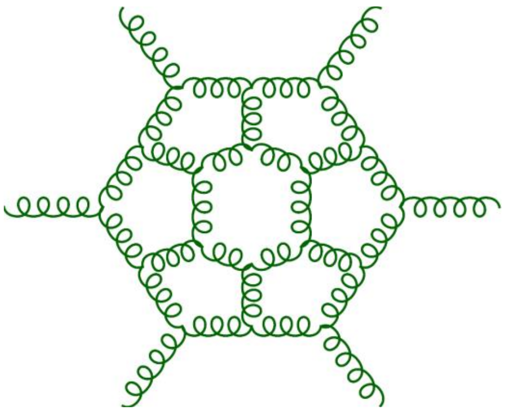
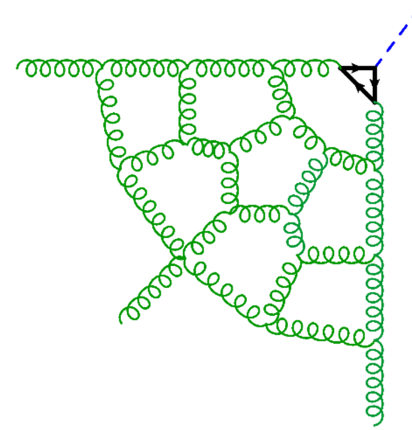
Antipodal (Self-)Duality in Planar N=4 Super-Yang-Mills Theory





Lance Dixon (SLAC)

LD, Ö. Gürdoğan, Y.-T.Liu, A. McLeod, M. Wilhelm

2112.06243, 2204.11901, 2212.02410

Workshop on "Mathematical Structures in Feynman Integrals"
University of Siegen
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Amplitudes and Integrals

- **Both** can have rich and surprising mathematical structure
- Although amplitudes are “just” linear combinations of integrals, they can have more unique properties than the integrals they are composed from
- Antipodal (self-)duality might be the tip of an iceberg

Consider the Harmonic Polylogarithms in one variable (HPL{0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize classical polylogs
- Define HPLs by iterated integration:

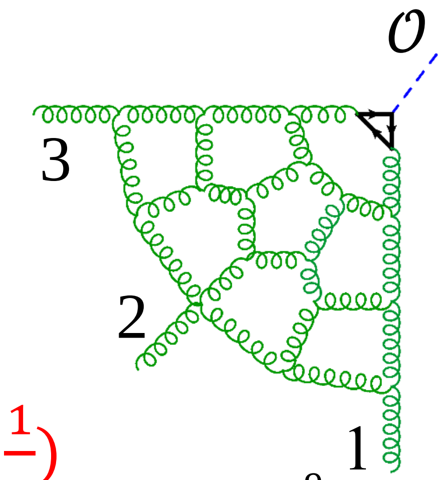
$$H_{0,\vec{w}}(x) = \int_0^x \frac{dt}{t} H_{\vec{w}}(t), \qquad H_{1,\vec{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives:

$$dH_{0,\vec{w}}(x) = H_{\vec{w}}(x) d\ln x \qquad dH_{1,\vec{w}}(x) = -H_{\vec{w}}(x) d\ln(1-x)$$

- Symbol alphabet: $\mathcal{L} = \{x, 1-x\}$
- Weight n = length of binary string $\vec{w}$
- Number of functions at weight $\boldsymbol{n = 2L}$ is number of binary strings: $\mathbf{2^{2L}}$
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol

A three-gluon form factor in planar N=4 SYM



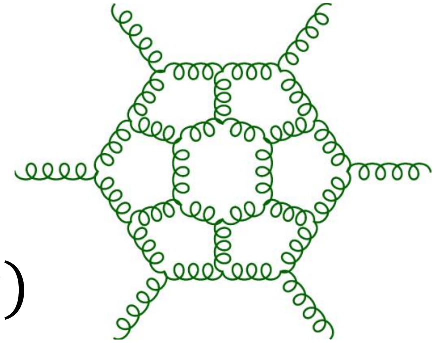
$u = \frac{s_{12}}{s_{123}}$, $v = \frac{s_{23}}{s_{123}}$, $w = \frac{s_{31}}{s_{123}} = 1 - u - v$

