Computing intersection numbers with a rational algorithm

Mathematical structures in Feynman integrals

Siegen
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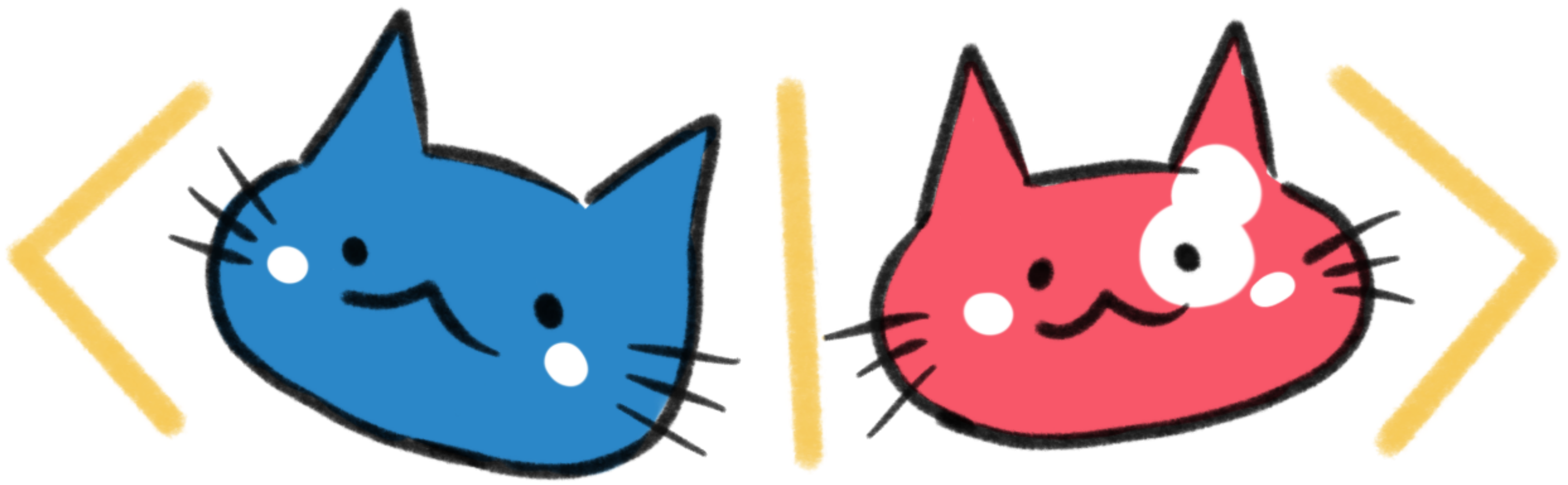

Gaia Fontana
University of Zürich
In collaboration with Tiziano Peraro

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Universität Zürich[UZH]

