Recent progress in the reconstruction of multi-loop amplitudes

---

Mathematical Structures in Feynman Integrals
Siegen, 14 February 2023

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Theory and Phenomenology
of Fundamental Interactions
UNIVERSITY AND INFN · BOLOGNA

INFN
Istituto Nazionale di Fisica Nucleare

erc

Tiziano Peraro - University of Bologna and INFN

Motivation

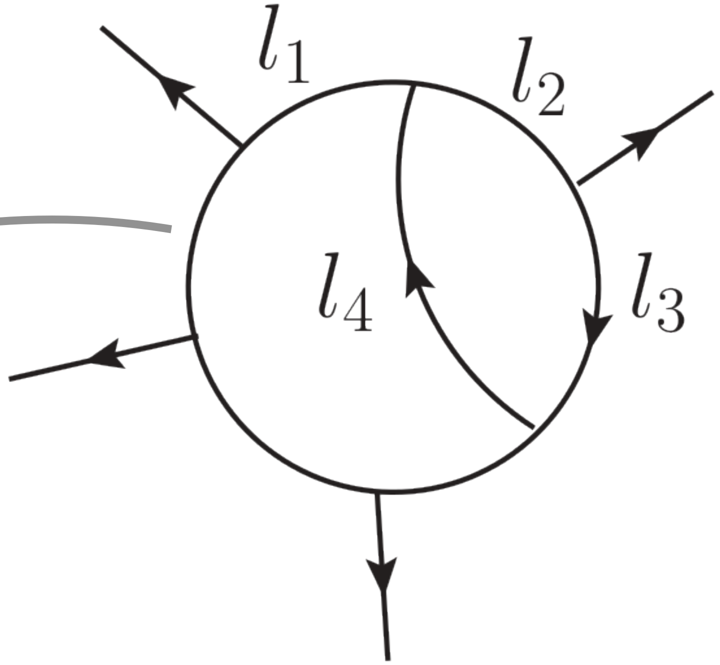
- Theoretical predictions for observables at %-level accuracy
  - search of new physics
  - test SM symmetry breaking mech.
  - high-multiplicity and masses increasingly important

- Crucial understanding of **amplitudes** and **Feynman integrals**
  - %-level ~ at least NNLO ~ 2 loops or more
- Exploiting **physical** and **mathematical structures**
  - many connections with fields of math.s and computing

Scattering amplitudes

- At the core of theoretical predictions
- Exhibit rich and interesting mathematical structures
- At the loop level, a combination of

$$\mathscr{A} = \int_{-\infty}^{\infty} \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{\mathscr{N}}{D_1 \, D_2 \, D_3 \cdots}$$



Inverse propagators

$$D_j = l_j^2 - m_j^2$$

Computing loop amplitudes

- Write **amplitudes** as a linear combination of **Feynman integrals**

$$\mathscr{A} = \sum_j a_j \; I_j$$

Rational coefficients

Integrals in a "nice/standard" form

- **Reduction** into a basis of linearly independent **master integrals** $\{G_j\} \subset \{I_j\}$

$$I_j = \sum c_{jk} G_k$$

rational coefficients

master integrals

- Compute the **master integrals** (**MIs**)

Analytic vs Algebraic complexity

- Analytic complexity

- understanding space of special functions for amplitudes
- appears in computation of MIs

- Algebraic complexity

- huge intermediate expressions
- appears in most steps if we have “many” loops, legs or scales

➡ *this talk*

From amplitudes to Feynman integrals

Integrand reduction

Ossola, Papadopoulos, Pittau (2007)

- Integrands of amplitudes as sums of irreducible contributions (at the integrand level)

$$\frac{\mathcal{N}(k)}{\prod_j D_j(k)} = \sum_T \sum_\alpha c_{T,\alpha} \frac{\mathbf{m}_T(k)^\alpha}{\prod_{j\in T} D_j(k)}$$

- the **“on-shell” integrands** $\mathbf{m}_T(k)^\alpha$
  - form a complete **integrand basis**
  - are in the “nice” form we want
- solve for unknown $c_{T,\alpha}$
  - on multiple cuts $\{D_j = 0\}_{j\in T}$ (linear system)
  - black-box polynomial reconstruction in D_j **[T.P. (2019)]**