$v \to \infty$ → Harmonic polylogarithms $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{u})$

$$
\begin{aligned}
F_3^{(1)}(v \to \infty) &= 2H_{0,1} + 6\zeta_2 \\
F_3^{(2)}(v \to \infty) &= -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4 \\
F_3^{(3)}(v \to \infty) &= 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1} \\
&\quad + 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1} \\
&\quad + 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1} \\
&\quad - \zeta_2(32H_{0,0,0,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1}) \\
&\quad - \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2
\end{aligned}
$$

8 loop result has $\sim 2^{2\times 8-2} = 16{,}384$ terms

6-gluon amplitude in planar N=4 SYM



$+\sim 10^9$ more

Depends on 3 “dual-conformal cross-ratios, $(\hat{u}, \hat{v}, \hat{w})$

Simplest for $(\hat{u}, \hat{v}, \hat{w}) = (1, \hat{v}, \hat{v})$

Let $H_{\vec{w}} \equiv H_{\vec{w}}(1 - \frac{1}{\hat{v}})$

$$A_6^{(1)}(1,\hat{v},\hat{v}) = 2H_{0,1}$$

$$A_6^{(2)}(1,\hat{v},\hat{v}) = -8H_{0,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2 H_{0,1} - 9\zeta_4$$

$$\begin{aligned} A_6^{(3)}(1,\hat{v},\hat{v}) = {} & 96H_{0,1,1,1,1,1} + 16H_{0,1,0,1,1,1} + 16H_{0,0,0,1,1,1} + 16H_{0,1,1,0,1,1} + 8H_{0,0,1,0,1,1} \\ & + 8H_{0,1,0,0,1,1} + 16H_{0,1,1,1,0,1} + 8H_{0,0,1,1,0,1} + 12H_{0,1,0,1,0,1} + 4H_{0,0,0,1,0,1} \\ & + 8H_{0,1,1,0,0,1} + 4H_{0,0,1,0,0,1} + 4H_{0,1,0,0,0,1} + 24H_{0,0,0,0,0,1} \\ & + \zeta_2(8H_{0,0,0,1} + 8H_{0,1,0,1} + 48H_{0,1,1,1}) \\ & + 42\zeta_4 H_{0,1} + 121\zeta_6 \end{aligned}$$

Exact map at symbol level ($\zeta_n \to 0$), with $\frac{1}{\hat{v}} = 1 - \frac{1}{u}$, $0 \leftrightarrow 1$,
if you also reverse order of HPL indices!!!
Works to 7 loops, where $\sim 2^{2\times 7-2} = 4{,}096$ terms agree

Planar N=4 SYM, “hydrogen atom” of amplitudes

- QCD’s maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in large N_c (planar) limit
- Structure very rigid:

$$\text{Amplitudes} = \sum_i rational_i \times transcendental_i$$

- For planar N=4 SYM, rational structure well understood, focus on transcendental functions.
- Furthermore, at least three dualities hold:

1. AdS/CFT
2. Amplitudes dual to Wilson loops
3. New “antipodal” duality between amplitudes and form factors (or among form factors?)

Transcendental Structure

- N=4 SYM amplitudes have "uniform weight" (transcendentality) $2L$ at loop order L

- Weight ~ number of integrations, e.g.

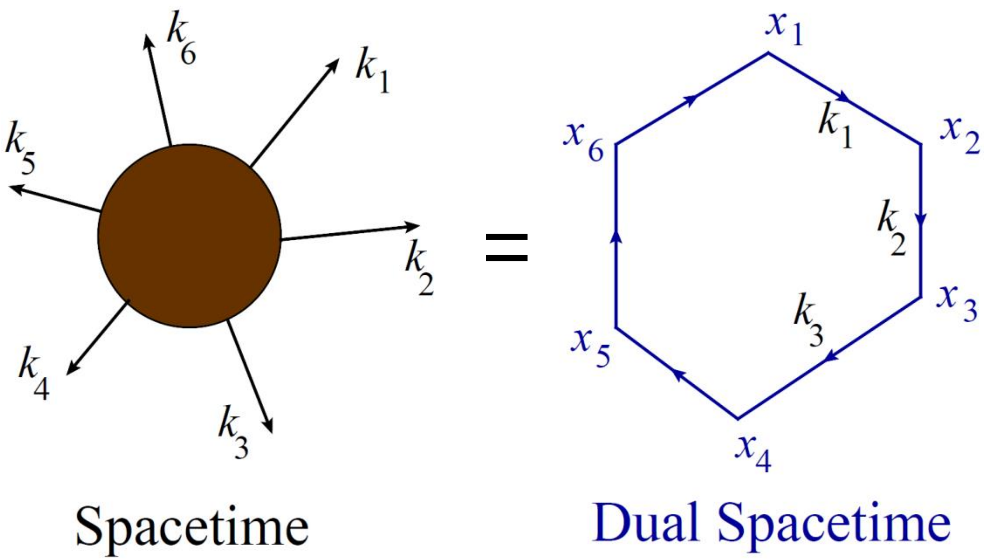
$$\ln(s) = \int_1^s \frac{dt}{t} = \int_1^s d\ln t \qquad 1$$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)] \qquad 2$$

