2D fishnet integrals and Calabi-Yau geometries

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based on work in collaboration with
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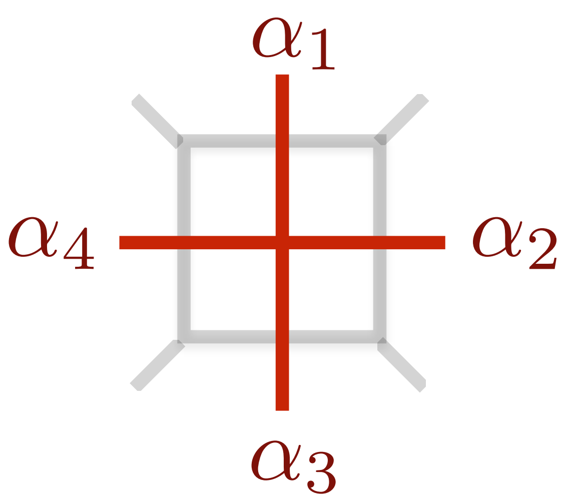
[2209.05291], + work in preparation.

Workshop on "Mathematical Structures in Feynman Integrals"
Siegen 13 February 2023

Feynman integrals

- Feynman integrals are the building blocks for multi-loop scattering amplitudes.
  - ➡ Important both for collider and gravitational wave phenomenology.
  - ➡ A window into the mathematical structure of pQFT.
- They exhibit a rich mathematical structure, with connections to (algebraic) geometry.
- We would like to understand the underlying mathematics as well as we can!

- **Example:** All 1-loop integrals compute the volumes of some polytope in hyperbolic space. [Davydychev, Delbourgo; Schnetz; …]

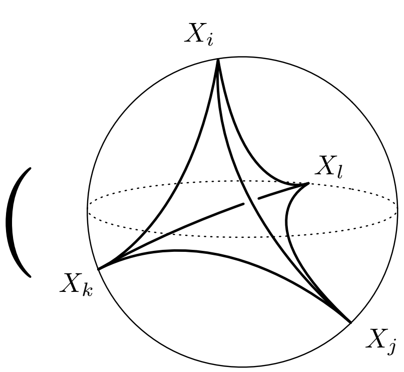


$$= \int \frac{d^4\xi}{i\pi^2} \prod_{i=1}^{4} \frac{1}{(\xi - \alpha_i)^2} = -\frac{4}{\alpha_{13}^2 \alpha_{24}^2} \frac{D(z)}{z - \bar{z}}$$

$$z\bar{z} = \frac{\alpha_{12}^2 \alpha_{34}^2}{\alpha_{13}^2 \alpha_{24}^2} \qquad (1-z)(1-\bar{z}) = \frac{\alpha_{14}^2 \alpha_{23}^2}{\alpha_{13}^2 \alpha_{24}^2}$$

Bloch-Wigner Dilogarithm:

$$D(z) = \mathrm{Im}\left[\mathrm{Li}_2(z) + \log|z| \log(1-z)\right] \sim \mathrm{Vol}(\quad)$$

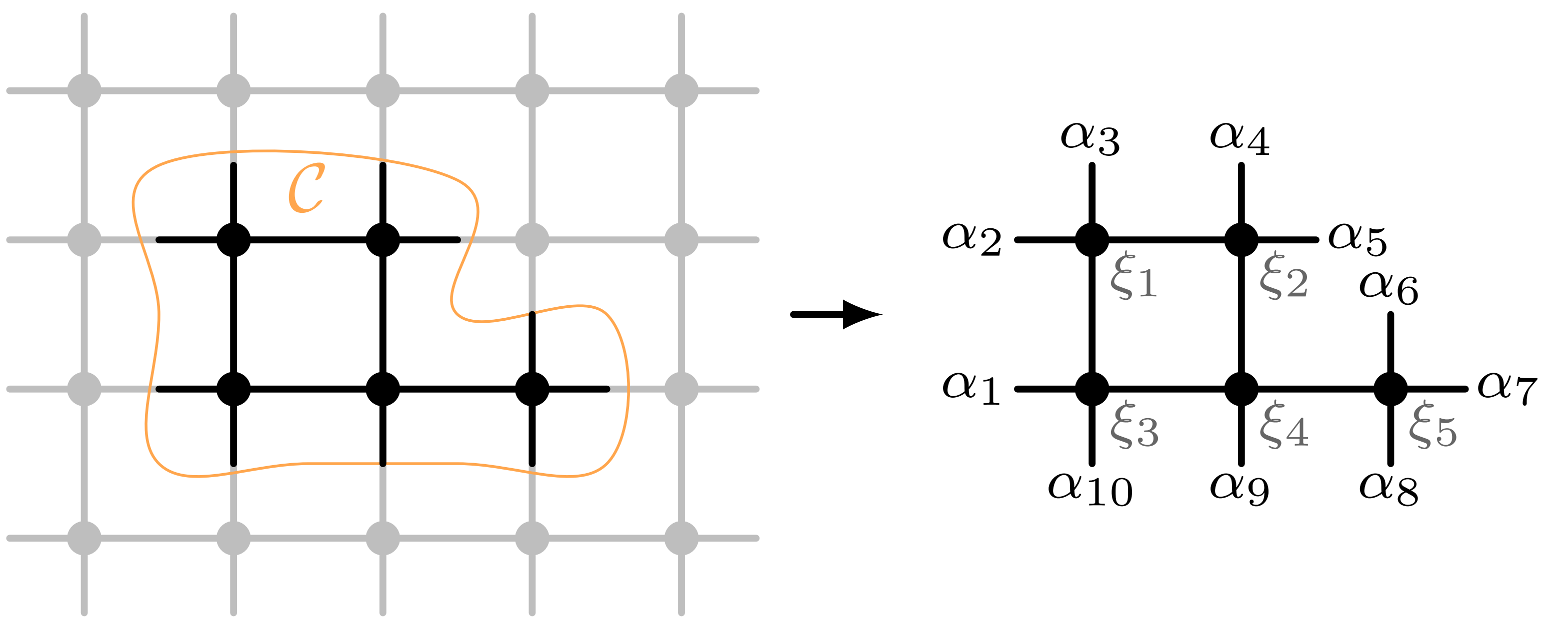


Pure function

[Picture taken from 1004.3498]

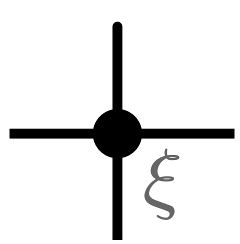
Fishnet integrals

- **Here:** We focus on **fishnet integrals** (in D dimensions).

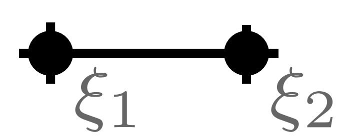


[Gürdogan, Kazakov; Chicherin, Kazakov, Löbbert, Müller, Zhong]

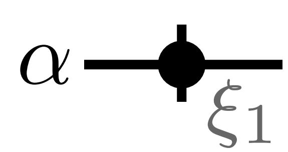
- **Feynman rules:**



$$= \int \mathrm{d}^D \xi$$



$$= \frac{1}{[(\xi_1 - \xi_2)^2]^{D/4}}$$



$$= \frac{1}{[(\xi - \alpha)^2]^{D/4}}$$

Fishnet integrals

- They are invariant under the Yangian $Y(\mathfrak{so}(1,D+1))$ of the conformal group in D (Euclidean) dimensions (with conformal weight 1 at each external point).

