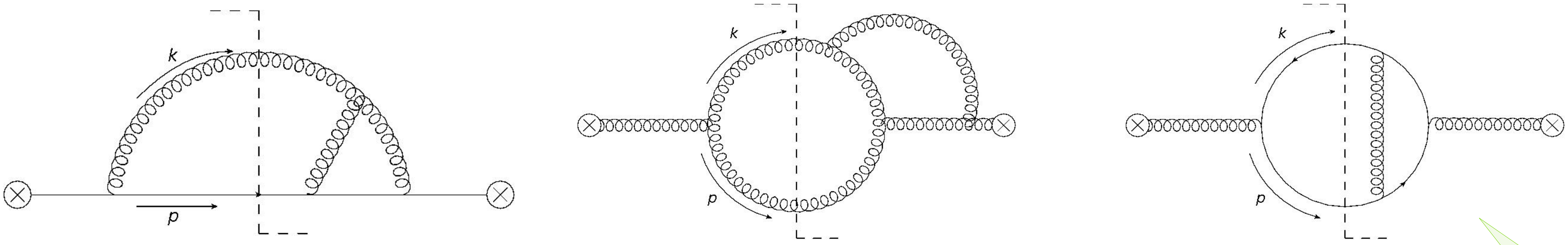
- **Master Formula**

$$J^{(1)}(\tau, \mu) \sim \Gamma\left(\frac{-2\epsilon}{1+n}\right) \int_0^1 \mathrm{d}z_k \mathrm{d}t_k \, z_k^{-1-2\frac{n}{1+n}\epsilon} \bar{z}_k^{-1-2\frac{n}{1+n}\epsilon} \left(z_k \bar{z}_k \left(P_{q^*}^{(0)}(z_k), P_{g^* \to gg}^{(0)}(z_k), P_{g^* \to q\bar{q}}^{(0)}(z_k)\right)\right) (4 t_k \bar{t}_k)^{-\frac{1}{2}-\epsilon} f(z_k, t_k)^{\frac{2}{1+n}\epsilon}$$

All singularities are factorised !

NNLO real-virtual contribution

- **Matrix Element: NLO splitting functions** $P^{(1)}_{q^*\to gq}(z_k)$ $P^{(1)}_{g^*\to gg}(z_k)$ $P^{(1)}_{g^*\to q\bar{q}}(z_k)$ [Bern,et.al., 95,99] [Kosower,Uwer,99]



$\sim \mathcal{O}(20)$

- **Phase space & measurement function follow NLO**
- **Master formula**

$$J^{(2),\mathrm{RV}}(\tau,\mu) \sim V(\epsilon)\,\Gamma\left(\frac{-4\epsilon}{1+n}\right)\int_0^1 \mathrm{d}z_k \mathrm{d}t_k\, z_k^{-1-4\frac{n}{1+n}\epsilon}\,\bar{z}_k^{-1-4\frac{n}{1+n}\epsilon}\left(z_k\bar{z}_k\left(\tilde{P}^{(1)}_{q^*\to qg}(z_k),\tilde{P}^{(1)}_{g^*\to gg}(z_k),\tilde{P}^{(1)}_{g^*\to q\bar{q}}(z_k)\right)\right)(4t_k\bar{t}_k)^{-\frac{1}{2}-\epsilon} f(z_k,t_k)^{\frac{4}{1+n}\epsilon}$$

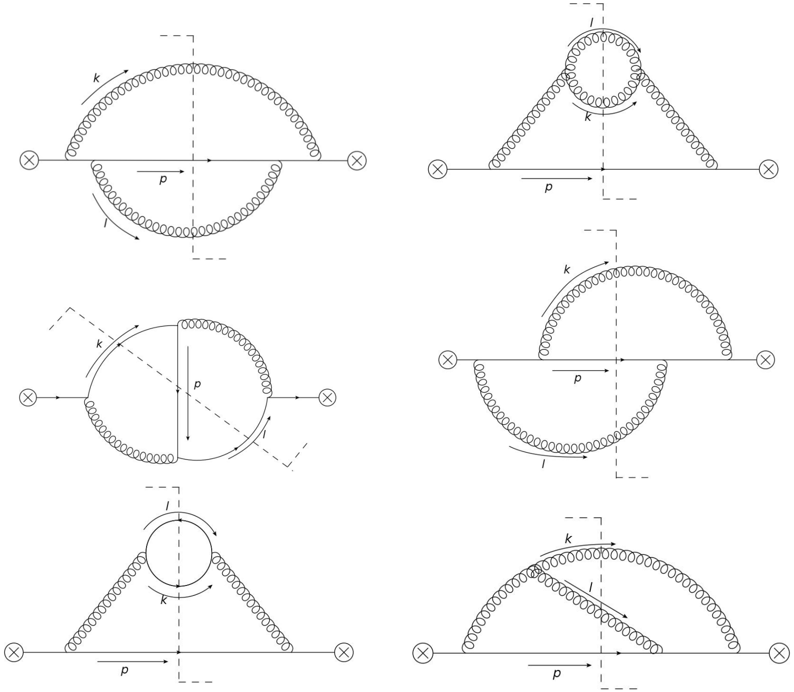
$\mathcal{O}(\epsilon^{-2})$

All singularities are factorised !