Outline

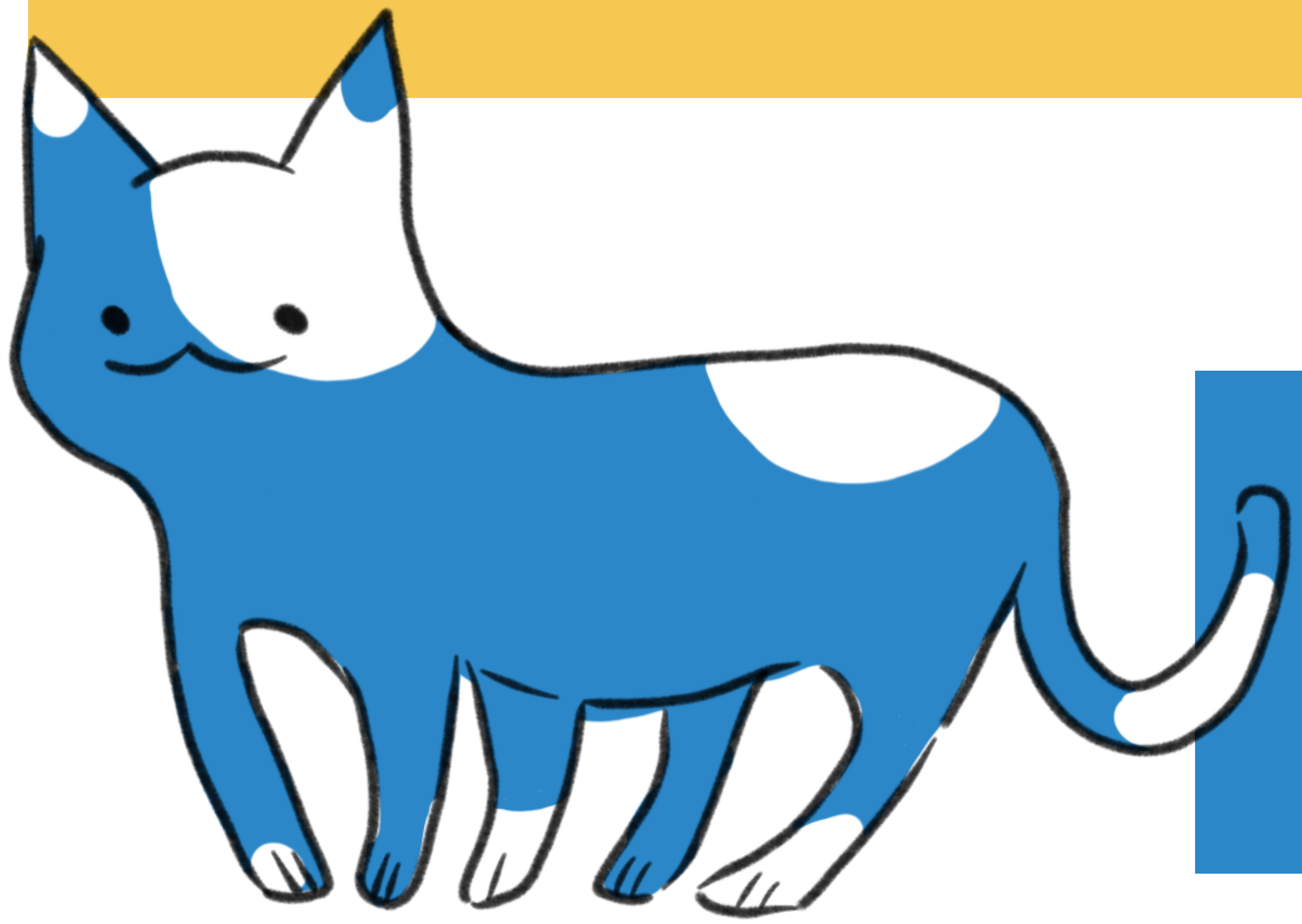
- ☐ why we do what we do (that is, decomposing Feynman integrals)
- ☐ new ways of doing that: a fast dive into intersection theory
- ☐ how can we make it better? rational algorithms!
- ☐ proof-of-concept implementation over finite-fields
- ☐ some examples that it actually works