[Chicherin, Kazakov, Löbbert, Müller, Zhong; Löbbert, Miczajka, Müller, Münkler]

➡ Conformal (level 0) generators: $\mathrm{J}^A = \sum_{j=1}^{n} \mathrm{J}_j^A$

$$\mathrm{P}_j^\mu = -i\partial_j^\mu, \qquad \mathrm{L}_j^{\mu\nu} = ix_j^\mu\partial_j^\nu - ix_j^\nu\partial_j^\mu,$$
$$\mathrm{D}_j = -ix_{j\mu}\partial_j^\mu - i\Delta_j, \qquad \mathrm{K}_j^\mu = -i\left(2x_j^\mu x_j^\nu - \eta^{\mu\nu}x_j^2\right)\partial_{j,\nu} - 2i\Delta_j x_j^\mu,$$

➡ Level 1 generators: $\widehat{\mathrm{J}}^A = \frac{1}{2} f^A{}_{BC} \sum_{k=1}^{n}\sum_{j=1}^{k-1} \mathrm{J}_j^C \mathrm{J}_k^B + \sum_{j=1}^{n} s_j \mathrm{J}_j^A$

➡ Invariance of the integral associated to the fishnet graph G :

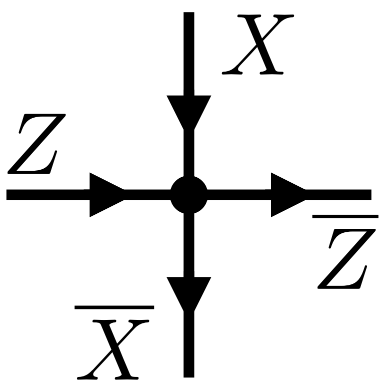
$$\mathrm{J}^A I_G = \widehat{\mathrm{J}}^A I_G = 0$$

Fishnet integrals

- They compute correlators in the D - dimensional fishnet theory:

$$\mathcal{L}_{\mathrm{FN},D} = N_c \,\mathrm{tr}\left[\overline{X}(-\Box)^{D/4}X + \overline{Z}(-\Box)^{D/4}Z + \lambda^2\, \overline{X}\,\overline{Z}XZ\right]$$

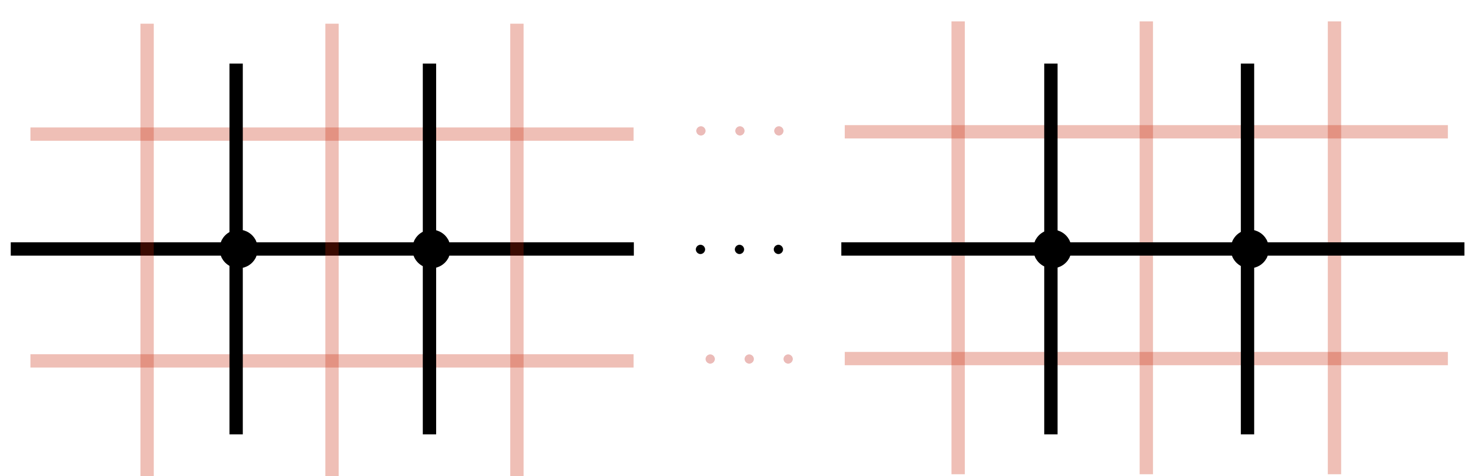
[Gürdogan, Kazakov; Kazakov, Olivucci]



$$= \lambda^2 \int \mathrm{d}^D\xi \qquad\qquad = \frac{1}{[(\xi_1-\xi_2)^2]^{D/4}}$$

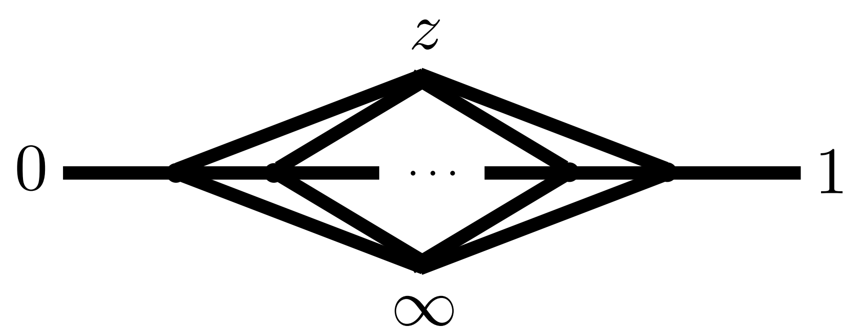
- Traintrack integrals compute specific supercomponents of $\mathcal{N}=4$ amplitudes in $D=4$.

[Caron-Huot, Larsen; Bourjaily, He, McLeod, von Hippel, Wilhelm]



Results in $D = 4$

- **Ladder integrals:** (single-valued) classical polylogs (for all loops).



[Davydychev, Ussyukina]

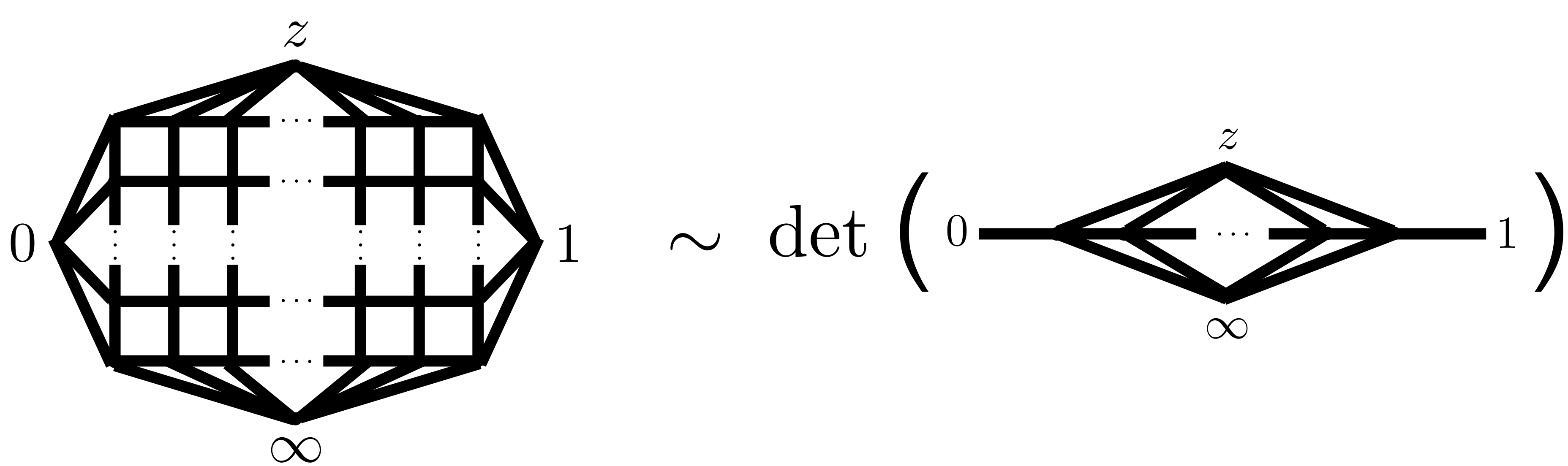
- **Two-loop train track:** elliptic polylogs.

