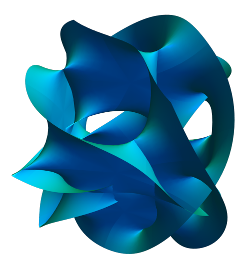
The Ice Cone Family and Iterated Integrals on Calabi-Yau Varieties



Christoph Nega

Joint work with:
Kilian Bönisch, Claude Duhr, Fabian Fischbach, Albrecht Klemm & Lorenzo Tancredi

"The Ice Cone Family and Iterated Integrals for Calabi-Yau Varieties" [1], "Feynman Integrals in Dimensional Regularization and Extensions of Calabi-Yau Motives" [2]

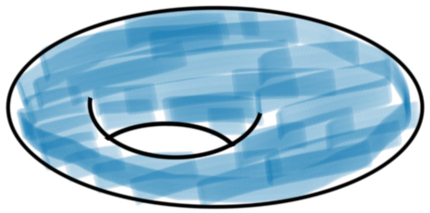
Mathematical Structures in Feynman Integrals
Siegen
February 14, 2023

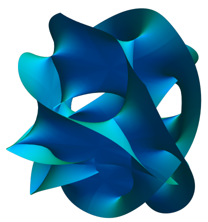
Motivation

- **Feynman integrals** are cornerstone of perturbative QFT and necessary for predictions in collider and gravitational wave experiments.

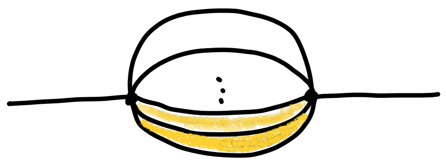
- High precision measurements require **multi-loop** Feynman integral computations.

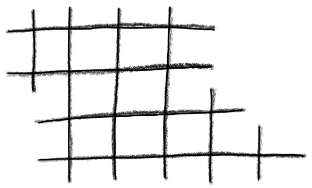
- There are many examples at two-loop order where **elliptic functions** show up. This means that these Feynman integrals have an associated non-trivial geometry.





- At higher loops we have examples where even **more complicated geometries** appear.





- Another family of Feynman integrals with Calabi-Yau geometry are the **ice cone integrals**.

Table of Content

Recap Calabi-Yau Geometries

- **Calabi-Yau manifolds** are natural generalizations of elliptic curves:

Calabi-Yaus are complex n-dim Kähler manifolds which have a unique holomorphic $(n,0)$-form

$$(\mathcal{E},\, \mathrm{d}x/y,\, \mathrm{d}x \wedge \mathrm{d}y) \qquad (X, \Omega, \omega)$$

CYs are defined via polynomial constraints

$$\{Y^2 Z - 4X^3 + g_2(t) X Z^2 + g_3(t) Z^3 = 0\} \subset \mathbb{P}^2 \qquad \{\sum_{i=0}^{4} X_i^5 - \Psi X_0 \cdots X_4 = 0\} \subset \mathbb{P}^4$$

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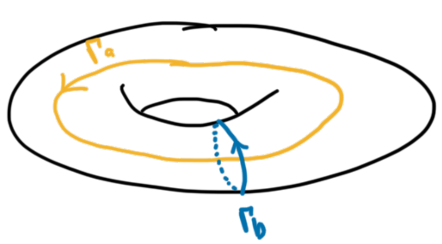
- **Period integrals** on Calabi-Yaus can be used to describe their shape and properties:

$$\begin{aligned}\Pi:\quad & H_n(X)\times H^n_{\mathrm{dR}}(X)\longrightarrow\mathbb{C}\\ &(\Gamma,\alpha)\longmapsto\int_\Gamma\alpha\end{aligned}\ ,$$



$$\alpha=\frac{\mathrm{d}X}{Y}\qquad\beta=\frac{X\mathrm{d}X}{Y}$$

$$,\qquad \Pi=\begin{pmatrix}\int_{\Gamma_a}\alpha & \int_{\Gamma_a}\beta\\ \int_{\Gamma_b}\alpha & \int_{\Gamma_b}\beta\end{pmatrix}$$

- On Calabi-Yaus we have a **monodromy invariant intersection pairing** Σ between periods:

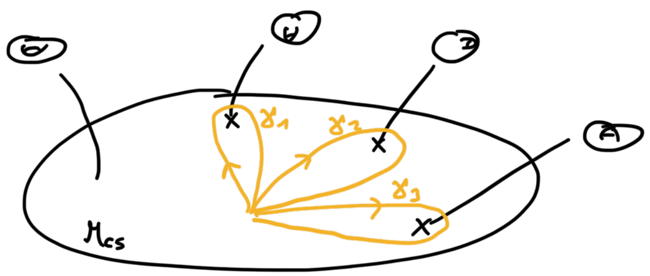
$$\underline{\Pi}^T\Sigma\underline{\Pi}\qquad\text{or}\qquad\underline{\Pi}^T\Sigma\partial_z^k\underline{\Pi}$$