Precision calculations in HEP

LHC High-Luminosity upgrade → % level precision

- @ least NNLO to match with experimental precision
- High precision calculation of perturbative scattering amplitudes

algebraic bottlenecks

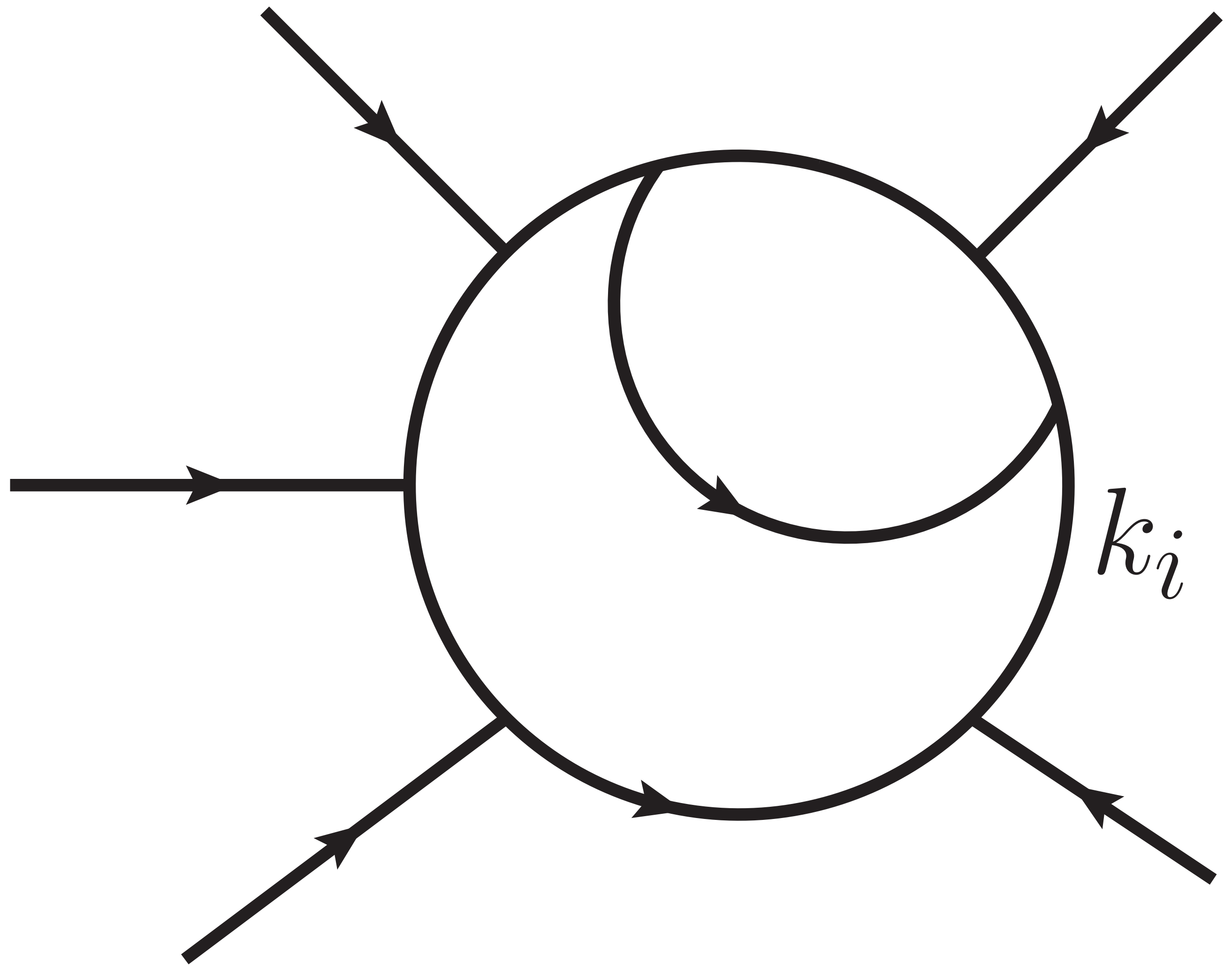


many legs

many loops

Reduction to master integrals

- scattering amplitudes → linear combination of Feynman integrals



$$A = \sum_j c_j I_j$$

with

$$I_j = \int_{-\infty}^{+\infty} \left(\prod_{i=1}^{L} \mathrm{d}^d k_i \right) \frac{1}{z_1^{\alpha_1} z_2^{\alpha_2} \dots} \qquad \alpha_i \gtrless 0$$

- not all are linearly independent!

- reduction to a minimal linearly independent set of **master integrals**

$$I_i = \sum_j c_{ij} G_j$$

Laporta algorithm

- reduction as solution of a large and sparse IBP system

$$\int \left(\prod_i \mathrm{d}^d k_i \right) \frac{\partial}{\partial k_i^{\mu}} \left(\frac{v_j^{\mu}}{z_1^{\alpha_1} z_2^{\alpha_2} \dots} \right) = 0$$

Drawbacks

- very large and sparse system
- algebraic structure not manifest

Intersection theory

- allows for a direct decomposition
- exploits the vector space structure obeyed by Feynman integrals

we consider n-folds integrals in $\mathbf{z} = (z_1, \ldots, z_n)$

integrals

$$|\varphi_R\rangle = \int dz_1 \cdots dz_n \, \frac{1}{u(\mathbf{z})} \, \varphi_R(\mathbf{z})$$

"dual" integrals

$$\langle\varphi_L| = \int dz_1 \cdots dz_n \, u(\mathbf{z}) \, \varphi_L(\mathbf{z})$$

φ_L / φ_R rational functions

$$u(\mathbf{z}) = \prod_j B_j(\mathbf{z})^{\gamma_j}$$

- γ_j are generic exponents
- $B_j(\mathbf{z})$ are polynomials

Change of representation

Baikov change of vars

$$k_j \to z_j$$

Baikov (1996)

$$I[\alpha_1, \ldots, \alpha_n] = \int \prod_{i=1}^{L} \mathrm{d}^d k_i \prod_{j=1}^{n} \frac{1}{z_j^{\alpha_j}} \to \int \mathrm{d}z_1 \ldots \mathrm{d}z_n B^\gamma \prod_{j=1}^{n} \frac{1}{z_j^{\alpha_j}} = |\varphi_R\rangle$$

with

$$|\varphi_R\rangle = \int dz_1 \cdots dz_n \, \frac{1}{u(\mathbf{z})} \, \varphi_R(\mathbf{z})$$