[Kristensson, Wilhelm, Zhang; Morales, Spiering, Wilhelm, Yang, Zhang]

- **ℓ-loop traintrack:** geometry is Calabi-Yau $(\ell - 1)$-fold.

[Bourjaily, He, McLeod, von Hippel, Wilhelm]

- **Basso-Dixon Formula:**

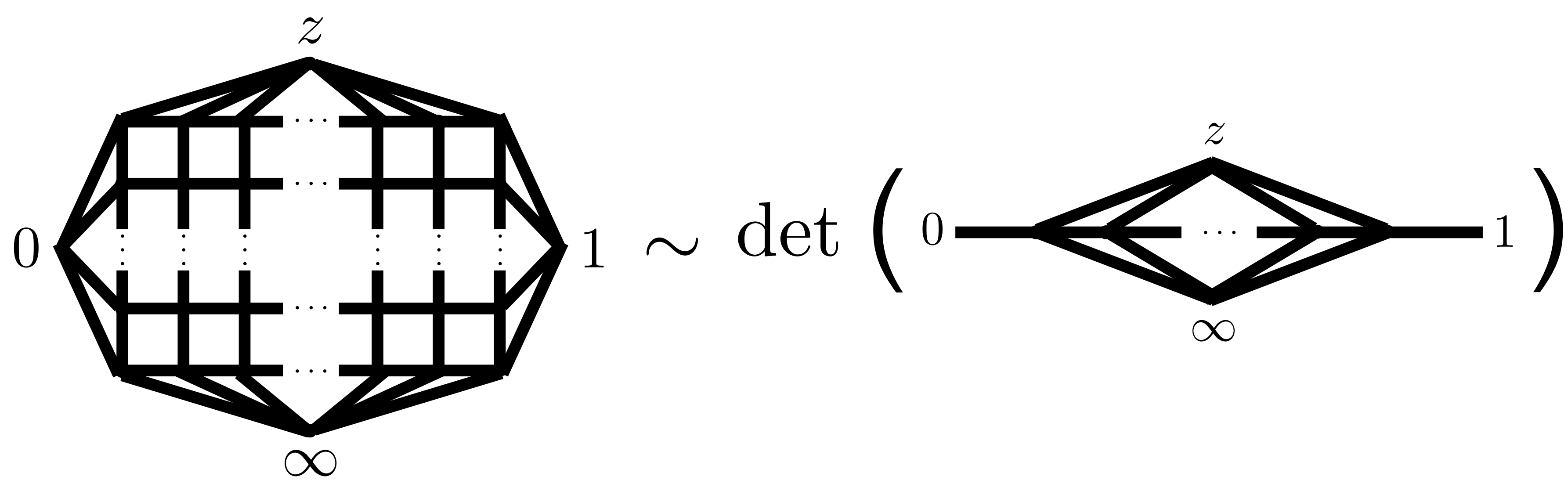


[Basso, Dixon]

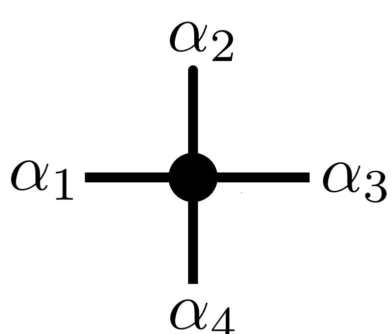
Results in $D = 2$

- ℓ -loop ladder integral: bilinear in ${}_{\ell+1}F_{\ell}$ - functions.
- Basso-Dixon formula in 2D:

[Derkachov, Kazakov, Olivucci]



- No known results for multi-variable traintracks.
- Observation: 1-loop can be expressed as elliptic integrals:



$$= \frac{4}{|a_{12}a_{34}|}\left[\mathrm{K}(z)\,\mathrm{K}(1-\bar{z}) + \mathrm{K}(1-z)\,\mathrm{K}(\bar{z})\right]$$

2 periods of an elliptic curve

$$\mathrm{K}(z) = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-zx^2)}} \qquad z = \frac{a_{23}a_{14}}{a_{21}a_{34}} \qquad a_k = \alpha_k^1 + i\,\alpha_k^2$$

[Corcoran, Löbbert, Miczajka]

Outline

1. Brief overview of Calabi-Yau geometry.
2. 2d fishnets and Calabi-Yau geometry.
3. Yangian-invariant Calabi-Yau periods
4. Fishnet integrals as volumes.

Brief overview of Calabi-Yau geometry

Calabi-Yau varieties

- A Calabi-Yau ℓ-fold is an ℓ-dimensional complex Kähler manifold with a unique nowhere vanishing $(\ell, 0)$-form.
  - ➡ (p, q)-form: p holomorphic and q anti-holomorphic differentials.
  - ➡ Uniqueness of $(\ell, 0)$-form is equivalent to vanishing of 1st Chern class.
- A Calabi-Yau ℓ-fold is uniquely defined by a triple

$$(M, \Omega, \omega)$$

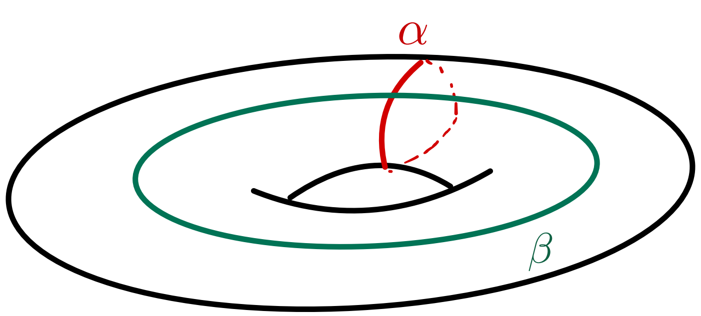
Complex manifold of dimension ℓ (M)

$(\ell, 0)$-form (defines complex structure) (Ω)

Kähler form; $(1,1)$-form (~metric) (ω)