$$\mathrm{Li}_n(x) = \int_0^x \frac{dt}{t} \mathrm{Li}_{n-1}(t) \qquad n$$

- QCD amps typically all weights from 0 to $2L$

Amplitudes = Wilson loops



- Polygon vertices x_i are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465

Duality holds at both strong and weak coupling

weak-weak duality, holds order-by-order

Dual conformal invariance

- Wilson n-gon invariant under inversion: $x_i^\mu \to \frac{x_i^\mu}{x_i^2}, \quad x_{ij}^2 \to \frac{x_{ij}^2}{x_i^2\, x_j^2}$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\cdots,j-1}$$

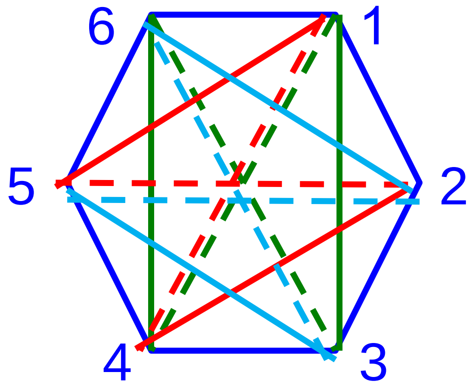
- Fixed, up to functions of invariant cross ratios:

$$\boxed{u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}}$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for n = 4,5

n = 6 → precisely 3 ratios:

$$\left\{\begin{aligned} \widehat{u} &= \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}} \\ \widehat{v} &= \frac{s_{23}s_{56}}{s_{234}s_{123}} \\ \widehat{w} &= \frac{s_{34}s_{61}}{s_{345}s_{234}} \end{aligned}\right.$$



Different routes to perturbative amplitudes

Draw all Feynman graphs G_i

Evaluate all Feynman rules: I_i

Perform all loop integrations: A_i

$$A = \sum_i A_i$$

Evaluate all unitarity cuts C_α

Construct **local**: integrand I

Perform all loop integrations: A_α

$$A = \sum_\alpha A_\alpha$$

Bootstrap: Guess $A = \sum_m c_m F_m$

F_m known functions

$c_m \in \mathbb{Q}$ unknown constants

Solve constraints, linear equations for $c_m \to r_m$

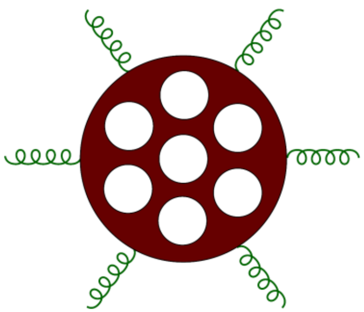
$$A = \sum_m r_m F_m$$

Hexagon function bootstrap

Loops

3	LD, Drummond, Henn, 1108.4461, 1111.1704;
4,5	Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
6,7	Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou, 1903.10890, 1906.07116; LD, Dulat, 23mm.nnnnn (NMHV 7 loop)

- Planar N=4 SYM: Make ansatz for space of multiple polylogarithms (MPLs) in which 6-point amplitude resides to determine it directly. No explicit Feynman integrands or integrals.
- Step toward doing this nonperturbatively (no loops to peek inside) for general kinematics
- Same method used for “Higgs” form factor; see below