$$u(\mathbf{z}) = B^{-\gamma}$$

$$\gamma = \frac{d - E - L - 1}{2}$$

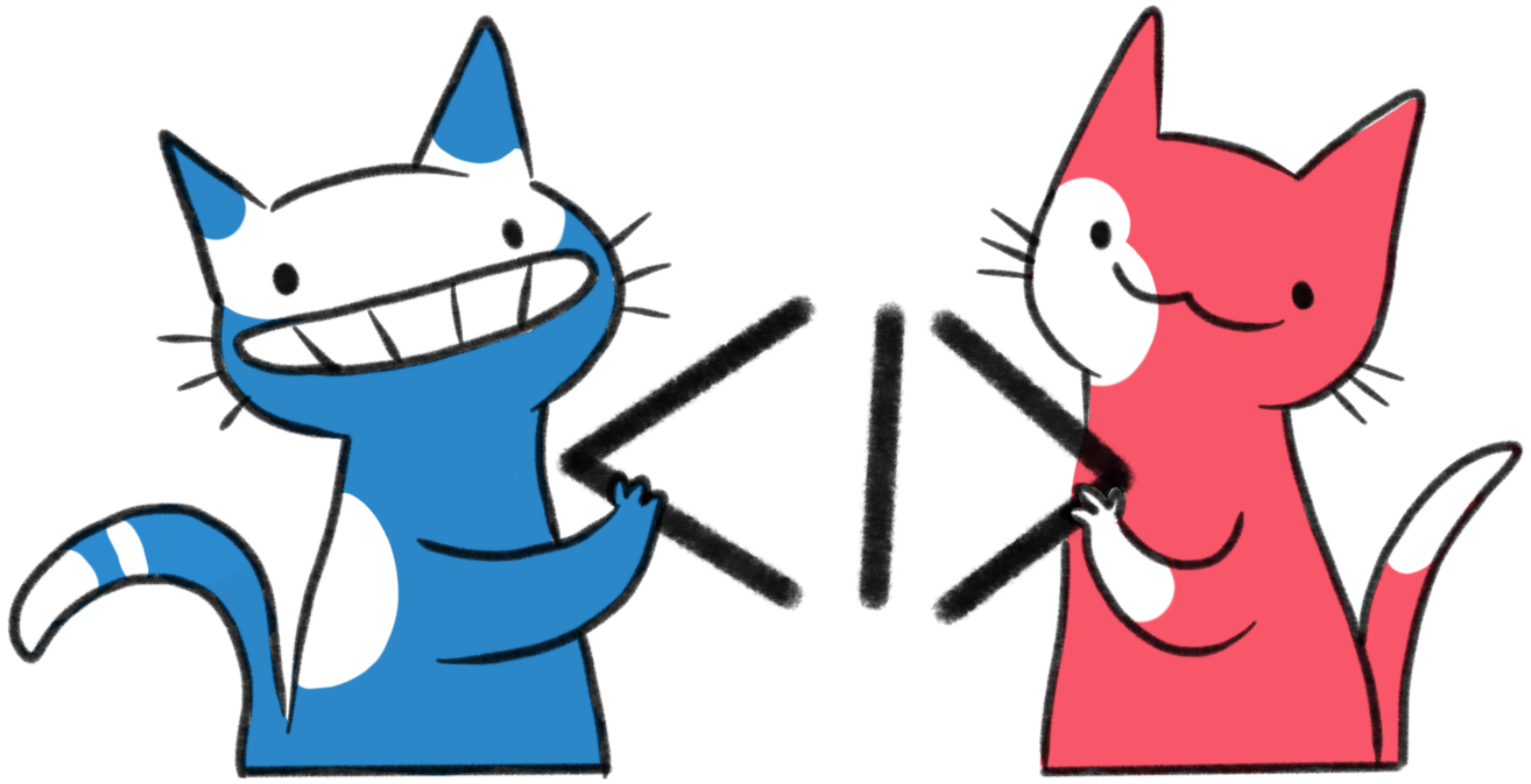
$$\varphi_R(\mathbf{z}) = \frac{1}{z_1^{\alpha_1} \ldots z_n^{\alpha_n}}$$

intersection numbers

calculation of scalar products between left and right integrals

Mastrolia, Mizera (2018) $\langle \varphi_L | \varphi_R \rangle$

they're rational!



Vector space characterized by:

- Dimension ν
- Basis $|e_i^{(R)}\rangle$ and dual basis $\langle e_i^{(L)}|$
- Scalar product: **intersection number**

identifications

- $|\varphi\rangle$ generic vector
  → Feynman integral to reduce
- $\{\,|\,e_i^{(R)}\rangle\}_{i=1}^{\nu}$ basis vectors
  → master integrals

decomposition of integrals as

$$|\varphi_R\rangle = \sum_{i=1}^{\nu} c_i^{(R)} |e_i^{(R)}\rangle \qquad c_i^{(R)} = \sum_{j=1}^{\nu} \left(\mathbf{C}^{-1}\right)_{ij} \langle e_j^{(L)} | \varphi_R\rangle$$

* similar formulae for dual integrals

where we introduced the metric

$$\mathbf{C}_{ij} = \langle e_i^{(L)} | e_j^{(R)} \rangle$$

Univariate algorithm

we have 1-folds integrals in the variable z

$$|\varphi_R\rangle = \int dz \, \frac{1}{u(z)} \, \varphi_R(z)$$

Frellesvig et al. (2019)

1-forms intersection numbers

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}_\omega} \mathrm{Res}_{z=p} (\psi \, \varphi_R)$$

where ψ is the local solution of

$$(\partial_z + \omega)\psi = \varphi_L \qquad \omega \equiv \frac{\partial_z u}{u}$$

around each $p \in \mathcal{P}_\omega$

$$\mathcal{P}_\omega = \{ \, z \, | \, z \, \text{is a pole of} \, \omega \, \} \bigcup \{\infty\}$$

Ansatz around p

$$\psi = \sum_{\min}^{\max} c_i (z-p)^i + O((z-p)^{\max+1})$$

$$(\partial_z + \omega)\psi = \varphi_L$$

- plug the ansatz in the differential equation
- solve for the c_i

Intersection numbers are always rational functions of the kinematic invariants and of the dimensional regulator (after sum over poles)