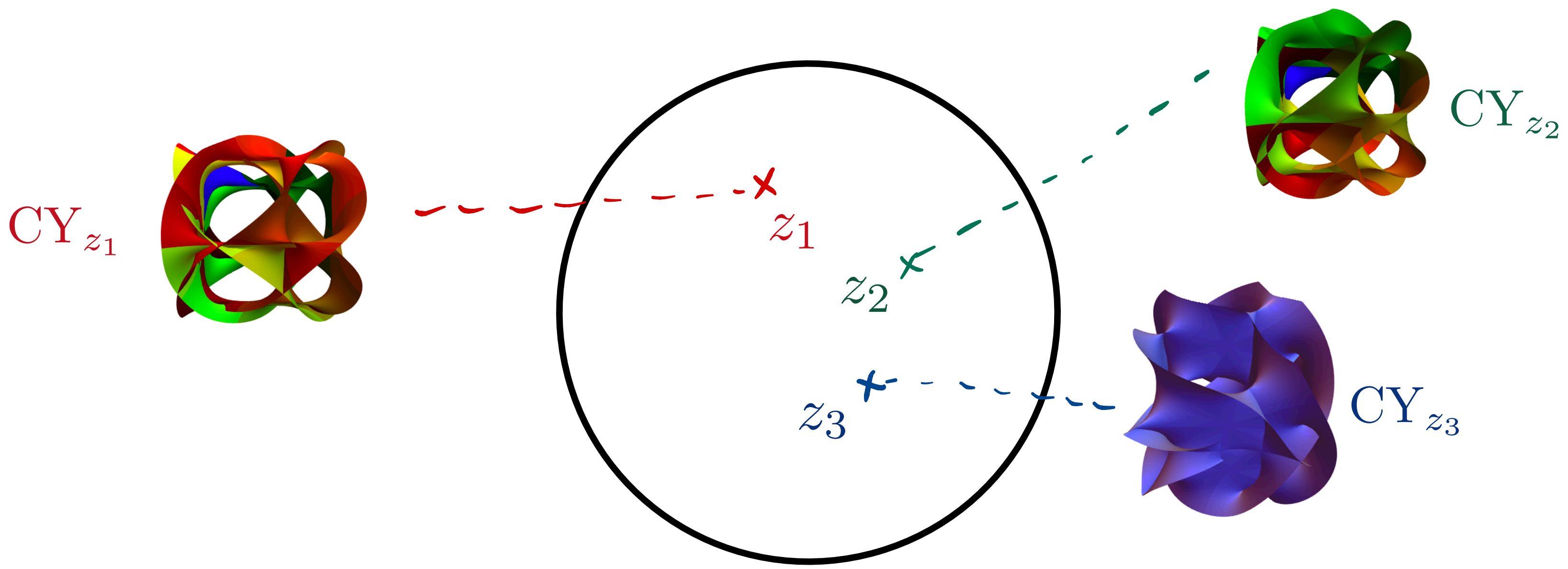
- Example: Calabi-Yau 1-fold = elliptic curve

$$\left(\mathcal{E}, \mathrm{d}z = \frac{\mathrm{d}x}{y}, A\, \mathrm{d}z \wedge \mathrm{d}\bar{z}\right)$$



Calabi-Yau varieties

- We are typically interested in families of CY varieties:



➡ For each z there is a CY variety M_z with a given top-form Ω_z .

- Example: Family of elliptic curves: $y^2 = x(1-x)(1-zx)$

➡ For each z there is $\mathcal{E}_z$ with $\Omega_z = \dfrac{dx}{y} = \dfrac{dx}{\sqrt{x(1-x)(1-zx)}}$.

➡ This does not fix the Kähler form! (We can still scale the torus, i.e., its area is not fixed!)

Periods

- We can integrate Ω_z over a basis of cycles of $H_\ell(M_z,\mathbb{Z})$.
  - ➡ Periods: $\Pi(z) = (\Pi_0(z),\ldots,\Pi_{b_\ell-1}(z))$, $\Pi_i(z) = \int_{\Gamma_i}\Omega_z$

- The periods are multivalued functions of z .
  - ➡ There is a monodromy-invariant bilinear pairing on periods:

$$\int_{M_z}\Omega_z\wedge\overline{\Omega}_z \sim \Pi(z)^\dagger\Sigma\Pi(z) \qquad \Sigma^T = (-1)^\ell\,\Sigma$$

- The periods are not all independent, but there is a quadratic relation among them:

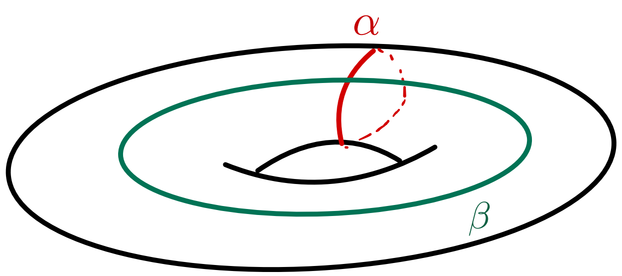
$$\Pi(z)^T\Sigma\Pi(z) \sim \int_{M_z}\Omega_z\wedge\Omega_z = 0$$

Picard-Fuchs ideal

- For every family of CYs, there is a set of differential operators whose solutions are precisely spanned by the periods!
  - ➡ They generate an ideal of differential operators, the Picard-Fuchs ideal (PFI) of the family of CYs.
  - ➡ For 1-parameter families, we only need a single operator, the Picard-Fuchs operator.
- Advantage: We can obtain the periods as solutions of certain differential equations.
- For certain 1-parameter families, the corresponding differential operators have been studied extensively.
  - ➡ Calabi-Yau operators. [van Straten; Bogner; …]

The Legendre family

- Example: Family of elliptic curves: $y^2 = x(1-x)(1-zx)$



$$\Pi_\alpha(z) = \int_\alpha \frac{\mathrm{d}x}{y} = 2\,\mathrm{K}(z) \qquad \Pi_\beta(z) = \int_\beta \frac{\mathrm{d}x}{y} = 2i\,\mathrm{K}(1-z)$$

➡ Bilinear pairing: $\Sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$\Pi(z)^T \Sigma \Pi(z) = 4i\,[\mathrm{K}(z)\mathrm{K}(1-z) - \mathrm{K}(z)\mathrm{K}(1-z)] = 0$$

$$\Pi(z)^\dagger \Sigma \Pi(z) = 4i\,[\mathrm{K}(z)\mathrm{K}(1-\bar{z}) + \mathrm{K}(\bar{z})\mathrm{K}(1-z)]$$

1-loop 2D fishnet integral!

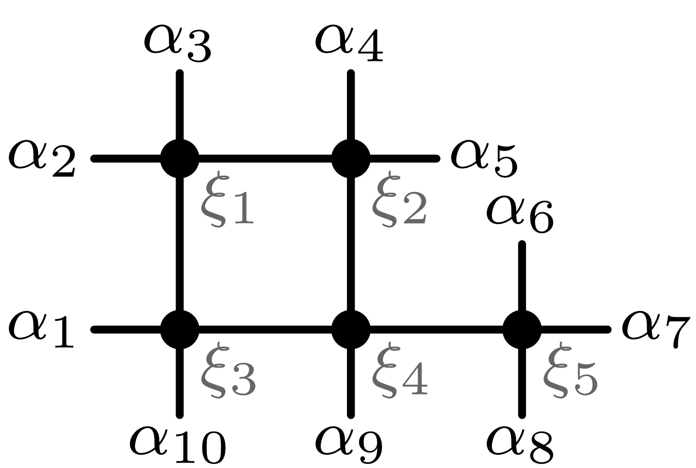
➡ Picard-Fuchs operator: $L = z^2(1-z)\partial_z^2 + z(1-2z)\partial_z - \frac{z}{4}$

$$Lf(z) = 0 \leftrightarrow f(z) = A\,\Pi_\alpha(z) + B\,\Pi_\beta(z) \qquad A, B \in \mathbb{C}$$