Recap Calabi-Yau Geometries

- Periods are governed by differential equations: **Picard-Fuchs equation** or **Gauss-Manin system**:
  - **Point of maximal unipotent monodromy**:

    hierarchic logarithmic structure

$$\begin{aligned}\varpi_0 &= \text{power series in z}\\ \varpi_1 &= \varpi_0 \log(z) + \Sigma_1\\ \varpi_2 &= \frac{1}{2}\varpi_0 \log(z)^2 + \Sigma_1 \log(z) + \Sigma_2\\ &\vdots\end{aligned}$$



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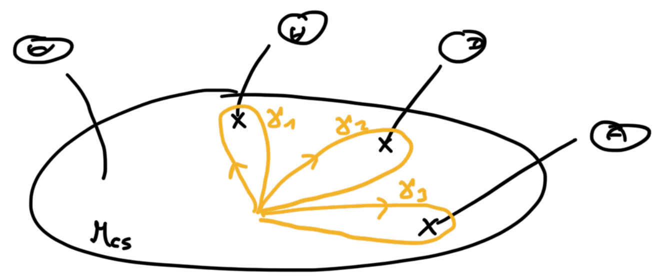

- On Calabi-Yaus there exists the phenomenon **Griffiths transversality**:

$$\begin{aligned} \Omega &\in H^{n,0}(X) \\ \partial_z \Omega &\in H^{n,0}(X) \oplus H^{n-1,1}(X) \\ &\vdots \\ \partial_z^n \Omega &\in H^{n,0}(X) \oplus \ldots \oplus H^{0,n}(X) \end{aligned}$$

  - There are **quadratic relations** between periods:

$$\int_X \Omega \wedge \partial_z^k \Omega = \Pi^T \, \Sigma \, \partial_z^k \Pi = \begin{cases} 0, & k < n \\ C_n, & k = n \end{cases}$$

  - We can simplify the **inverse Wronskian:**

$$\mathbf{W}(z)_{i,j} = \{\partial_z^i \varpi_j\}$$

$$\mathbf{W}(z)^{-1} = \Sigma \mathbf{W}(z)^T \mathbf{Z}(z)$$

Recap Calabi-Yau Geometries

- **Mirror symmetry** exchanges the complex and Kähler structure spaces of pairs of mirror Calabi-Yaus (M, W):

$$h^{n-1,1}(M) = h^{1,1}(W)$$

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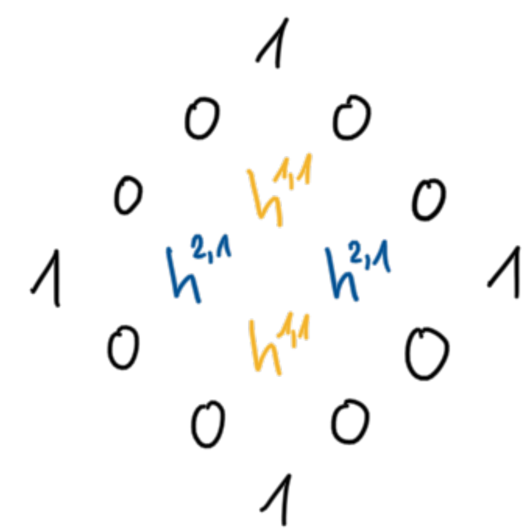

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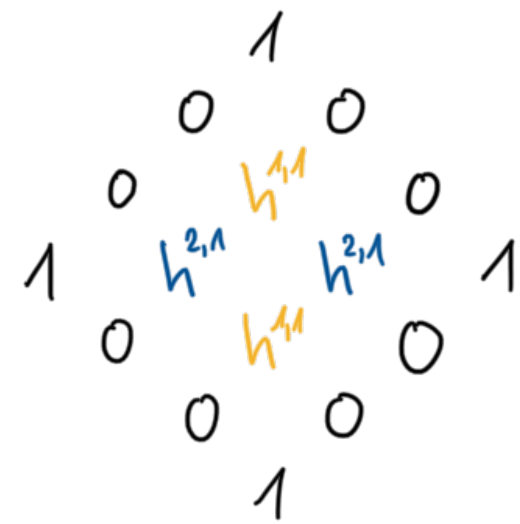
$$t(z) = \frac{\varpi_1}{\varpi_0} = \log(z) + \mathcal{O}(z)$$

W $\qquad\qquad$ M



Generalization of τ from an elliptic curve "B/A-period"

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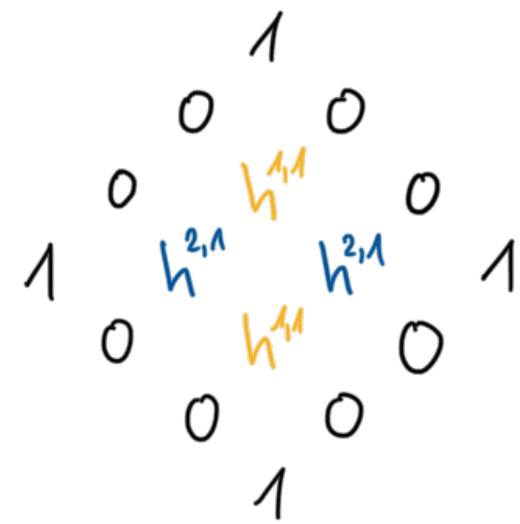