Multivariate algorithm

we have n-folds integrals in the variables $\mathbf{z} = (z_1, \ldots, z_n)$

$$|\varphi_R\rangle = \int dz_n \, |\varphi_R\rangle_{n-1}$$

$|\varphi_R\rangle_{n-1}$ is an $(n-1)$-fold integral in $z_1, \ldots z_{n-1}$

$$|\varphi_R\rangle_{n-1} = \sum_{j=1}^{\nu_{(n-1)}} \varphi_{R,j} \, |e_j^{(R)}\rangle_{n-1}$$

basis of master forms for the $(n-1)^{th}$ layer

we write n-forms as a function of $(n-1)$-forms by projecting each integral in the $(n-1)$-forms basis.

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p \in \mathcal{P}_{\mathbf{\Omega}}} \mathrm{Res}_{z_n = p} \Big(\psi_j \, \langle e_j^{(L)} | \varphi_R \rangle_{n-1} \Big)$$

- where ψ_j is the local solution of

$$\partial_{z_n} \psi_j + \psi_k \, \mathbf{\Omega}_{kj} = \varphi_{L,j} , \qquad (j = 1, \ldots, n)$$

- solved locally with the **ansatz**

$$\psi_j = \sum_{k=\min}^{\max} c_{jk} \, (z_n - p)^k + O((z_n - p)^{\max+1})$$

- $\mathbf{\Omega}$ solves the differential equation

$$\partial_{z_n} \langle e_j^{(L)} |_{n-1} = \sum_k \mathbf{\Omega}_{jk} \, \langle e_k^{(L)} |_{n-1}$$

- and

$$\mathcal{P}_{\mathbf{\Omega}} = \{ \, z \, | \, z \, \text{is a pole of} \, \mathbf{\Omega} \, \} \bigcup \{ \infty \}$$

- integrands are rational
- intersection numbers are rational

BUT

- non-rational contributions in intermediate stages
- after taking the sum over all residues we see cancellations

non-rational terms in the poles of ω and $\mathbf{\Omega}$

$$\mathcal{P}_{\omega} \qquad \mathcal{P}_{\mathbf{\Omega}}$$

- **computational bottleneck**
- **non-suitable for applications with finite-fields**

$p(z)$-adic series expansion

p-adic numbers

expansion of a rational number as series expansion of a prime number p with coefficients given by remainder of integer division

$p(z)$-adic functions

expansion of a rational function as series expansion of a prime polynomial $p(z)$ with coefficients given by remainder of polynomial division

def

$$f(z) = \sum_{i=\min}^{\infty} c_i(z) p^i(z), \quad c_i(z) = \sum_{j=0}^{\deg p - 1} c_{ij} z^j, \quad c_{ij} \in \mathbb{Q}$$

shortcut

$$p(z) \equiv \delta$$

- polynomial remainder w.r.t. $p(z) - \delta$, i.e. substituting $p(z)$ with δ

$$\lfloor f(z) \rfloor_{p(z)-\delta} \equiv f(z) \bmod p(z) - \delta$$

- expansion for $\delta \to 0$

$$\lfloor f(z) \rfloor_{p(z)-\delta} \Big|_{\delta \to 0} = \sum_{i=\min}^{\max} \sum_{j=0}^{\deg p - 1} c_{ij} z^j \delta^i + O(\delta^{\max+1})$$

Example: univariate algorithm

GF, Peraro (2022)

$$\langle \varphi_L | \varphi_R \rangle = \sum_{p(z) \in \mathcal{P}_\omega[z]} \langle \varphi_L | \varphi_R \rangle_{p(z)}$$