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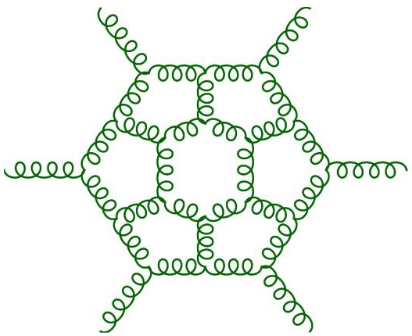
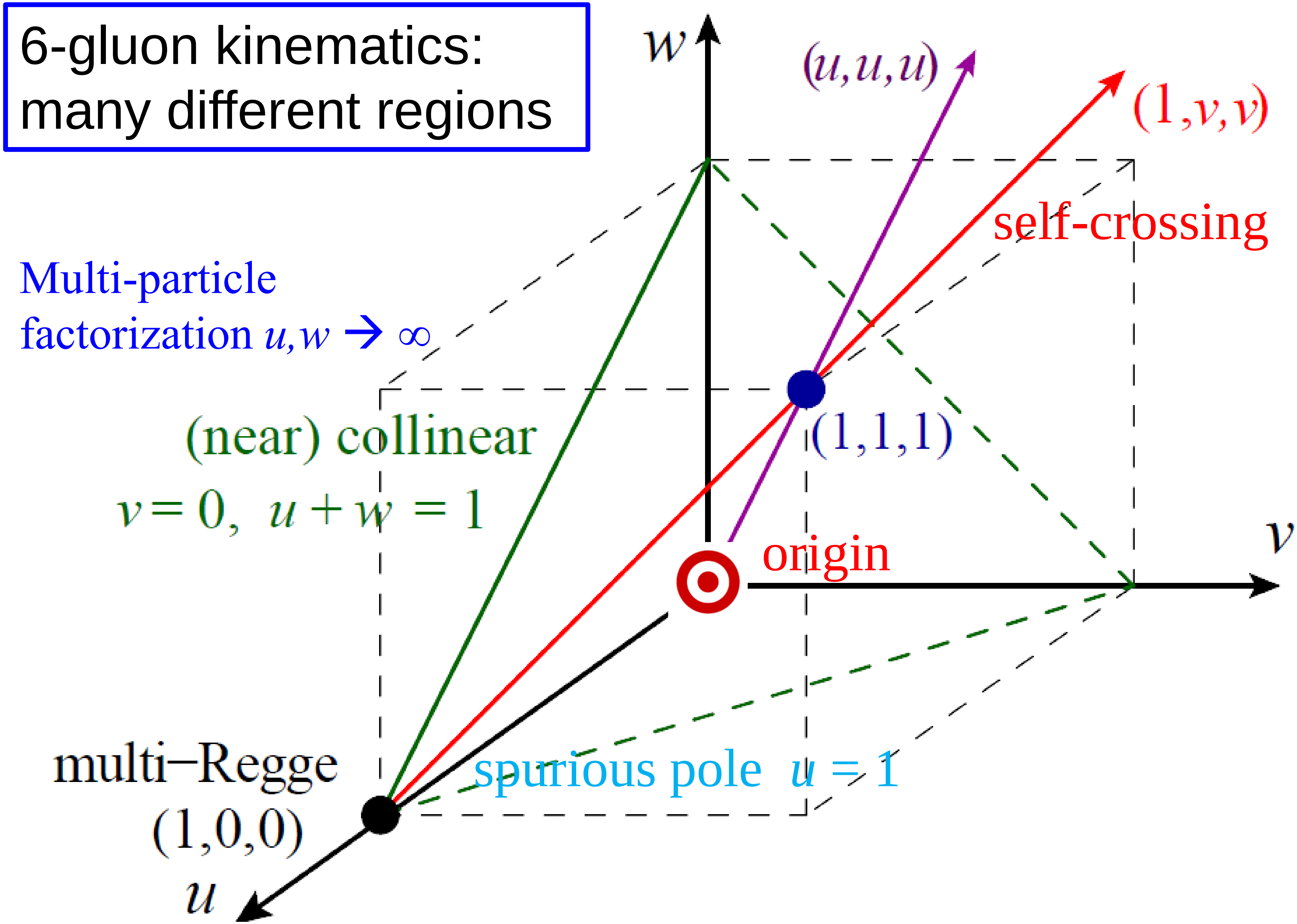
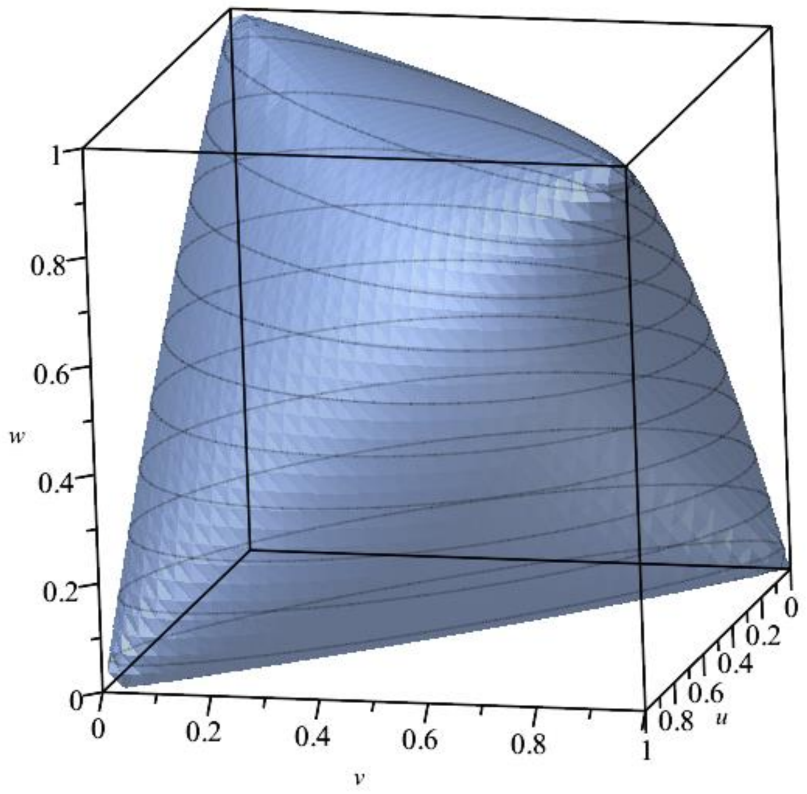