2D fishnets and Calabi-Yau geometry

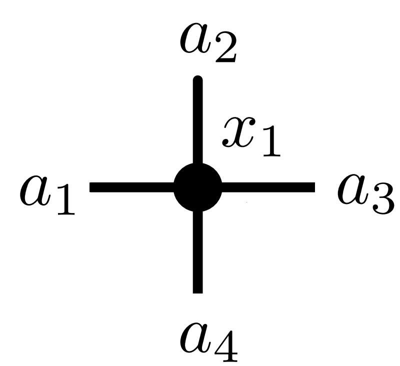
2D fishnets

- Holomorphic factorisation: $x_k = \xi_k^1 + i\xi_k^2 \qquad a_k = \alpha_k^1 + i\alpha_k^2$



$$\frac{1}{\sqrt{(\xi_i - \xi_j)^2}} = \frac{1}{|x_i - x_j|} \qquad \mathrm{d}^2\xi_k = \frac{i}{2}\mathrm{d}x_k \wedge \mathrm{d}\bar{x}_k$$

$$I_G(a) = \int \left(\prod_{j=1}^{\ell} \frac{\mathrm{d}x_j \wedge \mathrm{d}\bar{x}_j}{-2i} \right) \frac{1}{\sqrt{|P_G(x,a)|^2}}$$

- Example: $G =$  $\quad P_G(x,a) = (x_1 - a_1)\cdots(x_1 - a_4)$

- The Yangian also splits holomorphically:

$$Y(\mathfrak{so}(1,3)) = Y(\mathfrak{sl}_2(\mathbb{R})) \oplus \overline{Y(\mathfrak{sl}_2(\mathbb{R}))}$$

2D fishnets and CYs

- When does $y^2 = P_G(x,a)$ define a CY ℓ-fold?
    - ➡ The 1st Chern class must vanish.
    - ➡ This happens if $P_G(x,a)$ has degree 4 in each x_k .
    - ➡ This conditions is always satisfied for fishnet graphs, because all vertices are 4-valent!

Conclusion:

To every ℓ-loop fishnet graph G we can associate a family M_G of CY ℓ-folds parametrised by $a = (a_1, \ldots, a_n)$ and holomorphic $(\ell, 0)$-form

$$\Omega_G = \frac{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_\ell}{\sqrt{P_G(x,a)}}$$

[CD, Klemm, Löbbert, Nega, Porkert]

2D fishnets and CYs

- The Feynman integral is related to the periods:

$$I_G(a) = \int \left(\prod_{j=1}^{\ell} \frac{\mathrm{d}x_j \wedge \mathrm{d}\bar{x}_j}{-2i} \right) \frac{1}{\sqrt{|P_G(x,a)|^2}}$$

$$\sim \int_{M_G} \Omega_G \wedge \overline{\Omega}_G \quad \sim \Pi_G(a)^\dagger \Sigma \Pi_G(a)$$

➡ Generalises the bilinear expression at 1-loop:

$$\Pi(z)^\dagger \Sigma \Pi(z) = 4i \left[\mathrm{K}(z)\mathrm{K}(1-\bar{z}) + \mathrm{K}(\bar{z})\mathrm{K}(1-z) \right]$$

1-loop 2D fishnet integral!

- The periods are obtained by solving the PF differential equations.

➡ How can we find them?

2D fishnets and CYs

- Let $J \in Y(\mathfrak{sl}_2(\mathbb{R}))$. Yangian-invariance implies:

  $$0 = J\left[I_G(a)\right] \sim J\left[\Pi_G(a)^\dagger \Sigma \Pi_G(a)\right] \sim \Pi_G(a)^\dagger \Sigma J\left[\Pi_G(a)\right]$$

  ➡ $J\left[\Pi_G(a)\right] = 0$

  ➡ $Y(\mathfrak{sl}_2(\mathbb{R})) \subset \mathrm{PFI}$

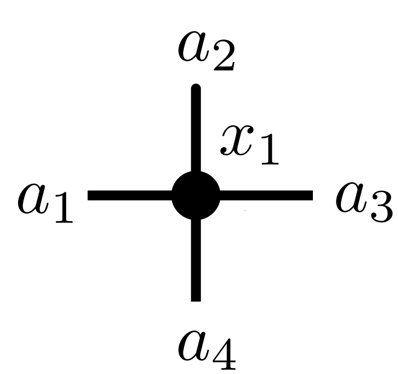
- The PFI contains more operators…
- Issue:

➡ The Yangian is built on a cyclic ordering of the external points.

➡ Fishnet graphs have more symmetries.

➡ Example:

$G =$



$\mathrm{Aut}(G) = S_4$

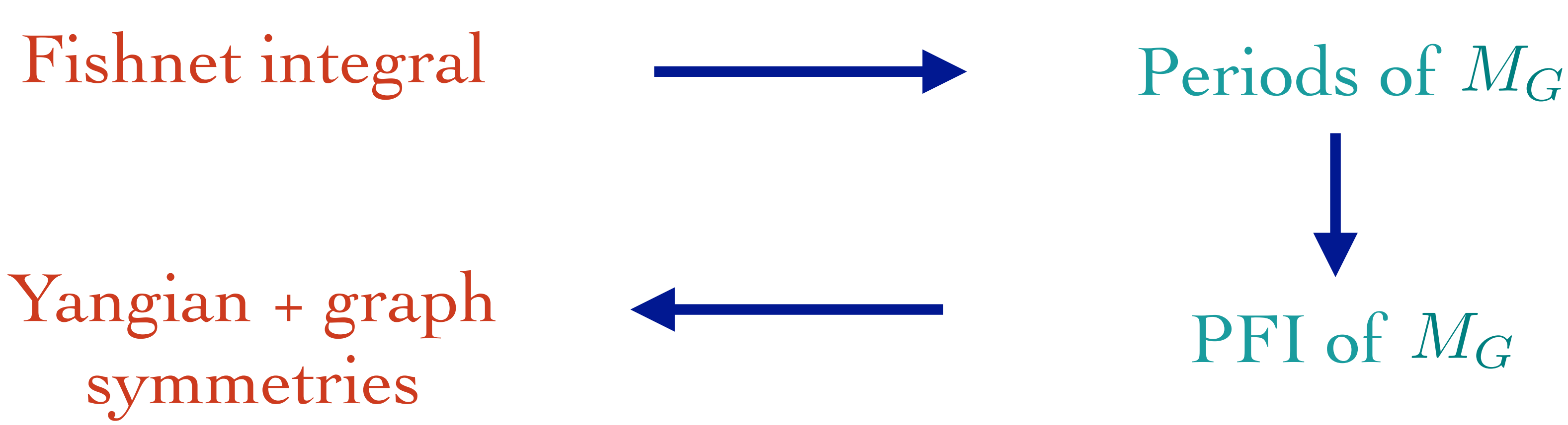
2D fishnets and CYs

- For all examples we studied: we obtain the complete PFI if we add to the Yangian all its $\mathrm{Aut}(G)$ permutations.

Conjecture:

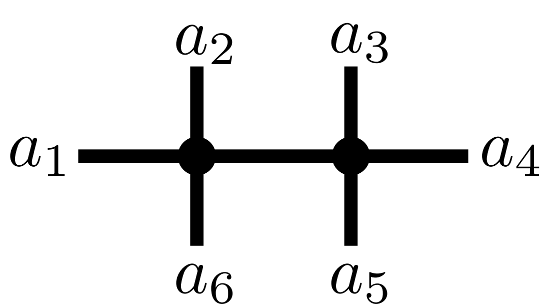
> The PFI of M_G is generated by the Yangian $Y(\mathfrak{sl}_2(\mathbb{R}))$ and all its $\mathrm{Aut}(G)$ permutations.
>
> [CD, Klemm, Löbbert, Nega, Porkert]

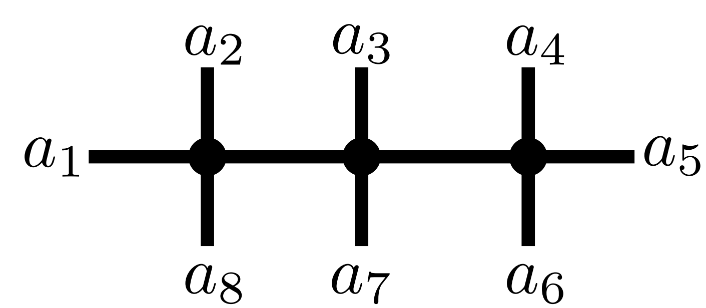
- Geometry informs physics, physics informs geometry!