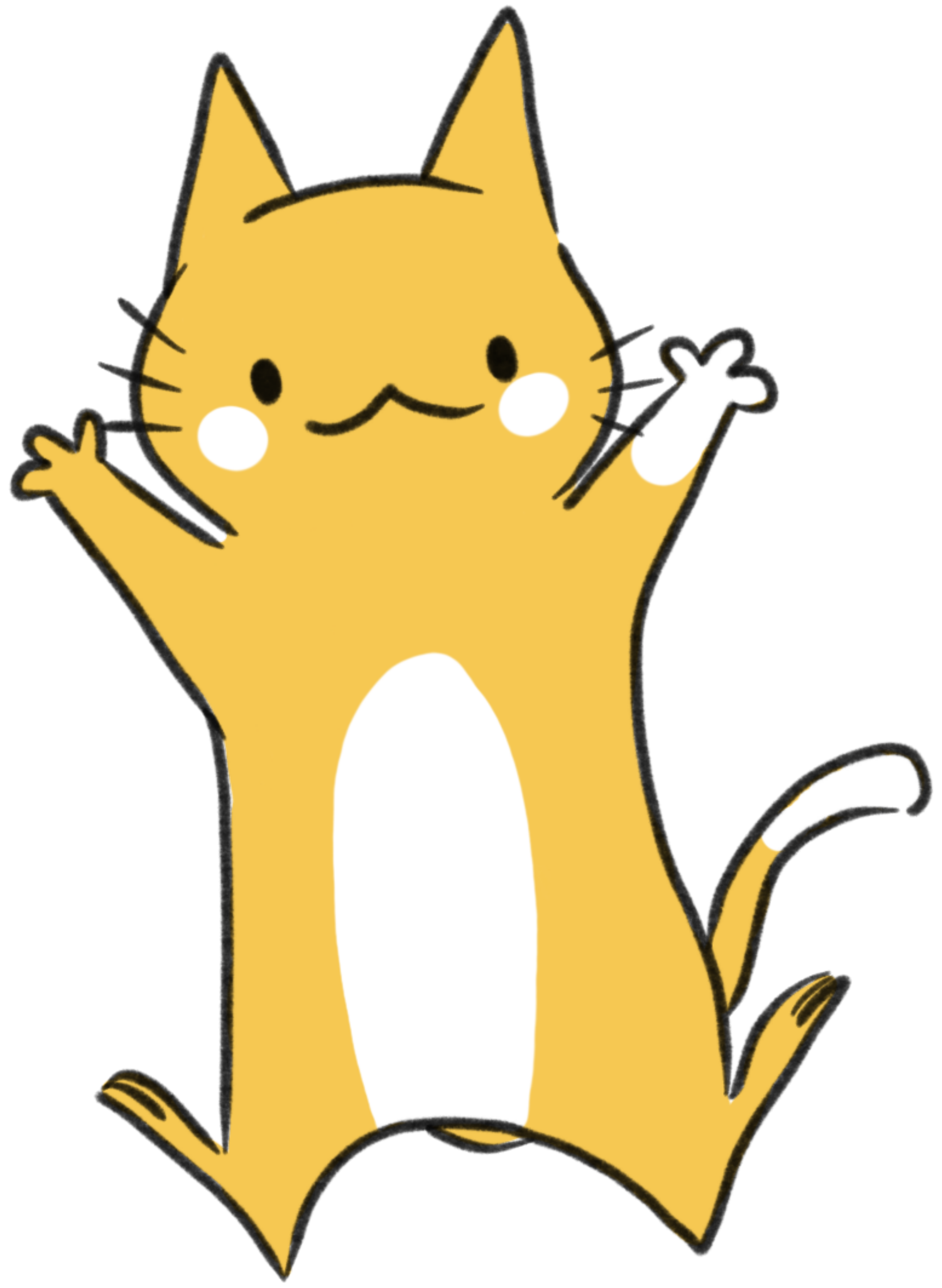
similar for multivariate case

summing over all $p(z) \in \mathcal{P}_\omega[z]$

$$\mathcal{P}_\omega[z] = \{\text{factors of the denominator of } \omega\} \bigcup \{\infty\}$$

Each addend of the form $\langle \varphi_L | \varphi_R \rangle_{p(z)}$ is the sum of all contributions to the intersection number coming from the roots of $p(z)$

$\langle \varphi_L | \varphi_R \rangle_\infty$ is computed as the contribution at $p = \infty$ with the "standard" algorithm to solve $(\partial_z + \omega)\psi = \varphi_L$



- we make an ansatz of the form

$$\psi = \sum_{i=\min}^{\max} \sum_{j=0}^{\deg p - 1} c_{ij} z^j p(z)^i + O(p(z)^{\max+1})$$

- we multiply the solution by φ_R

$$\psi\, \varphi_R = \sum_{i}^{-1} \sum_{j=0}^{\deg p - 1} \tilde{c}_{ij} z^j p(z)^i + O(p(z)^0)$$

- by the univariate global residue theorem

Weinzierl (2021)

$$\langle \varphi_L | \varphi_R \rangle_{p(z)} = \frac{\tilde{c}_{-1,\deg p - 1}}{l_c}$$

finite-fields implementation

GF, Peraro (to appear)

implementation on FiniteFlow of the **multivariate recursive** algorithm

method based on solution of linear systems and series expansions

⇒rational operations

input list of n-variate intersection numbers we want to reduce, e.g.:

$$\{\langle e_j^{(L)}|\varphi_R\rangle, \langle e_j^{(L)}|e_i^{(R)}\rangle\}$$

preliminary step

we can deduce the intersection numbers needed for each step

- $\langle \varphi_L | e_j^{(R)} \rangle_{n-1}$
- $\langle e_i^{(L)} | e_j^{(R)} \rangle_{n-1}$
- $(\partial_{z_n} \langle e_j^{(L)} |_{n-1}) | e_j^{(R)} \rangle_{n-1}$
- $\langle e_j^{(L)} | \varphi_R \rangle$

our implementation is an **iteration:**
starting from 1-forms, we compute all the necessary intersection numbers to get the n-forms given as input

$$z_1 \to \cdots \to z_{n-1} \to z_n$$

univariate algorithm

analytic input: $u(\mathbf{z})$

multivariate algorithm

needs as inputs

- denominator factors $p_i(z_n)$
- $(n-1)$-variate intersection numbers **reconstructed in z_n only**

dealing with poles

- $p = 0,\infty \to$ **Laurent expansion**
- all other factors $\to p(z)$-adic expansion

input for the n^{th}- step

* $\mathbf{\Omega}_{ij}$
* $\langle e_j^{(R)} | \varphi_R \rangle$
* $\langle \varphi_L | = \sum_{i=1}^{\nu_{n-1}} \varphi_{L,j} \langle e_j^{(L)} |$