$+ \ \sim 10^9$ more

6-gluon kinematics: many different regions



Parity-preserving surface



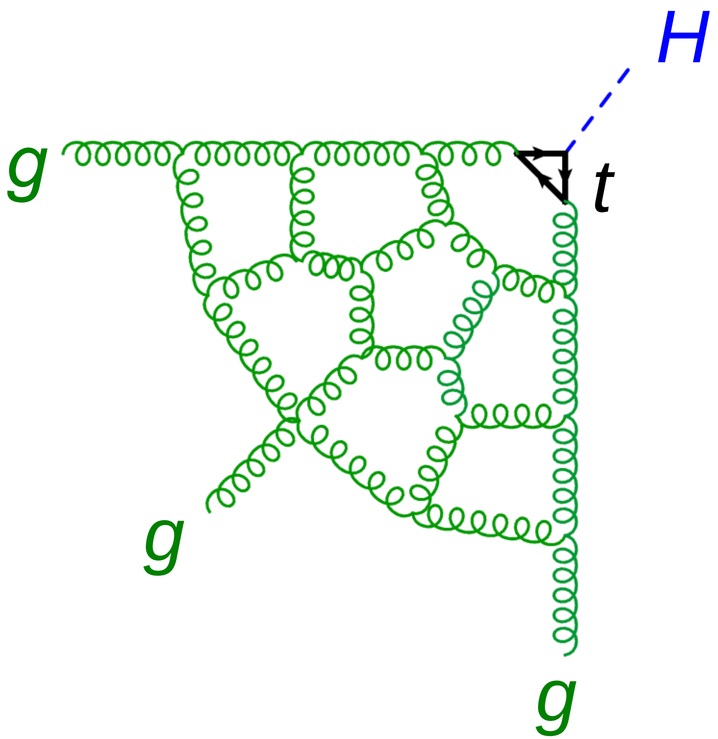
$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

where kinematics is in a 3d subspace of 4d spacetime
→ parity invariant

Bootstrap Goldilocks “Higgs” amplitude [planar N=4 form factor] to 8 loops

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2012.12286, 2204.11901

	Loops
2012.12286	3,4,5
2204.11901	6,7,8



- Matrix elements of operator $G^{a}_{\mu\nu\ SD}G^{\mu\nu\ a}_{SD}$ with n gluons in planar N=4 SYM
- *Hgg* form factor ($n = 2$) “too simple”, no kinematic dependence beyond overall $(-s_{12})^{-L\epsilon}$
- *Hggg* ($n = 3$) is “just right”, depends on only 2 dimensionless ratios
- 8 loop results for 2 variables are “data mine” for discovering e.g. antipodal duality