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W $\quad$ M

- CY differential operators have a special form in terms of the **canonical variable** t or rather $q = e^t$ [1]

$$\begin{aligned}
\widehat{\mathcal{L}}_{1,q} &= \theta_q^2\,, \\
\widehat{\mathcal{L}}_{2,q} &= \theta_q^3\,, \\
\widehat{\mathcal{L}}_{3,q} &= \theta_q^2 \frac{1}{Y_{3,1}} \theta_q^2\,, \\
\widehat{\mathcal{L}}_{4,q} &= \theta_q^2 \frac{1}{Y_{4,1}} \theta_q \frac{1}{Y_{4,1}} \theta_q^2\,, \\
\widehat{\mathcal{L}}_{5,q} &= \theta_q^2 \frac{1}{Y_{5,1}} \theta_q \frac{1}{Y_{5,2}} \theta_q \frac{1}{Y_{5,1}} \theta_q^2
\end{aligned}$$

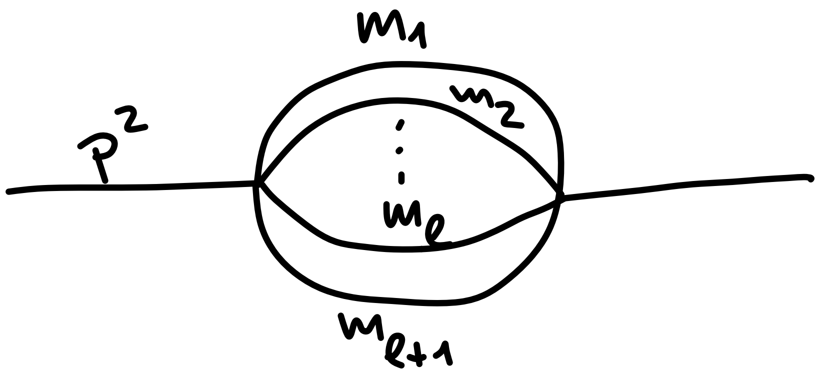
$Y_{n,k}$: invariants of CY n-fold

Normalized periods can be written as **iterated integrals**

$$\widehat{\varpi}_i = \frac{\varpi_i}{\varpi_0}$$

$$\begin{aligned}
\widehat{\varpi}_1(q) &= I(1;q) \\
\widehat{\varpi}_2(q) &= I(1, Y_{3,1};q) \\
\widehat{\varpi}_3(q) &= I(1, Y_{3,1}, 1;q)
\end{aligned}$$

Recap Banana Integrals



[2]

Two dimensions

Equal-mass and generic-mass case

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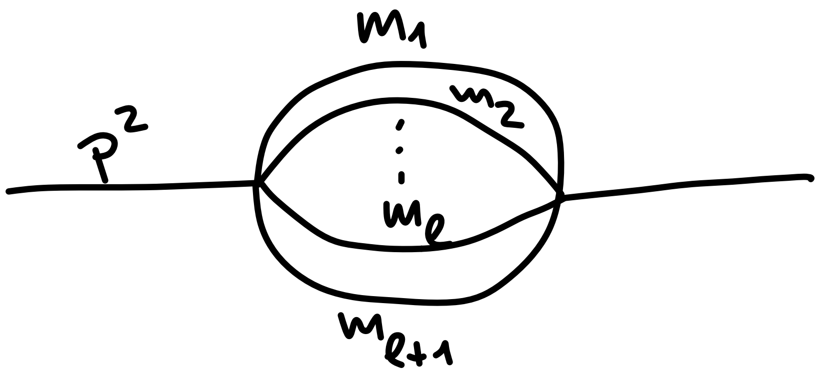

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Equal-mass and **generic-mass** case

- One can associate a **CICY geometry** to the maximal cuts: $M^{\mathrm{CI}}_{l-1} = \left\{ P_1 = P_2 = 0 \subset F_l \subset \bigtimes_{i=1}^{l+1} \mathrm{P}^1_{(i)} \right\}, \; z_i = m_i^2/p^2$

- From a GKZ approach one can construct **inhomogeneous differential eqs.**: $\mathcal{D}_r I(\underline{z}) = q_r(\underline{z}, \log(\underline{z}))$

- Banana integral is linear combination of **CY periods & special solutions**: $I(\underline{z}) = \sum_i \lambda_i \, \varpi_i(\underline{z})$

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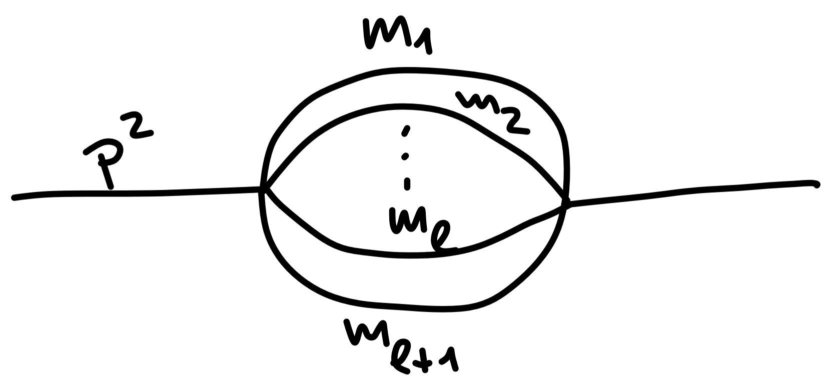