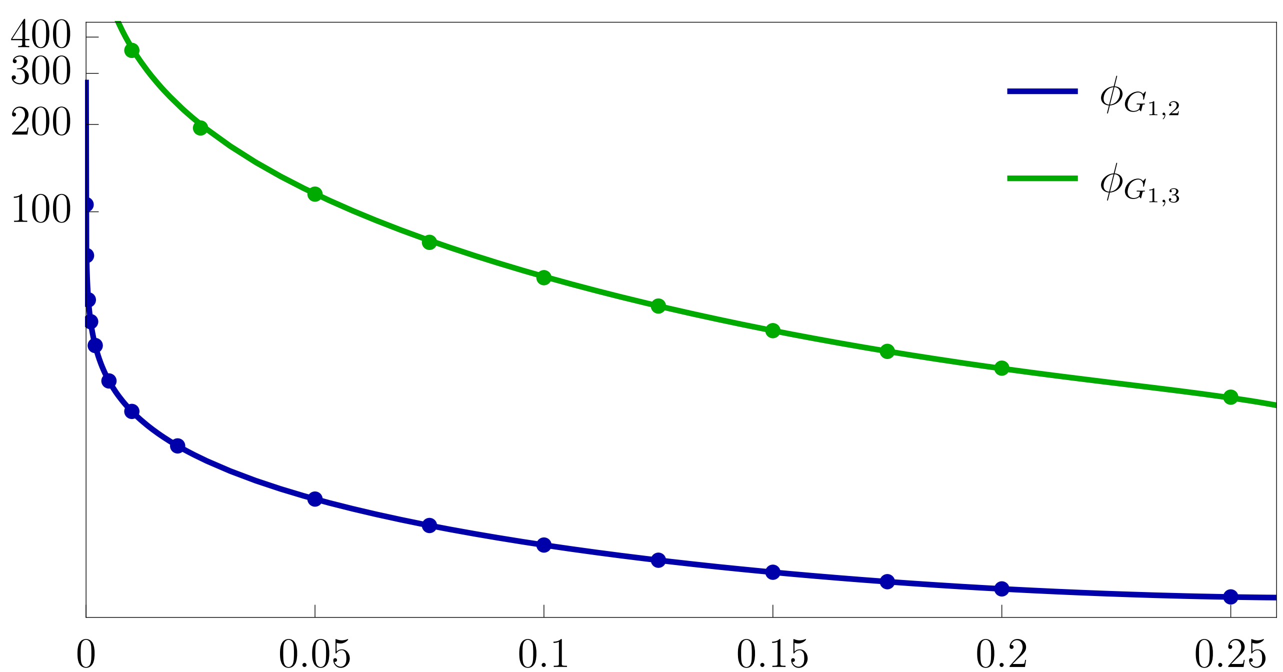


2- & 3-loop traintracks

- We reproduce in this way results for 1-parameter ladder and fishnet graphs to high loop order!
- New results: 2- and 3-loop traintrack in general kinematics:

$G_{1,2} =$ 

$G_{1,3} =$ 



$$I_G(a) = |F_G(a)|^2\, \phi_G(z)$$

cross ratios

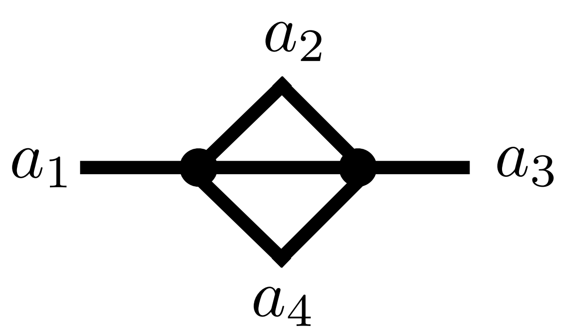
[CD, Klemm, Löbbert, Nega, Porkert]

2-loop ladder

- Periods of a 1-parameter family of CY 2-folds (K3 surfaces) can always be expressed in terms of products of elliptic integrals!

  [Doran; Bogner]

  ➡ The two-loop ladder integral can be expressed in terms of elliptic integrals!



$$= \frac{32}{|a_{12}a_{34}a_{24}|}\left(\mathrm{K}_+\overline{\mathrm{K}}_- + \mathrm{K}_-\overline{\mathrm{K}}_+\right)^2$$

$$\mathrm{K}_\pm = \mathrm{K}\left(\tfrac{1}{2}(1\pm\sqrt{1-z})\right) \qquad z = \frac{a_{23}a_{14}}{a_{21}a_{34}}$$

[CD, Klemm, Löbbert, Nega, Porkert]

- For higher loops, no representation in terms of elliptic integrals is expected to exist.

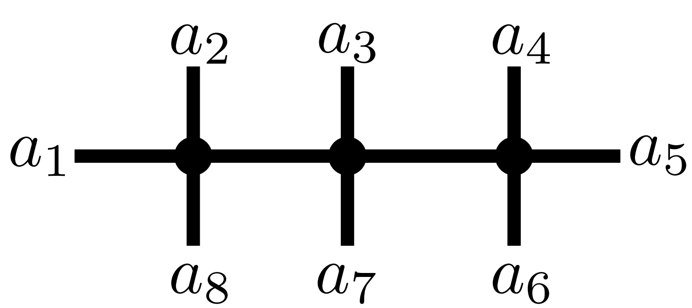
Yangian-invariant Calabi-Yau periods

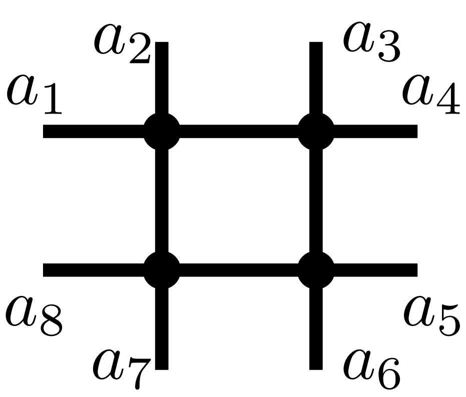
Yangian-invariant periods

- $Y(\mathfrak{sl}_2(\mathbb{R})) \subset \mathrm{PFI}$ implies that the periods are Yangian-invariants!

  ➡ Do we get all invariants (for this particular representation)?

- No! Example: The following 8-point integrals are both Yangian-invariant:



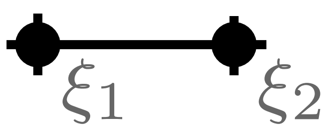


- We get all invariants with a prescribed symmetry group of the form $\mathrm{Aut}(G)$.

  ➡ Why all? The PFI is generated by $Y(\mathfrak{sl}_2(\mathbb{R}))$ and $\mathrm{Aut}(G)$, and the periods form a complete set of solutions.