Tensors and form factors

- Alt.: well-known decomposition of amplitudes

$$\mathscr{A} = \sum_j F_j T_j$$

T_j = **tensors** structures compatible with gauge, Lorentz and other symmetries, contracted with external polarization states

F_j = scalar **form factors**, computable at any loop order in perturbation theory

- Projecting out the form factors

$$F_j = P_j \cdot \mathscr{A}, \qquad P_j = \sum_k (T^\dagger \cdot T)^{-1}_{jk} T^\dagger_k$$

- Drawback: traditionally impractical with #legs ≥ 5

Physical tensors and projectors

A “physical” basis of tensors **[T.P., Tancredi (2019-20)]**:

$T_j \in$ set of tensor structures spanning the physical space of four-dimensional external momenta and polarizations (tHV scheme)

- Example: **5 gluon** amplitudes
  - 142 d-dimensional tensors $T_j = T_j^{\mu_1 \cdots \mu_5} \, \epsilon_{\mu_1}(p_1) \cdots \epsilon_{\mu_5}(p_5)$

combinations of $g^{\mu\nu}, p_j^{\mu}$

- Considering to 4-dim. momenta and polarizations

➡ only allow combinations of four indep. $p_j^{\mu} \longrightarrow$ 32 independent tensors !!!

➡ we can build simple projectors for the 32 helicity amplitudes

Physical tensors and projectors

- Another example: 4 fermion scattering $q\bar{q}Q\bar{Q}$
  - infinitely many tensor structures (they increase with the loop order)

Independent in four dimensions

$$T_1 = \bar{u}(p_2)\gamma_{\mu_1}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}u(p_3)\,,$$

$$T_2 = \bar{u}(p_2)\not{p}_3 u(p_1)\,\bar{u}(p_4)\not{p}_1 u(p_3)\,,$$

can be traded with orthogonal + evanescent

$$T_3 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}u(p_3)\,,$$

$$T_4 = \bar{u}(p_2)\gamma_{\mu_1}\not{p}_3\gamma_{\mu_3}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_3}u(p_3)\,,$$

$$T_5 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma_{\mu_5}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)\,,$$

$$T_6 = \bar{u}(p_2)\gamma_{\mu_1}\gamma_{\mu_2}\not{p}_3\gamma_{\mu_4}\gamma_{\mu_5}u(p_1)\,\bar{u}(p_4)\gamma^{\mu_1}\gamma^{\mu_2}\not{p}_1\gamma^{\mu_4}\gamma^{\mu_5}u(p_3)\,,$$

etc…

- Four dimensional external polarization states

➡ **only** T_1, T_2 are needed at **all loops**!

Axial couplings: $q\bar{q}gZ$

Gehrmann, T.P., Tancredi (2022)]

- **Physical tensor structures** for four-dimensional external states
  - spanned by p_i^μ $(i = 1,2,3)$ and $v_A^\mu = \epsilon^{\nu\rho\sigma\mu} p_{1\nu} p_{2\rho} p_{3\sigma}$
  - **parity-even** tensors contain even powers of v_A, which we can effectively replace using $v_A^\mu v_A^\nu \to g^{\mu\nu}$
  - **parity-odd** tensors contain one instance of v_A^μ

\# tensors = # helicity amplitudes = 12

$$
\begin{aligned}
A^{\mu\nu} = {} & \bar{u}(p_2) \not{p}_3 u(p_1) \left[F_1 p_1^\mu p_1^\nu + F_2 p_2^\mu p_1^\nu + F_3 g^{\mu\nu} + G_1 p_1^\mu v_A^\nu + G_2 p_2^\mu v_A^\nu + G_3 v_A^\mu p_1^\nu \right] \\
& + \bar{u}(p_2) \gamma^\nu u(p_1) \left[F_4 p_1^\mu + F_5 p_2^\mu \right] + \bar{u}(p_2) \gamma^\mu u(p_1) F_6 p_1^\nu \\
& + \bar{u}(p_2) \not{v}_A u(p_1) \left[G_4 p_1^\mu p_1^\nu + G_5 p_2^\mu p_1^\nu \right] + G_6 \left[\bar{u}(p_2) \gamma^\mu u(p_1) v_A^\nu + \bar{u}(p_2) \gamma^\nu u(p_1) v_A^\mu \right]
\end{aligned}
$$