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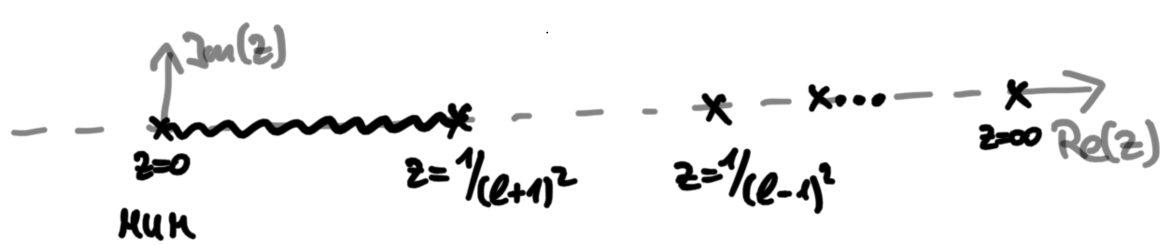
- **Boundary condition** follows from analytic properties:

$$\lambda_k^{(l)} = (-1)^k \binom{l+1}{k} \lambda_0^{(l-k)}$$



$$\sum_{l=0}^{\infty} \frac{\lambda_0^{(l)}}{(l+1)!} t^l = -\frac{\Gamma(1-t)}{\Gamma(1+t)} e^{-2\gamma t - i\pi t}$$

generating series

Recap Banana Integrals

- The **additional special solution** can be interpreted as **iterated Calabi-Yau period**:

Using variation of parameters/constants we find:

$$\underline{I}_{\mathrm{ban},l}(z) \sim \underline{\Pi}_l(z)^T \int_0^z \mathrm{d}z' \, \mathbf{W}_l(z')^{-1} \, \underline{\mathrm{Inhom}}_l(z') + \mathbf{W}_l \underline{\lambda}$$

use **quadratic relations** from Griffiths transversality

$$\sim \underline{\Pi}_l(z)^T \mathbf{\Sigma}_l \int_0^z \frac{\mathrm{d}z'}{z'^2} \, \underline{\Pi}_l(z') + \mathbf{W}_l \underline{\lambda}$$

$$\boxed{\text{Function space banana integrals} \quad \Longleftrightarrow \quad \text{iterated CY period integrals of } M_{l-1}}$$

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Function space banana integrals	$\Longleftrightarrow$	iterated CY period integrals of M_{l-1}

- We can also express everything through the **canonical variable**:

$$I_{\mathrm{ban},l} \sim \varpi_0(q) \left(\sum_{k=1}^{l} \lambda_k I(1, Y_1, \ldots, Y_{l-k-1}; q) + I(1, Y_1, \ldots, Y_1, 1, g_{\mathrm{ban}}; q) \right)$$

Pure function of weight l

Ice Cone Integrals

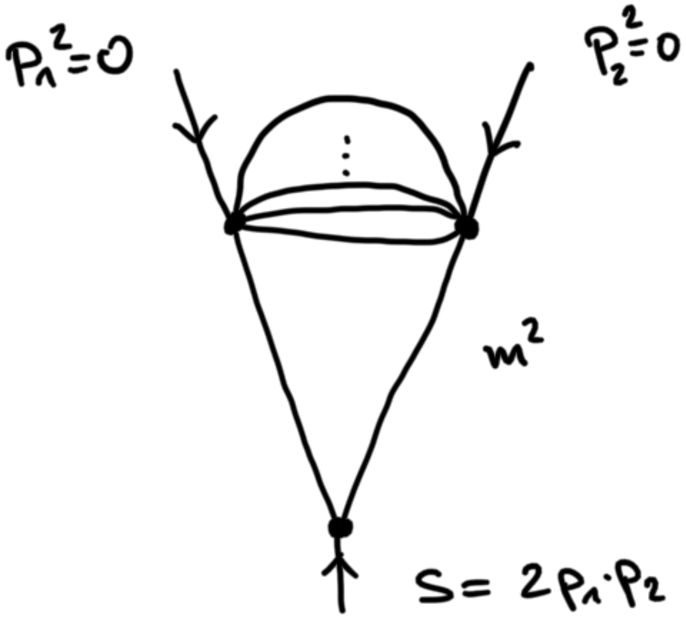
- Now we consider the family of **ice cone integrals**: [1]