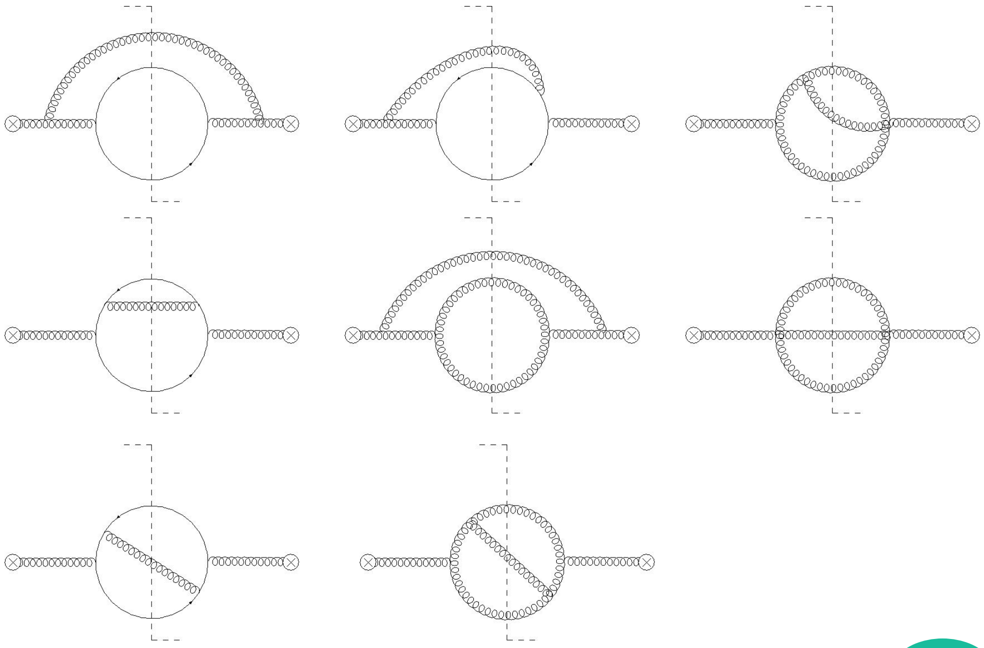
NNLO real-real contribution

- **Matrix element: LO triple collinear splitting function** [Catani,Grazzini,99]

$$J_q^{(2),\mathrm{RR}} \propto P^{(0)}_{q^*\to q'\bar{q}'q},\quad P^{(0)}_{q^*\to q\bar{q}q},\quad P^{(0)}_{q^*\to ggq}$$



$$J_g^{(2),\mathrm{RR}} \propto P^{(0)}_{g^*\to g\bar{q}'q'},\quad P^{(0)}_{g^*\to ggg}$$



