The Function Space of $\mathcal{N} = 4$ Scattering Amplitudes in the Regge Limit to all Orders

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The Goal of this Talk

Get a glimpse at the beautiful mathematical structure of scattering amplitudes in the Regge limit of $\mathcal{N} = 4$ SYM

The Planar Limit ('t Hooft limit) of $\mathcal{N}=4$

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$\mathcal{N}=4$ Amplitudes generally have nice properties

- Conformal symmetry
- Maximally transcendental
- $\mathcal{A}_{+\cdots+} = \mathcal{A}_{-+\cdots+} = 0$

$$\mathcal{A}_4^{(1)} \sim \frac{1}{2}\log^2\frac{s}{t} + \frac{2\pi^2}{3}$$

In the planar limit ($N_c \to \infty\,, a = \frac{N_c\, g^2}{8\pi^2}$ finite)

Define *dual coordinates* x_i, s.t. $p_i = x_i - x_{i-1}$

Dual conformal symmetry

[Drummond, Henn, Korchemsky, Sokatchev; Drummond, Henn, Smirnov, Sokatchev]

- Fixes all 4&5 point results
- $A_N = A_N^{\mathrm{BDS}} e^{R_N}$

[Anastasiou, Bern, Dixon, Kosower; Bern, Dixon, Smirnov]

[Drummond, Henn, Korchemsky, Sokatchev]

Multi Regge Kinematics (MRK)

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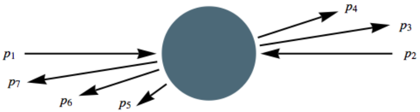
Forward scattering:

- $s \gg |t|$
- $\mathcal{A}_4^{(1)} \sim \frac{1}{2} \log^2 \frac{s}{t} + \frac{2\pi^2}{3}$

Multi Regge kinematics:

- Hierarchy in rapidity
- No hierarchy in transverse plane



- $$R_N \sim a^2 \left(\log \tau \, g^{(2)}_{N,\mathrm{LLA}} + g^{(2)}_{N,\mathrm{NLLA}} \right) + a^3 \left(\log^2 \tau \, g^{(3)}_{N,\mathrm{LLA}} + \log \tau \, g^{(3)}_{N,\mathrm{NLLA}} + g^{(3)}_{N,\mathrm{NNLLA}} \right) + \ldots \quad ,$$
  with $\log \tau \gg 1$

Multi Regge Kinematics

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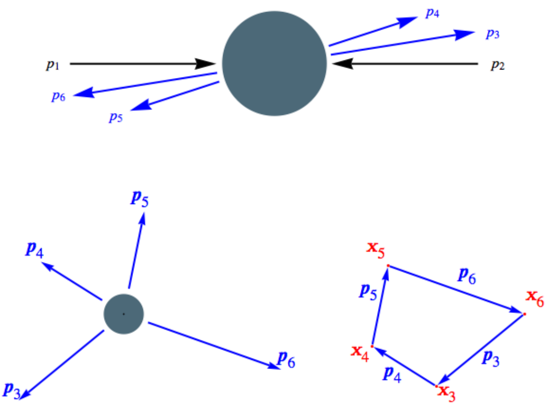
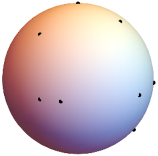
Kinematical dependence encoded in $N-2$ transverse momenta.



$$R_N \sim a^2 \left(\log\tau \, g^{(2)}_{N,\mathrm{LLA}}(\{\mathbf{x}_i\}) + g^{(2)}_{N,\mathrm{NLLA}}(\{\mathbf{x}_i\}) \right) + \dots$$

The Riemann Sphere With m Marked Points

Riemann sphere $\cong \mathbb{C}$ (stereographic projection)

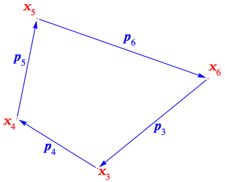


Riemann spheres with m marked points

$\Updownarrow$

'Phase space' of $N = m+2$-particle MRK amplitude.