Axial couplings: $q\bar{q}gZ$ @ 2 loops

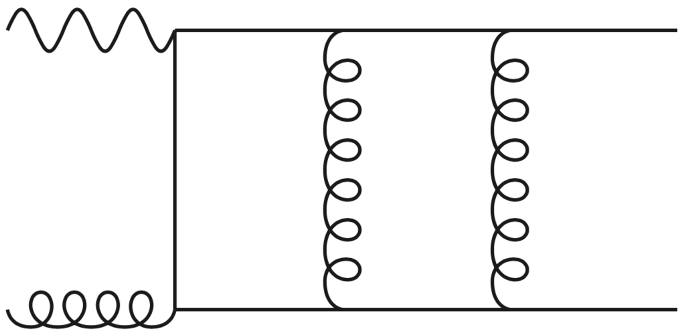
Gehrmann, T.P., Tancredi (2022)]

- Larin scheme

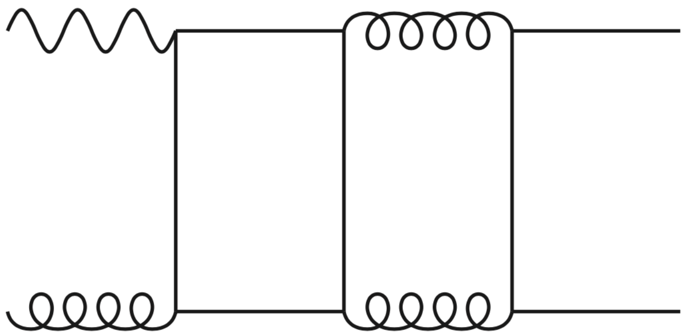
$$\gamma^{\mu}\gamma_5 \to \frac{i}{6}\epsilon^{\mu\nu\rho\sigma}\gamma_{\nu}\gamma_{\rho}\gamma_{\sigma}$$

  - Levi-Civita tensors “disappear” when contracted with the ones appearing in the projectors (inside the definition of v_A)

- First **explicit** calculation of axial non-singlet contributions
  - agreement with results derived from vector ones **[Garland, Gehrmann, Glover, Koukoutsakis, Remiddi (2002)]**



- **New results** for **axial singlet contributions**
  - include finite top-loop contributions in $m_t \to \infty$ limit
  - checked UV and IR consistency up to $\mathcal{O}(1/m_t^2)$



Finite fields and rational reconstruction

Finite fields and rational reconstruction

- A successful idea for dealing with **algebraic complexity** **[Kant (2014), von Manteuffel, Schabinger (2014), T.P. (2016)]**

- Reconstruct **analytic results** from **numerical evaluations**
  - intermediate steps are numbers instead of complicated expressions

- Evaluations over **finite fields** $\mathscr{Z}_p$ (computing modulo a **prime** p)

$$\mathscr{Z}_p = \{0,1,\ldots,p-1\}$$

- Use machine-size integers $p < 2^{64}$ (**fast** and **exact**)