external parameters: p_1 and p_2 with $p_1^2 = p_2^2 = 0$

so we have only $s = 2p_1 \cdot p_2$

internal masses: all equal to m

dimension: two



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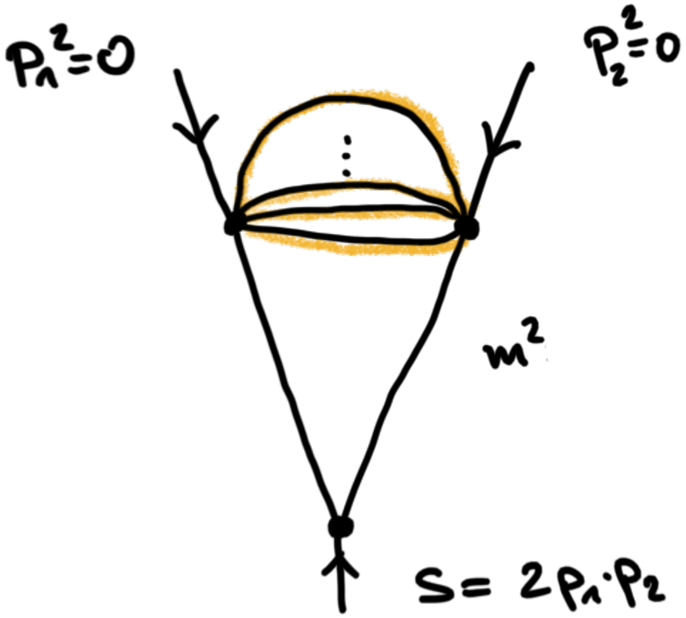

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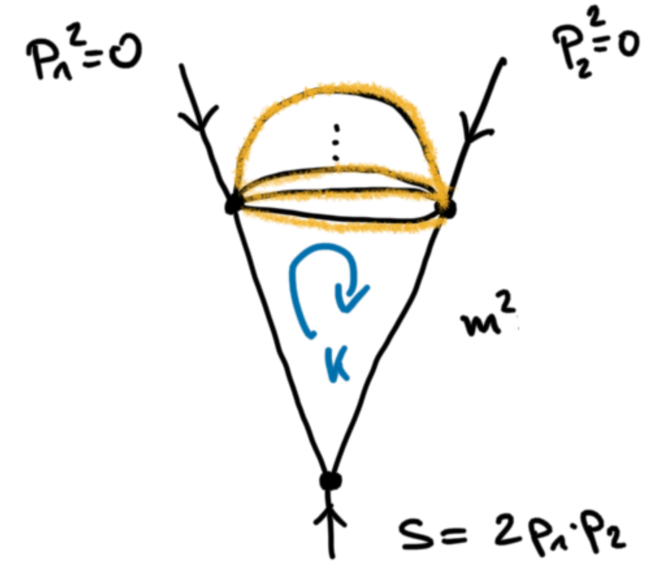
- Naively, we expect that the **banana integrals** play a prominent role for ice cone integrals since they explicitly appear in their diagrams.

- Our strategy to compute ice cone integrals has **three steps**:
  i) Find a good basis of master integrals such that the differential equations are simple.
  ii) Compute master integrals in terms of banana integrals.
  iii) Use monodromy considerations to obtain the correct linear combination.

Bananas in Ice Cones

- Consider the following representation of the ice cone:

$$I_{\text{ice}}^{(l)} = \int \frac{d^2k}{((k-p_1)^2 - m^2)((k+p_2)^2 - m^2)} I_{\text{ban}}^{(l-1)}(k^2)$$



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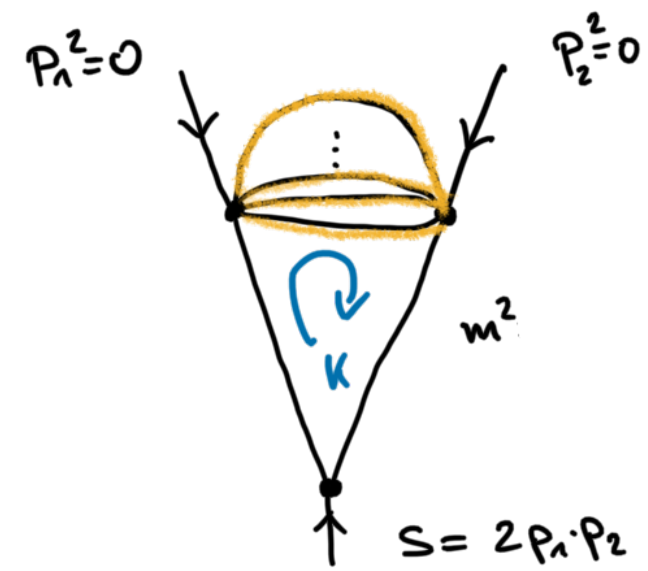

- We analyze the **maximal cuts** in with the Baikov representation:

$$I_{\text{ice, cut}}^{(l)} = \oint \frac{du}{(u - m^2 x)(u - m^2/x)} I_{\text{ban, cut}}^{(l-1)}(u)$$

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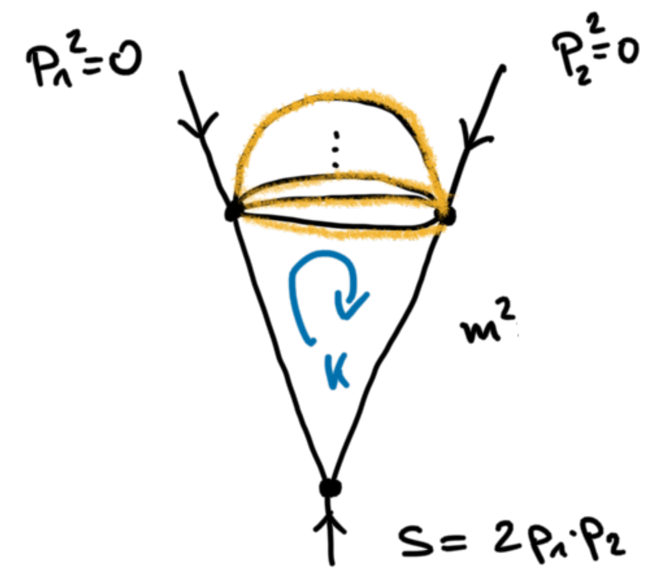