NNLO real-real contribution: CF TF nf

- **Sample divergence structure:**

$$P^{(0)}_{q^*\to q'\bar{q}'q} \in \frac{1}{s_{123}^2 s_{12}^2 (z_1+z_2)^2}$$

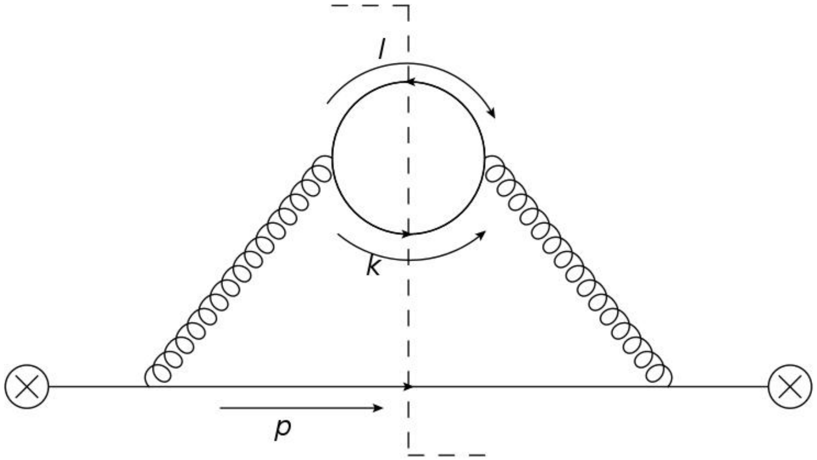
- **Phase space parametrisation**

$$z_{12} = \frac{k_- + l_-}{Q}, \quad b = \frac{k_T}{l_T}$$

$$a = \frac{k_- l_T}{k_T l_-}, \quad t_{kl} = \frac{1-\cos(\theta_{kl})}{2}$$

$$q_T = \sqrt{(k_- + l_-)(k_+ + l_+)}$$

- **Generic parametrisation of the measurement function in Laplace space**

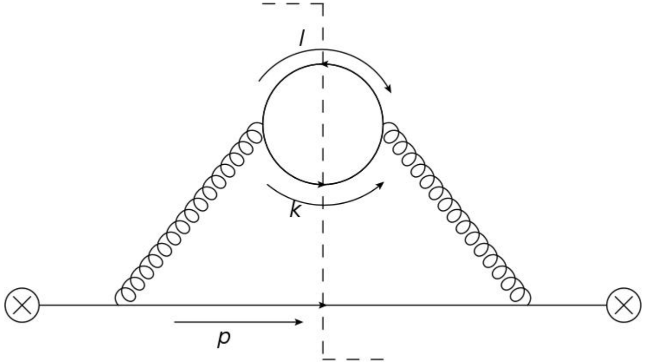


$$s_{123} = s_{12} + s_{13} + s_{23}, \quad s_{12} = (2k\cdot l), \quad s_{13} = (2k\cdot p), \quad s_{23} = (2l\cdot p)$$

$$z_1 = \frac{k_-}{Q}, \quad z_2 = \frac{l_-}{Q}, \quad z_3 = \frac{p_-}{Q}$$

$$\mathcal{M}_2(\tau,k,l,p) = \exp\left(-\tau q_T \left(\frac{q_T}{z_{12}Q}\right)^n F(z_{12}, b, a, t_{kl}, t_l, t_k)\right)$$

NNLO real-real contribution



- **Master formula:**

$$J^{(2),\mathrm{RR}}_{q'\bar{q}'q}(\mu,\tau) \sim \Gamma\left(\frac{-4\epsilon}{1+n}\right)\int_0^1 \mathrm{d}z_{12}\mathrm{d}u\mathrm{d}b\mathrm{d}v \;\; z_{12}^{-1-\frac{4n}{1+n}\epsilon}\, u^{-1-2\epsilon}\, \mathcal{W}(z_{12},u,v,b,t_l,t_k)\, F(z_{12},u,v,b,t_l,t_k)^{\frac{4}{1+n}\epsilon}$$

$$s_{12}^{-1} \sim (\bar{a}+4at_{kl})^{-1} \overset{\mathrm{NLT}}{\Longrightarrow} u^{-1}$$

Finite function

Non-zero in the singular limits of ME

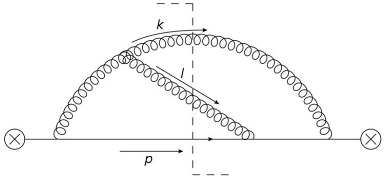
All singularities are factorised !

NNLO real-real contribution

- **Divergence structures**

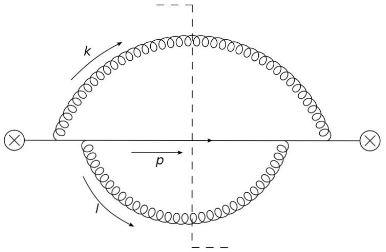
$$\frac{1}{z_1 z_2 s_{13} s_{23}}, \frac{1}{z_1 z_2 s_{13} s_{123}}, \frac{1}{z_1(1-z_1) s_{13} s_{23}} \ldots$$

Parametrisation similar depending on s_{ij}



- **Complications due to many overlapping singularities in ME**
- **Additional complications from measurements**
  - Thrust : $F \sim b + a(1 + z_{12})$
  - Angularity : $F \sim a^{\frac{1-n}{2}} (a^n + b) + a^{\frac{1+n}{2}} z_{12}^n$

Must stay non-zero in the limits
a→0,b→0,z_{12}→0



- **Strategy:**
  - Sector decomposition [Heinrich 08]
  - Selector functions
  - Non-linear transformation