Use $SL(2,\mathbb{C})$ symmetry to fix three points to get $N-5$ cross ratios



$$z_i = \frac{(\mathbf{x}_1 - \mathbf{x}_{i+3})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}{(\mathbf{x}_1 - \mathbf{x}_{i+1})(\mathbf{x}_{i+2} - \mathbf{x}_{i+1})}$$

Single-Valuedness

Optical theorem tells us that branch cuts start at

$$x_{ij}^2 = (x_i - x_j)^2 = 0$$

In MRK, this corresponds to branch cuts at

$$\mathbf{x}_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^2 = 0$$

In the transverse part, however

$$\mathbf{x}_{ij}^2 \geq 0$$

$\Rightarrow$ Scattering amplitudes in planar $\mathcal{N} = 4$ SYM in MRK are single-valued

Single-valued Polylogarithms

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Express amplitudes with *single-valued polylogarithms* $\mathcal{G}$ [Brown]

Single-valued polylogs are combinations of regular polylogs in z and $\overline{z}$ such that all branch cuts cancel

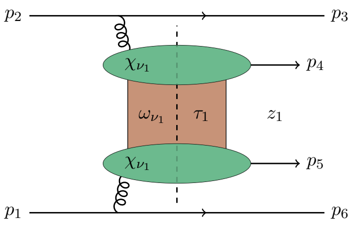
$$\mathcal{G}_0(z) = G_0(z) + G_0(\overline{z}) = \log z + \log \overline{z} = \log |z|^2$$

Fulfil same differential equation as regular Polylogs

$$\partial_z \mathcal{G}_{a_1,\ldots,a_n}(z) = \frac{1}{z - a_1} \mathcal{G}_{a_2,\ldots,a_n}(z)$$

The Remainder Function

The Remainder Function

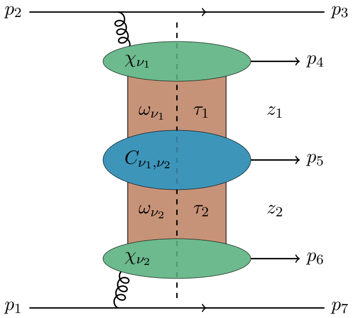


[Bartels, Lipatov, Sabio-Vera; Lipatov, Prygarin]

$$\sim \sum_{n_1} \int \frac{\mathrm{d}\nu_1}{2\pi} \left(\frac{z_1}{\bar{z}_1}\right)^{\frac{n_1}{2}} |z_1|^{2\mathrm{i}\nu_1} \chi_1^+ \tau_1^{-\omega_1} \chi_1^-$$

$$\equiv \mathcal{F}_1 \left[\chi_1^+ \tau_1^{-\omega_1} \chi_1^- \right]$$

The Remainder Function

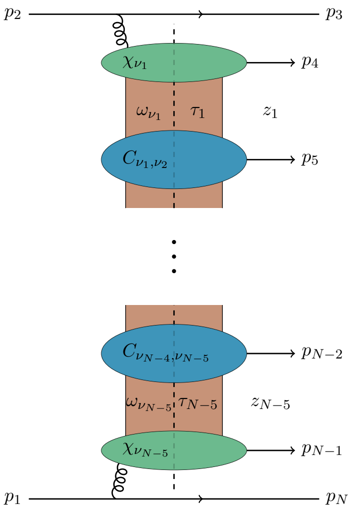


[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

$$\sim \mathcal{F}_2 \left[\mathcal{F}_1 \left[\chi_1^+ \tau_1^{-\omega_1} C_{12}^+ \tau_2^{-\omega_2} \chi_2^- \right] \right]$$

$$\equiv \mathcal{F}_{12} \left[\chi_1^+ \tau_1^{-\omega_1} C_{12}^+ \tau_2^{-\omega_2} \chi_2^- \right]$$

The Remainder Function



[Bartels, Lipatov, Sabio-Vera; Bartels, Kormilitzin, Lipatov, Prygarin]

The Remainder Function

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At LLA, replace all building blocks by their LO approximation