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Landau variable

have two choose two different residues

CY periods

We see now that **two copies** of the cut banana integrals appear in the cuts of ice cone:

$$\left\{I_{\text{cut, ice}}^{(l)}\right\} = \left\{I_{\text{ban, cut}}^{(l-1)}(m^2x), I_{\text{ban, cut}}^{(l-1)}(m^2/x)\right\}, \quad 2(l-1)$$

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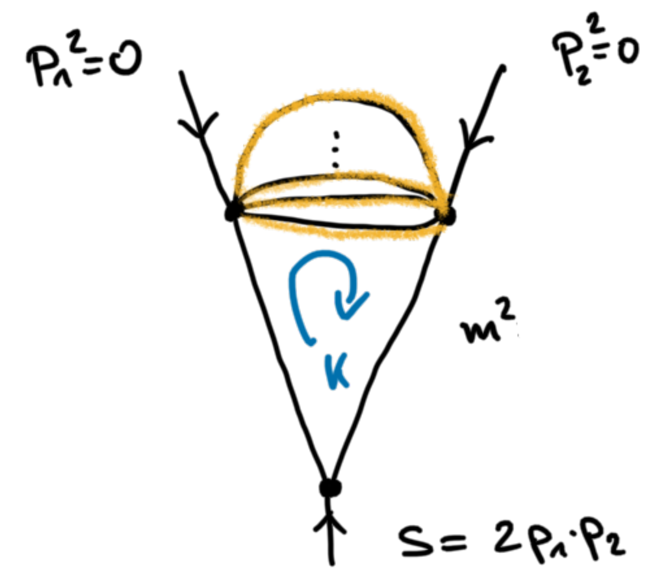

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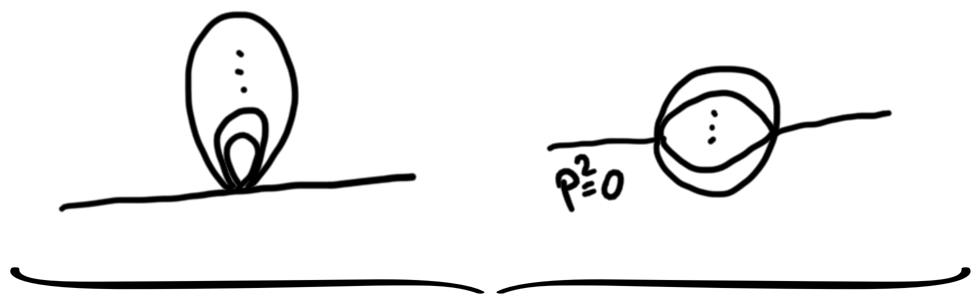
- For good basis of master integrals introduce appropriate numerators such that the **two residues decouple**.

Master Integrals and DEQs

- We found that **a good basis** of master integrals is given by: [1]

trivial master integrals:



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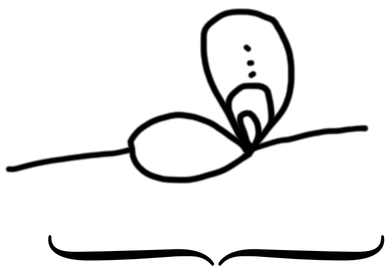

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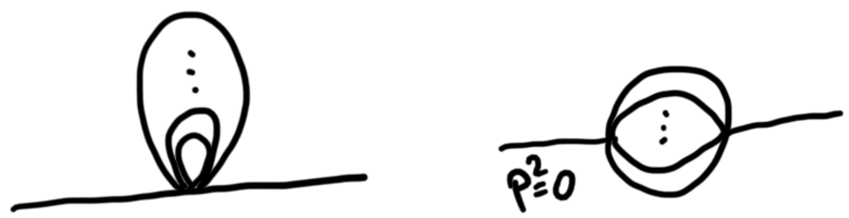

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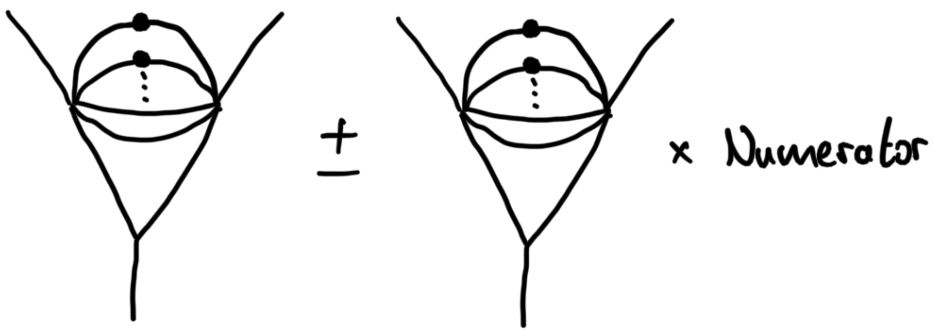