Factorised singularities in all regions

SCET renormalization

- **Jet function fulfils the RG equation**

$$\frac{\mathrm{d}}{\mathrm{d}\ \ln\mu} J_{q,g}(\tau,\mu) = \left[2g(n)\Gamma_{\mathrm{Cusp}}\left(\alpha_s\right)L + \gamma^J\left(\alpha_s\right)\right] J_{q,g}(\tau,\mu)$$

$$g(n) = \frac{1+n}{n},\ L = \ln\left(\frac{\mu\bar{\tau}}{(Q\bar{\tau})^{\frac{n}{1+n}}}\right),\ \bar{\tau} = \tau e^{\gamma_E}$$

- **Two loop jet function RGE solution**

$$J_{q,g}(\tau,\mu) = 1 + \left(\frac{\alpha_s}{4\pi}\right)\left\{g(n)\Gamma_0 L^2 + \gamma_0^J L + c_1^J\right\} + \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{g(n)^2 \frac{\Gamma_0^2}{2} L^4 + g(n)\left(\gamma_0^J + \frac{2\beta_0}{3}\right)\Gamma_0 L^3\right.$$

$$\left. + \left(g(n)\left(\Gamma_1 + \Gamma_0 c_1^J\right) + \gamma_0^J\left(\frac{\gamma_0^J}{2} + \beta_0\right)\right)L^2 + \left(\gamma_1^J + c_1^J\left(\gamma_0^J + 2\beta_0\right)\right)L + c_2^J\right\}$$

Extraction

$$\{\Gamma_0, \Gamma_1, \gamma_0^J, \gamma_1^J, c_1^J, c_2^J\}$$

- **Implementation in pySecDec** [Heinrich et.al. 17,18,21]

Preliminary

Thrust

$$\omega_T = k_+ + l_+ + p_+$$

$\gamma_1^{J_q}$	Analytical[1]	This work
$C_F T_F n_f$	-26.699	-26.699(5)
C_F^2	21.220	21.221(94)
$C_F C_A$	-6.520	-6.522(89)

$c_2^{J_q}$	Analytical[1]	This work
$C_F T_F n_f$	10.787	10.787(9)
C_F^2	4.655	4.658(117)
$C_F C_A$	2.165	2.167(132)

$\gamma_1^{J_g}$	Analytical[2]	This work
$(T_F n_f)^2$	0	$0 \pm 2 \cdot 10^{-4}$
$C_F T_F n_f$	-4	-3.999(13)
$C_A T_F n_f$	-9.243	-9.242(20)
C_A^2	9.297	9.297(55)

$c_2^{J_g}$	Analytical[2]	This work
$(T_F n_f)^2$	2.014	2.014(1)
$C_F T_F n_f$	0.900	0.904(50)
$C_A T_F n_f$	-13.725	-13.727(69)
C_A^2	3.197	3.195(168)

[[1]. Becher, Neubert 06,[2]. Becher, Bell 10]

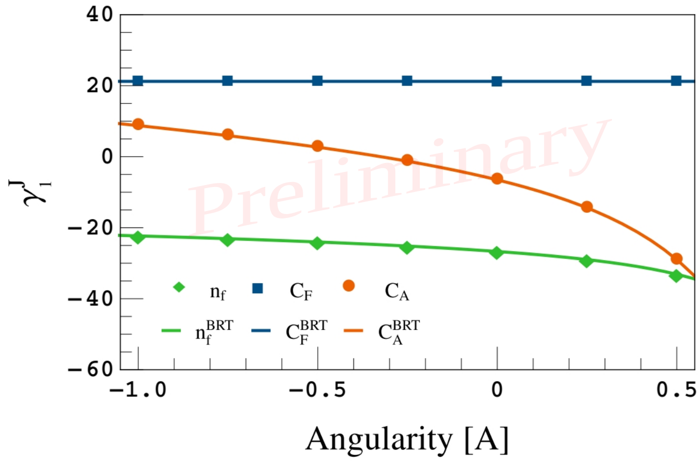
Angularities

Preliminary

- **Measurement function**

$$\omega_{Ang} = k_+^{1-A/2} k_-^{A/2} + l_+^{1-A/2} l_-^{A/2} + p_+^{1-A/2} p_-^{A/2}$$