$$\chi^+_{0,1}\tau_1^{aE_1}C^+_{0,12}\tau_2^{aE_2}\chi^-_{0,2}$$

$$\tau_k^{aE_k} = a\log\tau_k\,E_k + \frac{a^2}{2}\log^2\tau_k\,E_k^2 + \dots$$

At LLA: $\qquad \ell - 1 = \#E_k$

Define the *vacuum ladder*

$$\varpi_N = \chi^+_{0,1}C^+_{0,12}\cdots C^+_{0(N-6)(N-5)}\chi^-_{0,N-5}$$

An Example at 7 Points

A Graphical Representation - LLA

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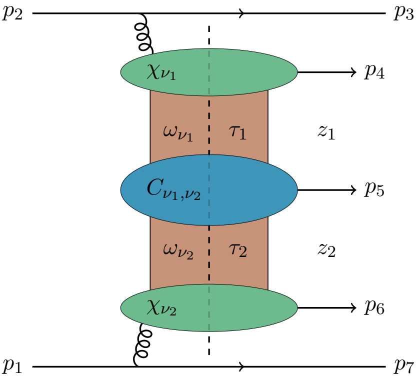
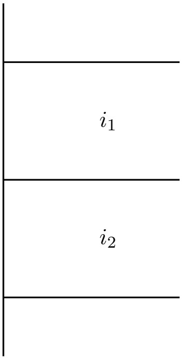
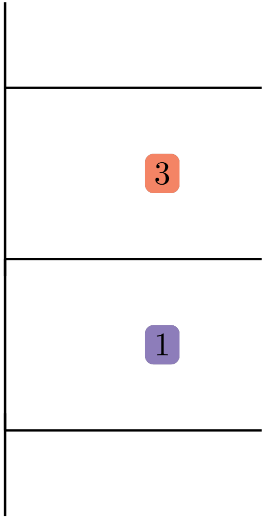




$$\mathcal{R}_{7,\mathrm{LLA}}^{(\ell)} \sim \sum_{i_1+i_2=\ell-1} \frac{a^\ell}{i_1! i_2!} \log^{i_1} \tau_1 \log^{i_2} \tau_2 \; g^{(i_1,i_2)}$$

A Graphical Representation - LLA

$$g^{(3,1)} = \mathcal{F}_{12}\left[\varpi_7 E_1^3 E_2\right] \equiv$$



Adding Loops

The number of E_k in the integrand is related to the loop order of LLA amplitudes

$\rightarrow$ Increase the loop order at a fixed logarithmic accuracy by inserting E_k into the integrand.

...but how?

Fourier-Mellin Convolutions

The Fourier-Mellin transform

$$\mathcal{F}[F(\nu,n)] = \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}\nu}{2\pi} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}} |z|^{2\mathrm{i}\nu} F(\nu,n)$$

Products are mapped to convolutions

$$\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G] = \frac{1}{\pi} \int \frac{\mathrm{d}^2 w}{|w|^2} \mathcal{F}[F](w) \mathcal{F}[G]\left(\frac{z}{w}\right)$$

Use this to raise loop order

$$g^{(i_1,i_2)} = g^{(i_1-1,i_2)} * \mathcal{F}_1[E_1] = g^{(0,0)} * \mathcal{F}[E_1]^{*i_1} * \mathcal{F}_2[E_2]^{*i_2}$$

Fourier-Mellin Convolutions

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$$\mathcal{F}_k[E_k] = -\frac{z_k + \overline{z}_k}{2|1 - z_k|^2}$$

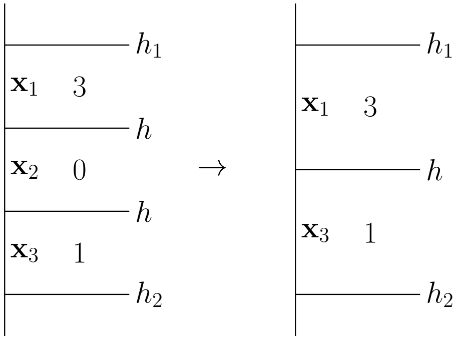
Single-valuedness allows us to solve the convolution integral by computing residues

[Schnetz]

$$\int \frac{\mathrm{d}^2 z}{\pi} f(z) = \mathrm{Res}_{z=\infty} F(z) - \sum_i \mathrm{Res}_{z=a_i} F(z) \qquad \partial_{\overline{z}} F(z) = f(z)$$

$\rightarrow$ Increasing loop order as simple as computing residues!

Faces with no Insertions



$$f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = f(\mathbf{x}_1, \mathbf{x}_3)$$

[Del Duca, Druc, Drummond, Duhr ,Dulat, RM, Papathanasiou, Verbeek]

Factorization of Remainder Functions - $N = 7$

$$R_{7,\mathrm{LLA}}^{(3)} \sim \log^2 \tau_1 \left|\begin{array}{l} \\ \hline \mathbf{x}_1 \;\; 2 \\ \hline \mathbf{x}_2 \;\; 0 \\ \hline \\ \end{array}\right. + \log^2 \tau_2 \left|\begin{array}{l} \\ \hline \mathbf{x}_1 \;\; 0 \\ \hline \mathbf{x}_2 \;\; 2 \\ \hline \\ \end{array}\right. + \log \tau_1 \log \tau_2 \left|\begin{array}{l} \\ \hline \mathbf{x}_1 \;\; 1 \\ \hline \mathbf{x}_2 \;\; 1 \\ \hline \\ \end{array}\right.$$