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"Gram master"

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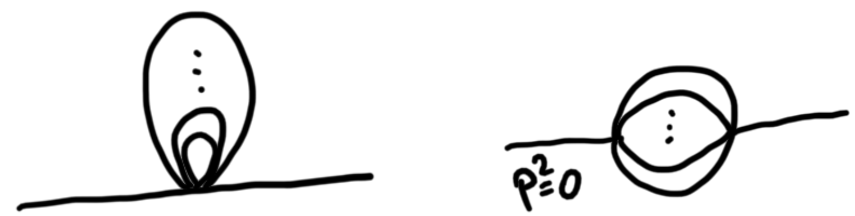

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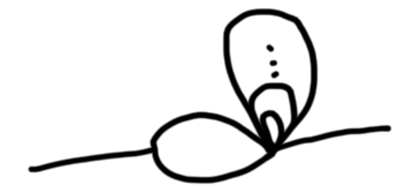

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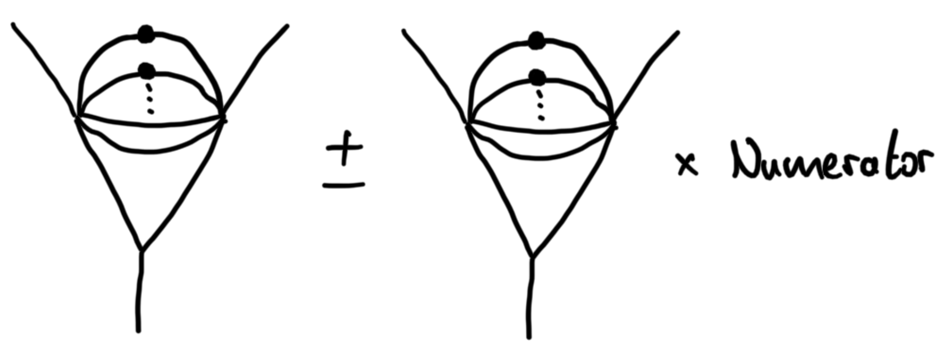

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- For this basis we can (conjecturally) write down the full GM system in two dimensions.

Master Integrals and DEQs

- The only non-trivial part of the GM system takes the simple form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{\mathcal{I}}_l^+ = \mathbf{GM}_{\mathrm{ban}}^{(l-1)}(x)\underline{\mathcal{I}}_l^+ + \underline{N}_l^+ I_0 + \mathcal{O}(d-2)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{\mathcal{I}}_l^- = \mathbf{GM}_{\mathrm{ban}}^{(l-1)}(1/x)\underline{\mathcal{I}}_l^- + \underline{N}_l^- I_0 + \mathcal{O}(d-2)$$

→ As in the banana case the master integrals of the ice cone family are **iterated CY period integrals**.

$$\underline{\mathcal{I}}_l^+ \sim \mathbf{W}_{l-1}^+ \mathbf{\Sigma}_{l-1} \int_0^x \frac{\log(x')}{x'^2} \underline{\Pi}_{l-1}(x')\mathrm{d}x' + \mathbf{W}_{l-1}^+ \underline{c}_l^+$$

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$$\mathcal{I}_{l,1}^+ \sim \varpi_0(q)\Bigg(\sum_{k=1}^{l-1} c_{l-k}^+ I(1, Y_1, \ldots, Y_{l-k-2}; q) - l!\, I(1, Y_1, \ldots, Y_1, 1, g_{\mathrm{ice}}; q)\Bigg)$$

Pure function of weight l

Master Integrals and DEQs

- The only non-trivial part of the GM system takes the simple form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{\mathcal{I}}_l^+ = \mathbf{GM}_{\mathrm{ban}}^{(l-1)}(x)\underline{\mathcal{I}}_l^+ + \underline{N}_l^+ I_0 + \mathcal{O}(d-2)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\underline{\mathcal{I}}_l^- = \mathbf{GM}_{\mathrm{ban}}^{(l-1)}(1/x)\underline{\mathcal{I}}_l^- + \underline{N}_l^- I_0 + \mathcal{O}(d-2)$$

→ As in the banana case the master integrals of the ice cone family are **iterated CY period integrals**.

$$\mathcal{I}_{l,1}^+ \sim \varpi_0(q)\left(\sum_{k=1}^{l-1} c_{l-k}^+ I(1,Y_1,\ldots,Y_{l-k-2};q) - l!I(1,Y_1,\ldots,Y_1,1,g_{\mathrm{ice}};q)\right)$$

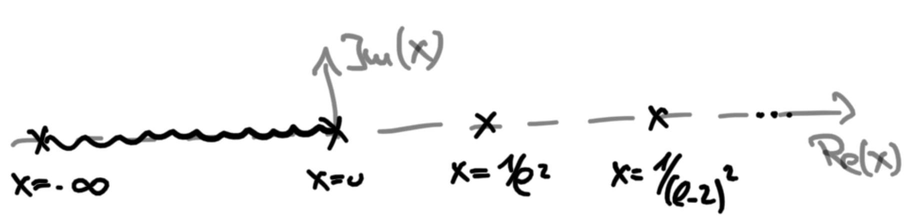
Pure function of weight l

- To fix the boundary condition we notice:

$$\mathcal{L}_{\mathrm{ice},l}\mathcal{I}_{l,1}^+ = (\theta-1)^2\mathcal{L}_{\mathrm{ban},l-1}\mathcal{I}_{l,1}^+ = 0$$