$\Big\} \mathscr{X}_n$

between two steps:

$$(n-1) \to n$$

- rational reconstruction of $\mathscr{X}_n$ **only** in z_n, with everything else set to a number mod p
- identify denominator factors of $\mathscr{X}_n$ in z_n, **fully** reconstruct a simple subset of them

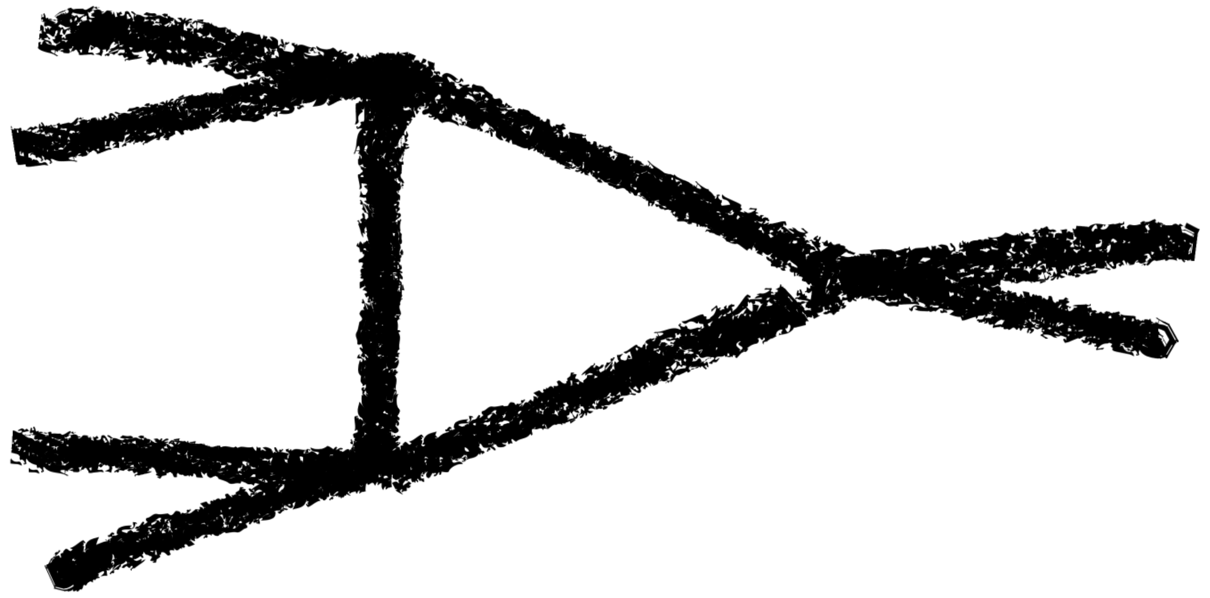
[avoid reconstructing large intermediate expressions in all variables

can be done in a small number of evaluations]