- Collect numerical evaluations and infer **analytic result** form them

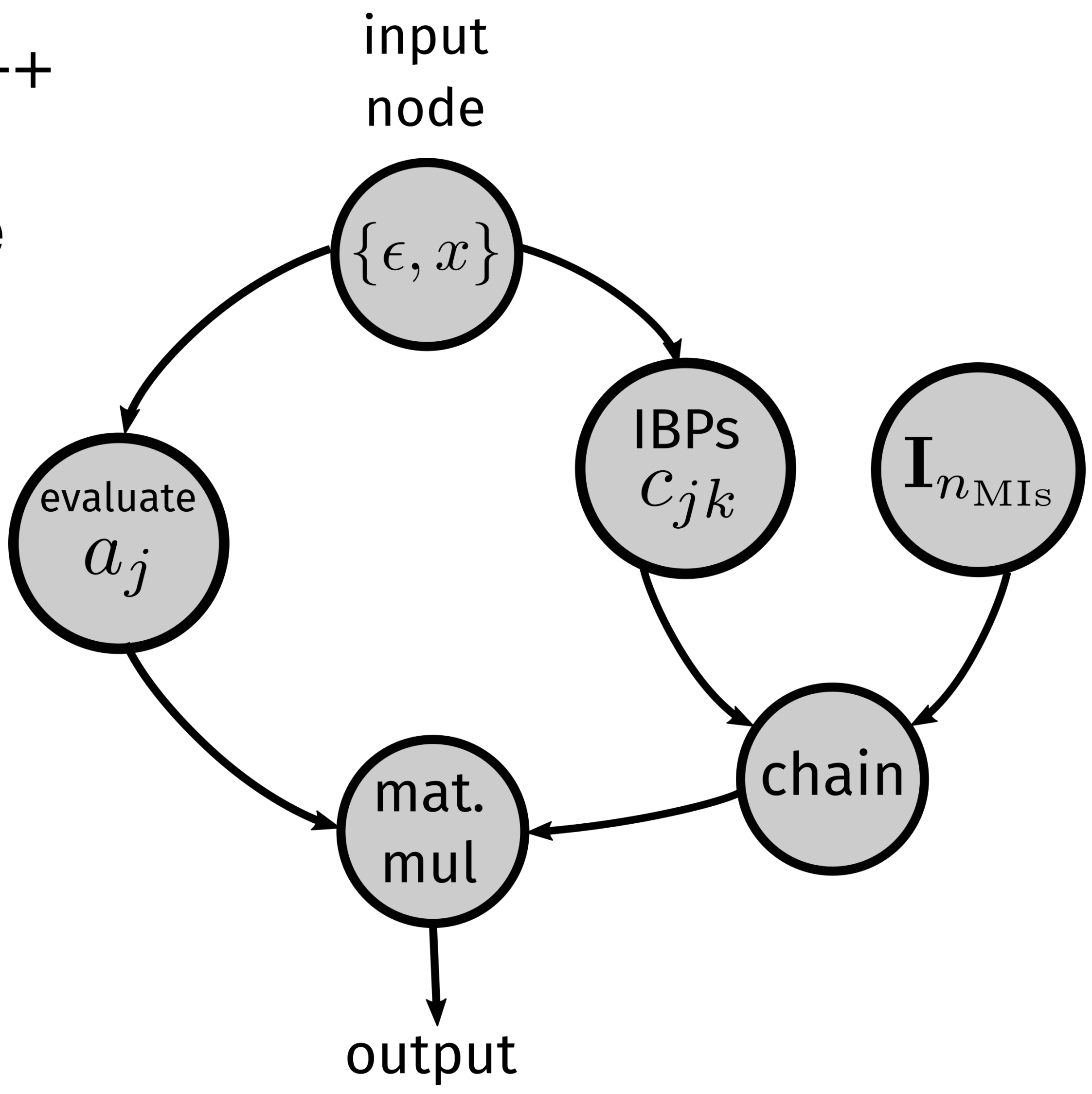
Finite fields and rational reconstruction

- Applicable to any **rational algorithm**
- Sidesteps appearance of large intermediate expressions
- Massively **parallelizable**
  - numerical evaluations are independent of each other
  - algorithm-independent parallelization strategy
- Yielded some of the most impressive multi-loop results to date
- Examples of known codes using it:
  FinRed, FiniteFlow, FireFly+Kira, Fire, Caravel

FiniteFlow

T.P. (2019)

- Builds numerical algorithms via a high-level interface
- Combines core algorithms into a computational graph
  - graph evaluation implemented in C++
- Usable as a *Mathematica* package
  - build efficient implementations of custom algorithms
  - reconstruct analytic results
- Produced many cutting-edge multi-loop results



IBP reduction to master integrals

- IBPs are **large** and **sparse** linear systems
- they reduce Feynman integrals I_j to a lin. indep. set of MIs G_j

$$I_j = \sum_{jk} c_k\, G_k$$

- amplitudes can be reduced mod IBPs

$$\mathscr{A} = \sum_j a_j\, I_j = \sum_{jk} a_j\, c_{jk}\, G_k = \sum_j A_j\, G_j \qquad \text{with } A_j = \sum_k a_k\, c_{kj}$$