Double extension of CY operator

- From analytic properties of ice cone integral we get a **generating series** for boundary constants $\underline{c}_l^+$:

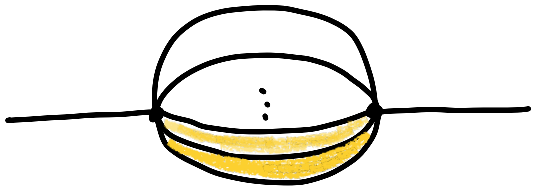


$$c_{l+1,k+1}^+ = (l+1)c_{l,k}^+$$

$$1 + \sum_{l+2}^{\infty}(-1)^{l+1}c_{l,1}^+\frac{t^l}{l!} = \Gamma(1-t)^2 e^{-2\gamma t}$$

$\hat{\Gamma}$-class ?

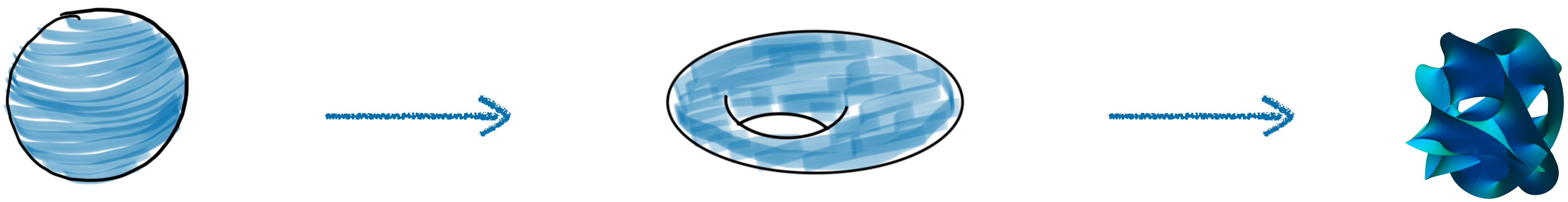
Bananas vs. Ice Cones



maximal cut geometry: $(l-1)$-dimensional CY M_{l-1}	maximal cut geometry: two $(l-2)$-dimensional CYs M_{l-2}
$$I_{\mathrm{ban},l} \sim \varpi_0(q)\Big(\sum_{k=1}^{l} \lambda_k I(1,Y_1,\ldots,Y_{l-k-1};q) + I(1,Y_1,\ldots,Y_1,1,g_{\mathrm{ban}};q)\Big)$$ pure function of weight l	$$\mathcal{I}^{+}_{l,1} \sim \varpi_0(q)\Big(\sum_{k=1}^{l-1} c^{+}_{l-k} I(1,Y_1,\ldots,Y_{l-k-2};q) - l!\,I(1,Y_1,\ldots,Y_1,1,g_{\mathrm{ice}};q)\Big)$$ pure function of weight l
$$\mathcal{L}_{\mathrm{ban},l} = (\theta-1)\mathcal{L}_{\mathrm{CY},l-1}$$ single extension	$$\mathcal{L}_{\mathrm{ice},l} = (\theta-1)^2\mathcal{L}_{\mathrm{CY},l-2}$$ double extension
generating series: $-\dfrac{\Gamma(1-t)}{\Gamma(1+t)}e^{-2\gamma t - i\pi t}$ + $\widehat{\Gamma}$-class	generating series: $\Gamma(1-t)^2 e^{-2\gamma t}$

Conclusion

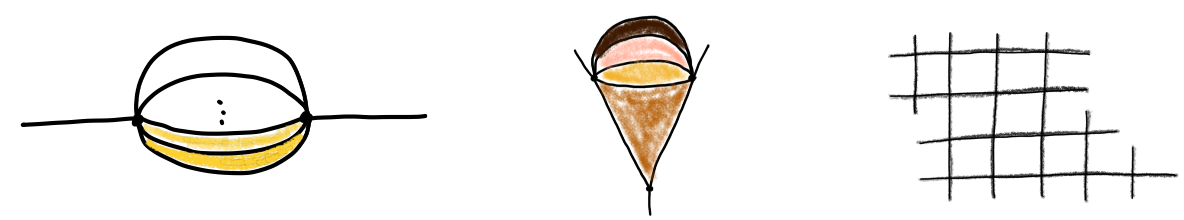
- Unterstanding CY geometries is essential for understanding higher loop Feynman integrals.



Riemann sphere — elliptic curve — Calabi-Yau

What about other geometries?

- Using CY techniques we can solve so far three different families of Feynman graphs:



- But still we have many open questions:
  - ϵ-factorized differential equation, uniform weight functions, integration kernels
  - Other families with underlying Calabi-Yau geometry?
  - Mathematical definition of iterated Calabi-Yau periods similar to elliptic polylogs

Thank you for your attention