Yangian-invariant periods

- The periods compute the 1D fishnet integrals!



$$= \frac{1}{[(\xi_1 - \xi_2)^2]^{D/4}} \xrightarrow{D=1} \frac{1}{\sqrt{|\xi_1 - \xi_2|}}$$

➡ Fishnet integral in 1D: $I_G^{D=1}(a) = \int_{\mathbb{R}^L} \Omega_G$

- 'Double-copy' formula for 2D fishnets?

$$I_G(a) \sim \int_{M_G} \Omega_G \wedge \overline{\Omega}_G \sim \Pi_G(a)^\dagger \Sigma \Pi_G(a)$$

$$\sim \int_{\Gamma_i} \Omega_G$$

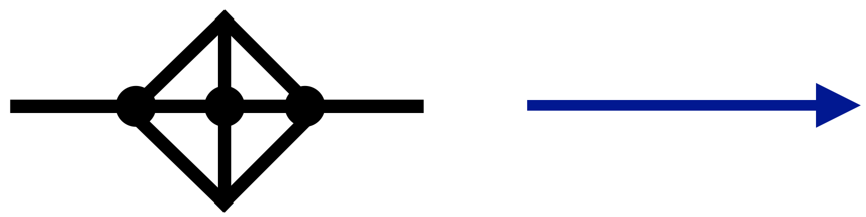
➡ Similar to KLT relation for string amplitudes.

➡ In fact, it is the single-valued map, just like in string theory!

[cf. Stieberger, Taylor; Schlotterer, Schnetz; Brown, Dupont; …]

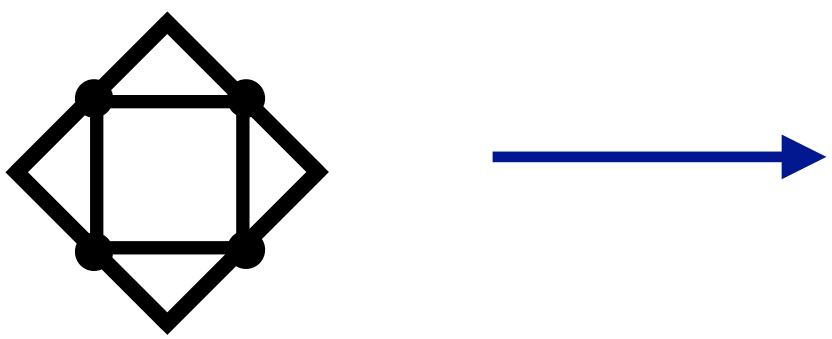
Yangian-invariant periods

- Basso-Dixon formula for 1D fishnet/ Yangian-invariant periods:



CY 3-fold / 4 Yangian-invariant periods:

$$\big(\Pi_0(z), \Pi_1(z), \Pi_2(z), \Pi_3(z)\big)$$



CY 4-fold / 5 Yangian-invariant periods:

$$\big(D_{01}(z), D_{02}(z), D_{03}(z), D_{12}(z), D_{23}(z)\big)$$

$$D_{ij}(z) = \det \begin{pmatrix} \Pi_i(z) & \Pi_j(z) \\ \partial_z \Pi_i(z) & \partial_z \Pi_j(z) \end{pmatrix}$$

1 relation from $\Pi(z)^T \Sigma \Pi(z) = 0$: $D_{13}(z) = D_{02}(z)$ [cf. Almkvist]

- In general: the periods associated to $M \times N$ fishnets $(M \leq N)$ are $M \times M$ determinants of of the (derivatives) periods of $(M+N-1)$-loop ladders graphs.

➡ Basso-Dixon formula for Yangian-invariant CY periods!

Yangian-invariant periods

- We can combine the 'double-copy' formula with the Basso-Dixon formula in 1D and 2D:

$$I_{\mathrm{FN}}^{D=2}(a) \overset{\textbf{DC}}{\sim} \Pi_{\mathrm{FN}}(a)^\dagger \Sigma_{\mathrm{FN}} \Pi_{\mathrm{FN}}(a)$$

$$\overset{\textbf{1D-BD}}{\sim} \det(\partial^k \Pi_{\mathrm{Lad}}(a))^\dagger \Sigma_{\mathrm{FN}} \det(\partial^l \Pi_{\mathrm{FN}}(a))$$

$$\overset{\textbf{2D-BD}}{\sim} \det\left(\partial^k I_{\mathrm{Lad}'}^{2D}(a)\right)$$

$$\overset{\textbf{DC}}{\sim} \det\left(\partial^k \Pi_{\mathrm{Lad}'}(a)^\dagger \Sigma_{\mathrm{Lad}'} \partial^l \Pi_{\mathrm{Lad}'}(a)\right)$$

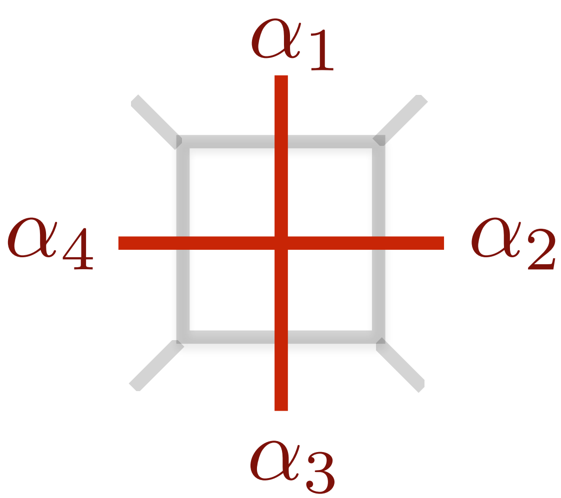
➡ Highly non-trivial relation between CY periods!

➡ Not even the loop orders of the ladder integrals involved are the same!

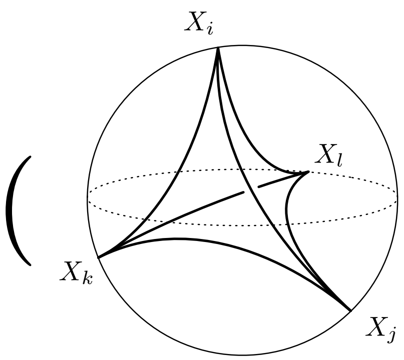
Fishnet integrals as volumes

Fishnets as volumes

- In 4D:



$$= -\frac{4}{\alpha_{13}^2\alpha_{24}^2}\frac{D(z)}{z-\bar{z}} \sim \mathrm{Vol}\Big($$



$$\Big)$$

➡ No known extension beyond 1-loop.

- In 2D: Which metric shall we use?

➡ Ω_G defines a complex structure on M_G, but no Kähler structure!

➡ We have no canonical choice of metric to compute a volume…

Mirror symmetry

- Mirror symmetry: CY ℓ-folds come in pairs (M, W) s.t.

$$\dim H^{p,q}(M) = \dim H^{\ell-p,q}(W)$$

$$H^{p,q}(M) = \text{cohomology classes of } (p,q)\text{-forms}$$

- Mirror symmetry exchanges complex and Kähler structures:

$$H^{\ell-1,1}(M) \overset{\text{MS}}{\longleftrightarrow} H^{1,1}(W)$$

Parametises complex structures on M — Parametises Kähler structures on W

$$\Omega_z \overset{\text{MS}}{\longleftrightarrow} \omega = \sum_i (\text{Im } t_i(z))\, e_i$$

e_i: Basis of $H^{1,1}(W)$

Mirror map: $$t_i(z) = \frac{\Pi_i(z)}{\Pi_0(z)} \sim \log z_i + \mathcal{O}(z^2)$$

$\Pi_i(z)$: log-divergent solutions

$\Pi_0(z)$: holomorphic solution

The classical volume

Fishnet graph G

Complex structure
Ω_G on M_G

Kähler structure
ω_G on W_G

- Classical volume:

$$\mathrm{Vol}_{\mathrm{cl.}}(W_G) = \int_{W_G} \frac{\omega_G^\ell}{\ell!} \qquad\qquad t_i^{\mathbb{R}}(z) = \mathrm{Im}\, t_i(z)$$

$$= \frac{1}{\ell!} \sum_{i_1,\cdots,i_\ell} C^{\mathrm{cl.}}_{i_1,\cdots,i_\ell}\, t^{\mathbb{R}}_{i_1}(z)\cdots t^{\mathbb{R}}_{i_\ell}(z)$$

➡ $C^{\mathrm{cl.}}_{i_1,\cdots,i_\ell}$ are (explicit computable) intersection numbers.