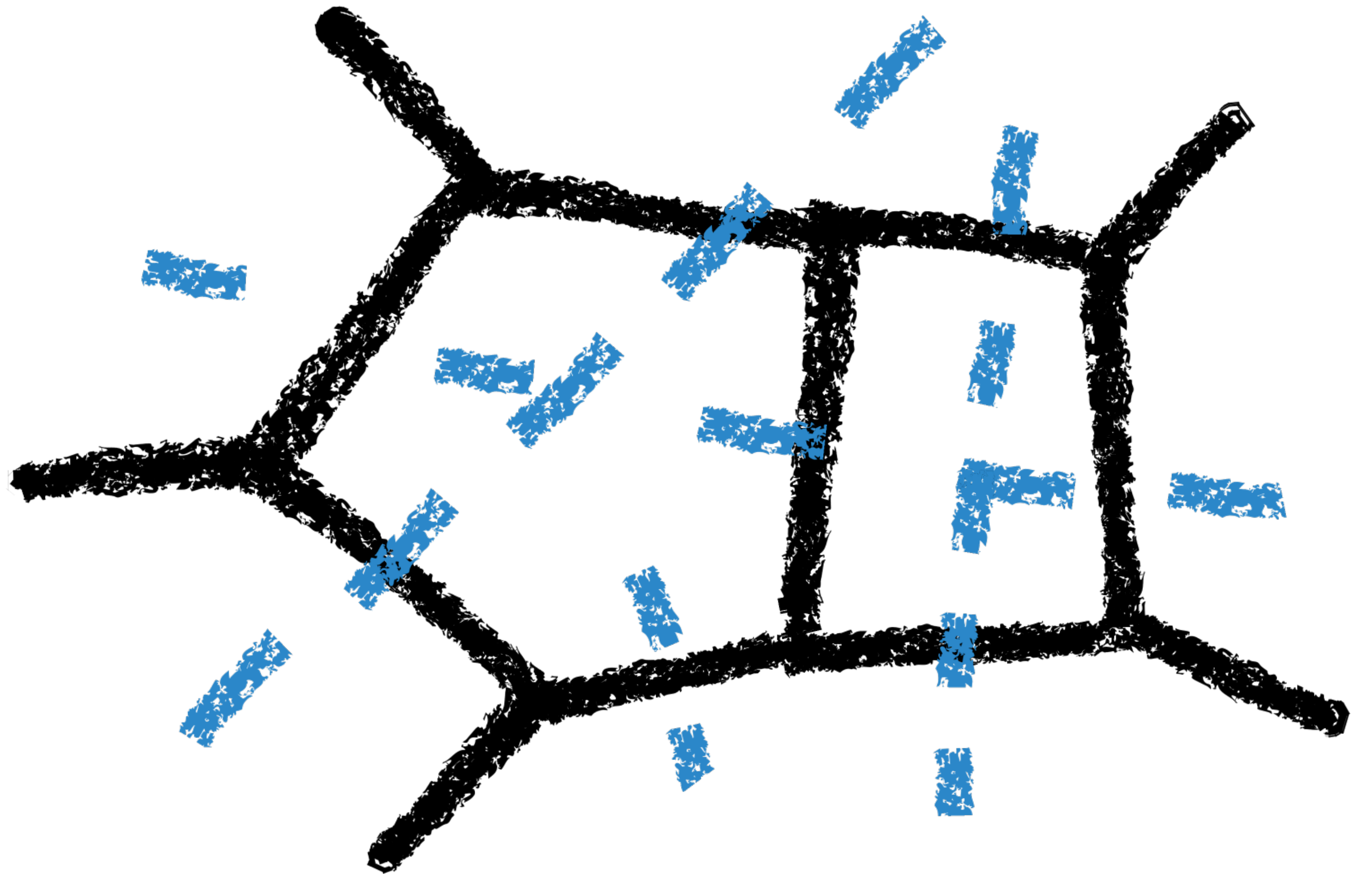
examples

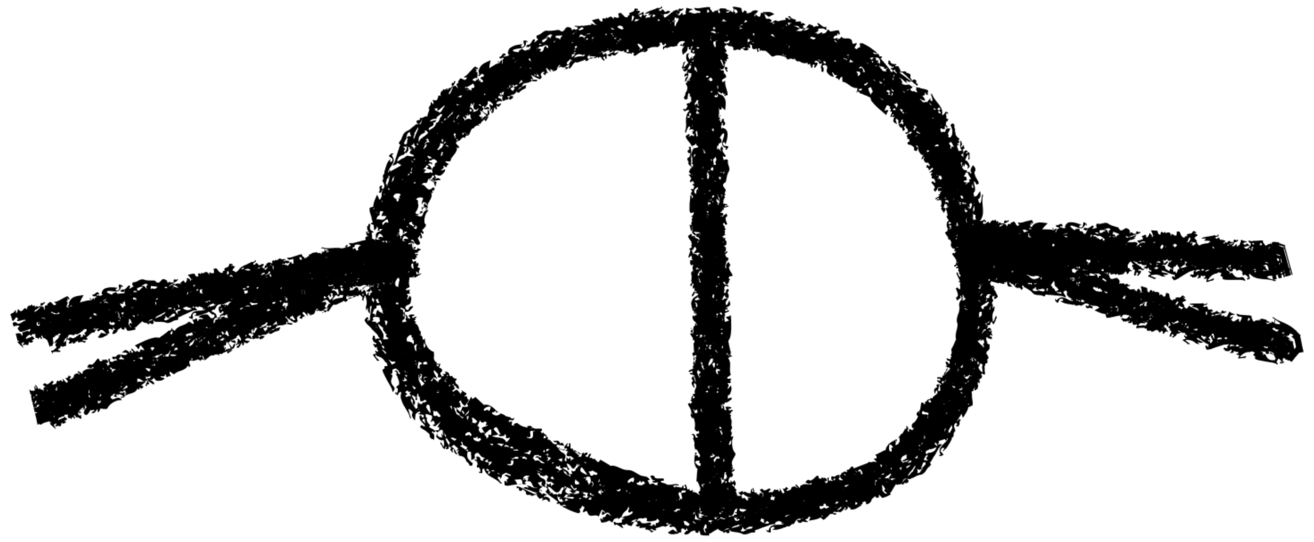
successfully reduced to master integrals the following topologies

one loop



two loops





Thank you for your attention!