$$= \log^2 \tau_1 \left|\begin{array}{l} \\ \hline \mathbf{x}_1 \;\; 2 \\ \hline \\ \end{array}\right. + \log^2 \tau_2 \left|\begin{array}{l} \\ \hline \mathbf{x}_2 \;\; 2 \\ \hline \\ \end{array}\right. + \log \tau_1 \log \tau_2 \left|\begin{array}{l} \\ \hline \mathbf{x}_1 \;\; 1 \\ \hline \mathbf{x}_2 \;\; 1 \\ \hline \\ \end{array}\right.$$

$$R^{(3)}_{8,\mathrm{LLA}} \sim \log^2\tau_1 \left|\begin{array}{l}\hline \mathbf{x}_1\ \ 2 \\ \hline \end{array}\right. + \log^2\tau_2 \left|\begin{array}{l}\hline \mathbf{x}_2\ \ 2 \\ \hline \end{array}\right. + \log^2\tau_3 \left|\begin{array}{l}\hline \mathbf{x}_3\ \ 2 \\ \hline \end{array}\right.$$

$$+ \log\tau_1\log\tau_2 \left|\begin{array}{l}\hline \mathbf{x}_1\ \ 1 \\ \hline \mathbf{x}_2\ \ 1 \\ \hline \end{array}\right. + \log\tau_1\log\tau_3 \left|\begin{array}{l}\hline \mathbf{x}_1\ \ 1 \\ \hline \mathbf{x}_3\ \ 1 \\ \hline \end{array}\right. + \log\tau_2\log\tau_3 \left|\begin{array}{l}\hline \mathbf{x}_2\ \ 1 \\ \hline \mathbf{x}_3\ \ 1 \\ \hline \end{array}\right.$$

Factorization of Remainder Functions

Reminder:

$$\text{LLA} \Leftrightarrow \sum i_k = \ell - 1$$

$\rightarrow$ can only have limited number of 'insertions' at a given order

At MHV, a finite number of building blocks are enough to describe the ℓ-loop amplitude for all numbers of particles

Beyond LLA

The Remainder Function

$$-\omega = aE - \frac{a^2}{4}\left(D^2E - 2VDE + 4\zeta_2 E + 12\zeta_3\right) + \mathcal{O}(a^3)\,,$$

$$\chi^+ = \chi_0^+\left[1 - \frac{a}{4}\left(E^2 + \frac{3}{4}N^2 - NV + \frac{\pi^2}{3}\right) + \mathcal{O}(a^2)\right],$$

$$\chi^- = \chi_0^-\left[1 - \frac{a}{4}\left(E^2 + \frac{3}{4}N^2 + NV + \frac{\pi^2}{3}\right) + \mathcal{O}(a^2)\right],$$

$$\begin{aligned} C_{12}^+ = C_{0,12}^+\Big[1 + a\Big(&\tfrac{1}{2}\big[DE_1 - DE_2 + E_1E_2 + \tfrac{1}{4}(N_1+N_2)^2 + V_1V_2 \\ &+ (V_1 - V_2)\big(M_{12} - E_1 - E_2\big) + 2\zeta_2 + i\pi(V_2 - V_1 - E_1 - E_2)\big] \\ &- \tfrac{1}{4}(E_1^2 + E_2^2 + N_1V_1 - N_2V_2) - \tfrac{3}{16}(N_1^2 + N_2^2) - \zeta_2\Big) + \mathcal{O}(a^2)\Big]\,, \end{aligned}$$

Define $X_i = X(\nu_i, n_i)$, $X_{ij} = X(\nu_i, n_i, \nu_j, n_j)$.

The Remainder Function

$$
\begin{aligned}
-\omega &= aE + a^2\, {}^{\mathsf{h}}P_3(D,E,V,N,\text{"}\pi\text{"}) + \mathcal{O}(a^3)\,,\\
\chi^{\pm} &= \chi_0^{\pm}\left[1 + a\, {}^{\mathsf{h}}P_2(D,E,V,N,\text{"}\pi\text{"}) + \mathcal{O}(a^2)\right]\,,\\
C_{12}^{+} &= C_{0,12}^{+}\left[1 + a\, {}^{\mathsf{h}}P_2(D,E,V,N,M,\text{"}\pi\text{"}) + \mathcal{O}(a^2)\right],
\end{aligned}
$$