Fishnets as volumes

- 1 loop:

$$I_{\text{1-loop}}(a) \sim \mathrm{K}(z)\mathrm{K}(1-\bar{z}) + \mathrm{K}(\bar{z})\mathrm{K}(1-z) = 2\,|\mathrm{K}(z)|^2\,\mathrm{Im}\,\tau(z)$$

$|\Pi_{\text{1-loop},0}(z)|^2$; overall scale

$\mathrm{Vol}_{\text{cl.}}(W_{\text{1-loop}})$

$$\tau(z) = i\mathrm{K}(1-z)/\mathrm{K}(z)$$

- 2 loops:

$$I_{\text{2-loop}}(a) \sim \left(\mathrm{K}_+\,\overline{\mathrm{K}}_- + \mathrm{K}_-\,\overline{\mathrm{K}}_+\right)^2 = 4\,|\mathrm{K}_-|^4\,(\mathrm{Im}\,\tau(z))^2$$

$|\Pi_{\text{2-loop},0}(z)|^2$ $\mathrm{Vol}_{\text{cl.}}(W_{\text{2-loop}})$

$$\tau(z) = i\mathrm{K}_+/\mathrm{K}_-$$

- 3 loops:

$$I_{\text{3-loop}}(a) \not\sim |\Pi_{\text{3-loop},0}(z)|^2\,\mathrm{Vol}_{\text{cl.}}(W_{\text{3-loop}})$$

The quantum volume

- For $\ell \geq 3$, the volume receives instanton corrections:

  ➡ Quantum volume:

$$\Pi_G(z)^\dagger \Sigma \Pi_G(z) \sim |\Pi_{G,0}(z)|^2 \, \mathrm{Vol}_{\mathrm{q}}(W_G)$$

$$\sim |\Pi_{G,0}(z)|^2 \, \mathrm{Vol}_{\mathrm{cl.}}(W_G) + \mathcal{O}(e^{-t_i(z)})$$

  ➡ The same notion of quantum volume appears in string theory and geometry. [cf. e.g. Lee, Lerche, Weigand]

- Instanton corrections are absent for elliptic curves and K3 surfaces.

  ➡ The classical and quantum volumes agree for $\ell = 1, 2$.

Pure functions?

- For 1-parameter families, the periods close to $z=0$ behave like:

$$\Pi_{G,k}(z) = \Pi_{G,1}(z)\,\frac{1}{(k-1)!}\log^{k-1} z + \mathcal{O}(z)$$

(For fishnets, z is the cross ratio)

- They can be written as [CD, Klemm, Nega, Tancredi]:

$$\Pi_{G,k}(z) = \Pi_{G,1}(z)\, I(Y_0, Y_1, \ldots, Y_{k-2}; q)$$

Iterated integrals

letters = Y-invariants of the CY [cf. Bogner]

$$I(Y_0, Y_1, \ldots, Y_{k-2}; q) = \frac{1}{(k-1)!}\log^{k-1} q + \mathcal{O}(q)$$

Pure functions?

- Interesting observation: quadratic relations among CY periods turn into simple shuffle relations among iterated integrals!

$$\Pi(z)^T \Sigma \Pi(z) = 0 \quad \longleftrightarrow \quad ⧢(\mathrm{id}\otimes S)\Delta_{\mathrm{dec}} = 0$$

[cf. Nega's talk]

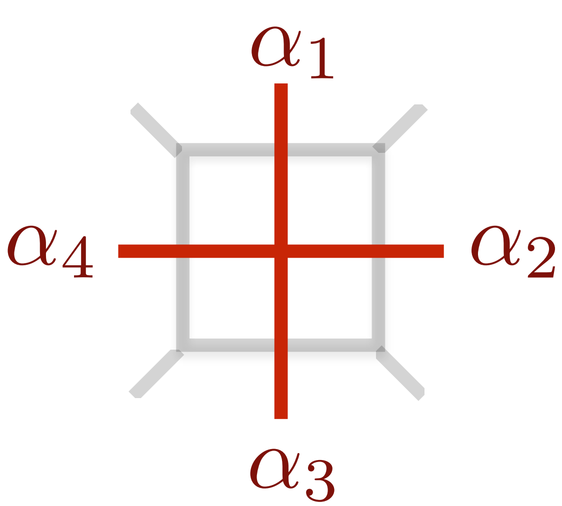
Pure functions?

$$\Pi_{G,k}(z) = \Pi_{G,1}(z)\, I(Y_0, Y_1, \ldots, Y_{k-2}; q)$$

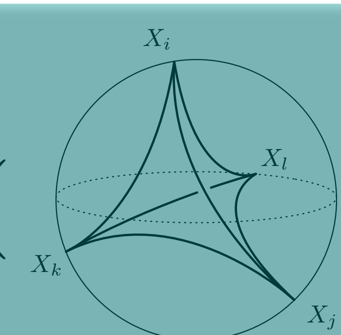
Pure functions?

$$\Pi_G(z)^\dagger \Sigma \Pi_G(z) \sim |\Pi_{G,0}(z)|^2\, \mathrm{Vol}_q(W_G)$$

- To be compared with 4D box:



$$= -\frac{4}{\alpha_{13}^2 \alpha_{24}^2} \frac{1}{z - \bar{z}} \mathrm{Vol}(\quad)$$



Bloch-Wigner dilog

Pure function

- Does the same work for more parameters?
  - ➡ Works for K3 / 2-loop, currently checking 3 loops!

Summary

Physics	CY geometry
– ℓ-loop fishnet graph G	– Family of CY ℓ-folds M_G
– Feynman integrand $\Omega_G \wedge \overline{\Omega}_G$	– $(\ell, 0)$-form Ω_G
– Cross ratios of external points	– Independent moduli
– Yangian and graph symmetries	– Picard Fuchs ideal
– Yangian invariants	– Periods
– Basso-Dixon formula	– Alternating products of PF operators.
– Feynman integral $I_G(a)$	– Quantum volume of W_G

Conclusions

- 2D Yangian-invariant fishnet integrals are closely related to Calabi-Yau geometries!

- This gives a new way to compute these integrals:

  ➡ Computation of these fishnet integrals is reduced to the computation of the periods.

  ➡ Periods are obtained form PF differential equations.

  ➡ PF differential equations are generated by Yangian and permutation symmetries.

- Bonus: first interpretation of a multi-loop integral as a volume.

  ➡ Receives instanton corrections starting from 3 loops.

Conclusions

- Implications of Yangian symmetry for geometry?
  - ➡ Basso-Dixon formula for periods?
- Implications for integrability of fishnet theories?
  - ➡ Explanation of instanton corrections at 3 loops?
  - ➡ Are there other Yangian invariants besides the periods?
- Is there a similar story for 4D fishnet integrals?
  - ➡ Volume interpretation in 4D?
  - ➡ Role of mirror symmetry in this context?

1-loop integrals as volumes