- final results for A_k often much simpler than c_{ij}

➡ solve IBPs numerically and compute A_j

Coefficients of the ϵ-expansion

- **If** MIs are known in terms of special functions f_k

$$G_j = \sum_k g_{jk}(\epsilon, x) f_k + \mathcal{O}(\epsilon)$$

ϵ = dimensional regulator

x = kinematic variables + masses

- we plug these into the amplitude

$$\mathscr{A} = \sum_k u_k(\epsilon, x) f_k + \mathcal{O}(\epsilon)$$

- we first reconstruct the coefficients **only** in ϵ (for numerical x) to evaluate the expansion of the amplitude (and then we reconstruct in x)

$$u_k(\epsilon, x) = \sum_{j=-p}^{0} u_k^{(j)}(x)\, \epsilon^j + \mathcal{O}(\epsilon)$$

Partial fractions

- Reconstructed results come out **collected** and **GCD-simplified**

- **partial fractioning** is known to yield simplifications

- multivariate partial fractions require some care
  (uniqueness of result, avoiding spurious denominators)

- modern implementations use some **algebraic geometry**
  [Abreu, Dormans, Cordero, Ita, Page, Sotnikov (2019)
  Boehm, Wittmann, Wu, Xu, Zhang (2020)
  Heller, von Manteuffel (MultivariateApart,2021)]

$1/d_j(x_i) \to q_j \quad \Rightarrow \quad$ reduction mod $\langle q_1 d_1(x_i) - 1, \ldots, q_n d_n(x_i) - 1 \rangle$

with an appropriate monomial ordering $(q_j > x_k)$

Partial fractions and reconstruction

Partial fractioned results
are often much simpler…

… but require prior
analytic knowledge
of full result

- Simplifying the reconstruction
  - guess denominator factors
    e.g. from the "letters" l_k from univariate slices $x_i = a_i \tau + b_i$ ($a_i, b_i =$ random integers)
  - reconstruct in one variable or two
  - univariate/bivariate partial fraction
  - reconstruct in all other variables
  - applied e.g. to $3g + 2\gamma$ @ 2 loops and other processes
    [Badger, Brønnum-Hansen, Chicherin, Gehrmann, Hartanto, Henn, Marcoli, Moodie, Zoia, T.P. (2021)]

$$u(x_i) = \frac{n(x_i)}{\prod_k l_k(x_i)^{\alpha_k}}$$

IBPs and syzygies

IBP reduction

Chetyrkin, Tkachov (1981), Laporta (2000)

- Feynman integrals obey linear relations, e.g. IBPs

$$\int \left(\prod_j d^d k_j \right) \frac{\partial}{\partial k_j^\mu} v^\mu \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots} = 0, \qquad v^\mu \in \{p_i^\mu, k_i^\mu\}$$

- Very **large** and sparse linear system
  ⇒ yields reduction to MIs

$$I_j = \sum_k c_{jk} G_k$$

- Often a **huge bottleneck**!

- Very active research on direct decomposition approaches

  ➡ see e.g. Gaia's and Pierpaolo's talks

Lowering the complexity of IBP systems

- IBP relations contain **higher-powers of propagators**

$$0 = \int \frac{\partial}{\underline{\partial k_j^\mu}} v^\mu \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots} = -\nu_1 \int \left(v^\mu \frac{\partial D_1}{\partial k_j^\mu} \right) \frac{1}{D_1^{\nu_1+1} D_2^{\nu_2} \cdots} + \cdots$$

- many of these don't contribute to the amplitude

- can we build a system without them? **[Gluza, Kajda, Kosower (2011)]**

$$\sum_j \int \frac{\partial}{\partial k_j^\mu} v_j^\mu \frac{1}{D_1^{\nu_1} D_2^{\nu_2} \cdots} = 0, \qquad v_j^\mu = \sum_m \alpha_{jm} p_m^\mu + \sum_n \beta_{jn} k_n^\mu$$

$$\sum_j v_j^\mu \frac{\partial D_i}{\partial k_j^\mu} = \gamma_i D_i, \qquad \text{for all } i \text{ with } \nu_i > 0$$