Corrections are homogeneous polynomials of D, E, V, N, M, π

$\rightarrow$ Show that each building block raises the weight by 1

The Weight of Elementary Building Blocks

Take a function

$$\mathcal{K}(z) = \frac{|z|^2}{(z-a)(\bar{z}-\bar{b})} \tag{1}$$

and consider

$$\mathcal{G}(a_1,\ldots,a_n;z) * \mathcal{K}(z) = \frac{1}{\pi}\int \mathrm{d}^2 w\, \mathcal{G}(a_1,\ldots,a_n;w)\frac{1}{(w-za)(\overline{w}-\overline{z}\overline{b})}$$

$\to$ Use residues to get

$$\mathrm{Res}_{\overline{w}=\overline{z}\overline{b}} \frac{\mathcal{G}(za,a_1,\ldots,a_n;w)}{(\overline{w}-\overline{z}\overline{b})} \sim \mathcal{G}(za,a_1,\ldots,a_n;zb)$$

The MRK Function Space in $\mathcal{N} = 4$

MHV Remainder functions are pure functions of weight 2ℓ to all orders in perturbation theory

Beyond MHV

Going beyond MHV - The Ratio Function

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Can flip helicities with convolutions, too!

$$\mathcal{F}_{12}\left[\chi_1^- C_{12}^+ \chi_2^-\right] = \mathcal{F}_{12}\left[\chi_1^+ C_{12}^+ \chi_2^-\right] * \mathcal{F}_1\left[\frac{\chi_1^-}{\chi_1^+}\right]$$

$$\mathcal{F}_1\left[\frac{\chi_1^-}{\chi_1^+}\right] = -\frac{z_1}{(1-z_1)^2}$$

Double pole in $z_1 \rightarrow$ introduces rational coefficients

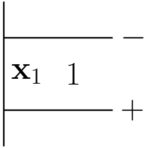
Going beyond MHV - The Ratio Function

Certain amplitudes will never factorize . . .

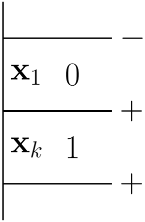
- $\mathcal{R}_{N,+-+-\ldots}$ won't simplify

. . . but others still do

$$\mathcal{R}^{(2)}_{N,-+\cdots+} \sim \log\tau_1$$



$$+\sum_{k=2}^{N-5} \log\tau_k$$



Conclusions

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- Developed a framework that allows us to compute virtually any amplitude in MRK in $\mathcal{N}=4$ SYM
- A finite set of building blocks encodes amplitudes for any number of particles.
- Computed the building blocks to yield:
  - All MHV 5-loop amplitudes at LLA
  - 8-point LLA amplitudes for any helicity configuration up to 4 loops
  - All MHV 3-loop amplitudes at NLLA
  - 7-point NLLA NMHV up to 3 loops
- Completely determined the function space of MRK amplitudes to all orders

Backup

Backup—Weight of Building Blocks

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$\mathcal{F}[E], \mathcal{F}[N], \mathcal{F}[V]$ have this form!

What about $D = -\mathrm{i}\partial_\nu$?

$$\mathcal{F}[DX] = -\mathrm{i} \sum_{n=-\infty}^{\infty} \int \frac{\mathrm{d}\nu}{2\pi} \left(\frac{z}{\bar{z}}\right)^{\frac{n}{2}} |z|^{2\mathrm{i}\nu} \partial_\nu X$$

$$\overset{\text{IBP}}{=} \mathcal{G}(0, z)\mathcal{F}[X]$$

$\rightarrow$ Raises weight by 1!

Backup—Weight of Building Blocks

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$M \equiv M(\nu_1, n_1, \nu_2, n_2) \rightarrow$ Things get more complicated.

Can write M_{12} as

$$M_{12} = \frac{D_1 C^+_{0,12}}{C^+_{0,12}} + F_1 = -\frac{D_2 C^+_{0,12}}{C^+_{0,12}} + F_2 - N_2$$

with a new building-block F and treat derivatives of C using IBP.

$\mathcal{F}[F]$ has the form (1) $\rightarrow$ raises weight by 1.

$\rightarrow$ MHV amplitudes are pure combinations of SVMPLs of maximal transcendentality

Backup—Helicity Flips Beyond LLA

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What happens beyond MHV?

Can flip helicities using (see Claude's talk)

$$H = \frac{\chi^-}{\chi^+} = H_0(1 + a\,{}^{\mathsf{h}}P_2(D, E, N, V, \text{"}\pi\text{"}) + \mathcal{O}(a^3))\,,$$

→ Pure functions of uniform transcendental weight up to LO helicity flips.

→ Leading singularities R_{bac}