- **Check jet anomalous dimensions @ NNLO (against SoftSERVE)**

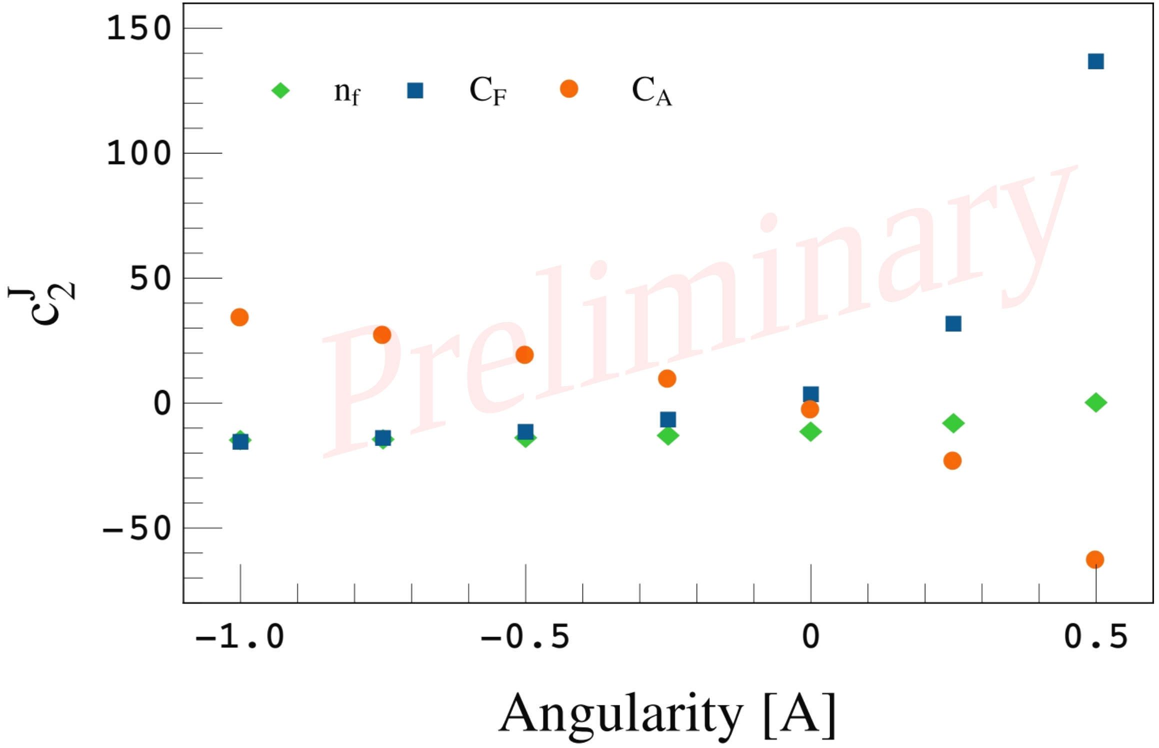


$$\gamma^H + \gamma^S + 2\gamma^J = 0$$

[Bell,Rahn,Talbert, 19]

Angularities

- **Matching coefficients at two loops**



Transverse Thrust

Preliminary

$$\omega_{TT} = 4\sin\theta_B \left[(|k_\perp| - \left|\vec{n}_\perp \cdot \vec{k}\right|) + (|l_\perp| - \left|\vec{n}_\perp \cdot \vec{l}\right|) + (|p_\perp| - |\vec{n}_\perp \cdot \vec{p}|) \right]$$

$\gamma_1^{J_q}$	Numerical[1]	Numerical[2]	This work
$C_F T_F n_f$	-41^{+2}_{-3}	-42.183(5)	-42.172(18)
C_F^2	21.220	21.220	21.610(338)
$C_F C_A$	157^{+20}_{-30}	167.54(6)	167.345(312)

$c_2^{J_q}$	This work
$C_F T_F n_f$	-5.911(34)
C_F^2	42.548(592)
$C_F C_A$	116.663(607)

$\gamma_1^{J_g}$	Numerical[1]	Numerical[2]	This work
$(T_F n_f)^2$	0	0	0 ± 10^{-4}
$C_F T_F n_f$	-4	-4	-3.997(25)
$C_A T_F n_f$	$-16.3^{+1.5}_{-1.0}$	-16.985(2)	-16.953(43)
C_A^2	91^{+15}_{-10}	96.329(30)	96.329(208)

$c_2^{J_g}$	This work
$(T_F n_f)^2$	7.862(1)
$C_F T_F n_f$	-47.210(116)
$C_A T_F n_f$	30.691(192)
C_A^2	172.918(817)

[[1]. Becher, Tormo, Piclum 16,[2]. Bell, Rahn, Talbert 19]

Conclusion

- **Developed an automated framework to calculate Jet for a wide class of observables at NNLO.**

- **Using a suitable phase-space parametrisation we are able to completely disentangle IR divergences into monomial form.**

- **We have presented the first results for event shape observables Thrust, Angularities and Transverse Thrust.**

- **Future plans:**
  - Extend framework to SCET II observables & jet algorithms
  - Development of a automated C++ code.