- **syzygy equations** for polynomials
$\alpha_{jm} = \alpha_{jm}(D_i),\ \beta_{jm} = \beta_{jm}(D_i),\ \gamma_j = \gamma_j(D_i)$

Syzygy equations

- A syzygy equation has the form

$$\mathbf{f}(\mathbf{z}) \cdot \mathbf{g}(\mathbf{z}) = \sum_{j=1}^{n} \underbrace{f_j(\mathbf{z})}_{\text{known polynomials}} \, \underbrace{g_j(\mathbf{z})}_{\text{unknown polynomials}} = 0$$

- can be solved via linear algebra by making an ansatz for g_j
  [see also Schabinger (2012)]

- if $\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(M)}$ are generators of the solutions, then any solution

$$\mathbf{g}(\mathbf{z}) = \sum_{j=1}^{M} \underbrace{p_j(\mathbf{z})}_{\text{arbitrary polynomials}} \, \mathbf{g}^{(j)}(\mathbf{z})$$

IBPs in the Baikov representation

$$I = \int \left(\prod_{i=1}^{\ell} d^d k_i \right) \frac{1}{D_1^{\nu_1} \cdots D_n^{\nu_n}} = C \int dz_1 \cdots dz_n \frac{B(z_1, \ldots, z_n)^\gamma}{z_1^{\nu_1} \cdots z_n^{\nu_n}}$$

B = Baikov polynomial, $\gamma = (d - \ell - e - 1)/2$

- **Integration by Parts**

$$0 = \sum_j \int \frac{\partial}{\partial z_j} \left(B^\gamma \frac{a_j(z_1, \ldots, z_n)}{z_1^{\nu_1} \cdots z_n^{\nu_n}} \right)$$

$$0 = \sum_j \int \left(\frac{\partial a_j}{\partial z_j} + \frac{\gamma}{B} a_j \frac{\partial B}{\partial z_j} - \nu_j \frac{a_j}{z_j} \right) \frac{B^\gamma}{z_1^{\nu_1} \cdots z_n^{\nu_n}}$$

dim. shifted

higher powers

IBPs in the Baikov representation

Ita (2016), Larsen, Zhang (2016)

$$0 = \sum_j \int \left(\frac{\partial a_j}{\partial z_j} \boxed{+ \frac{\gamma}{B} a_j \frac{\partial B}{\partial z_j}} \boxed{- \nu_j \frac{a_j}{z_j}} \right) \frac{B^\gamma}{z_1^{\nu_1} \ldots z_n^{\nu_n}}$$

Syzygy eq.s $(i) \quad \sum_j a_j \frac{\partial B}{\partial z_j} = b_0 \, B$ $\qquad (ii) \quad a_j = z_j \, b_j$

(i) and (ii) have simple closed-form solutions
[Böhm, Georgoudis, Larsen, Schulze, Zhang (2018)]

Three alternative approaches:

1. plug (ii) into (i) and solve
2. combine (i) and (ii) with alg. geometry (module intersections)
   [Böhm, Georgoudis, Larsen, Sch¨onemann, Zhang]
3. put solutions of (i) in a **matrix** and **Gauss-eliminate** higher-powers
   [von Manteuffel]
   - exploit **fast linear solvers over f.f.** + **rational reconstruction**
   - avoid reconstruction of complicated solutions

IBPs in the Baikov representation

- Syzygies yield new **parametric identities** for each sector
  - then proceed as in traditional Laporta alg.
- Identities can be used in **integrand bases** (see e.g. numerical unitarity **[Ita et al. (2016)]**)
- Can be combined with traditional Laporta identities
  - e.g. to fix incomplete reductions or avoid complex syzygy solutions
- **Up to ~ 10x improvements in efficiency**

Conclusions

- **Amplitudes** and **loop integrals**
  - are at the core of theoretical predictions
  - exhibit rich and interesting **mathematical structures**
- Math. structures are exploited by **modern methods**
  - integrand reduction, tensor decomposition, new reduction techiques, finite fields and rational reconstruction…
- Many interesting future directions
  - direct decomposition of integrals, ansatzes for amplitudes, bases of functions, improvements to reconstruction…