Hggg kinematics is two-dimensional

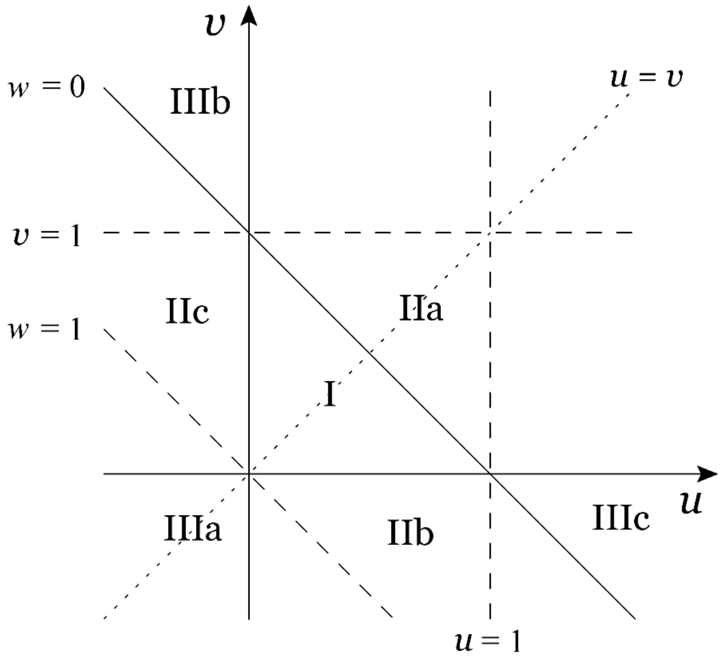
$$k_1 + k_2 + k_3 = -k_H$$

$$s_{123} = s_{12} + s_{23} + s_{31} = m_H^2$$

$$s_{ij} = (k_i + k_j)^2 \qquad k_i^2 = 0$$

$$u = \frac{s_{12}}{s_{123}} \qquad v = \frac{s_{23}}{s_{123}} \qquad w = \frac{s_{31}}{s_{123}}$$

$$u + v + w = 1$$



I = decay / Euclidean

IIa,b,c = scattering / spacelike operator

IIIa,b,c = scattering / timelike operator

N=4 amplitude is S_3 invariant

$D_3 \equiv S_3$ dihedral symmetry generated by:

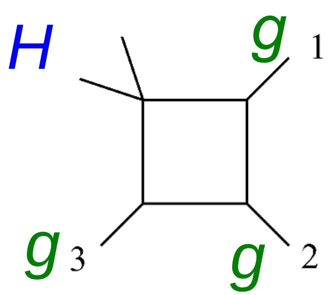
a. cycle: $i \to i + 1 \pmod 3$, or

$$u \to v \to w \to u$$

b. flip: $u \leftrightarrow v$

One loop

scalar box integral:



$$= \int \frac{d^4p}{p^2(p-p_1)^2(p-p_1-p_2)^2(p-p_1-p_2-p_3)^2}$$

$$= \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{12}}\right) + \mathrm{Li}_2\left(1-\frac{s_{123}}{s_{23}}\right) + \frac{1}{2}\ln^2\left(\frac{s_{12}}{s_{23}}\right) + \cdots$$

$$= \mathrm{Li}_2\left(1-\frac{1}{u}\right) + \mathrm{Li}_2\left(1-\frac{1}{v}\right) + \frac{1}{2}\ln^2\left(\frac{u}{v}\right) + \cdots$$

→ Symbol is $u \otimes (1-u) + v \otimes (1-v) - u \otimes v - v \otimes u$

Including cycles, $u \to v \to w \to u$, symbol alphabet at one loop is

$$\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Two loops

- *Hggg* computed in QCD at 2 loops

  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Stress tensor 3-point form factor $\mathcal{F}_3$ in N=4 SYM computed next Brandhuber, Travaglini, Yang, 1201.4170
- Symbol alphabet **still**

$$\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$$

2d HPLs

Gehrmann, Remiddi, hep-ph/0008287

$\mathcal{L} = \{u, v, w, 1-u, 1-v, 1-w\}$

Space graded by weight. Every weight n function F obeys:

$$\boxed{\begin{aligned} \frac{\partial F(u,v)}{\partial u} &= \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w} \\ \frac{\partial F(u,v)}{\partial v} &= \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w} \end{aligned}}$$

$w = 1 - u - v$

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight n-1 2d HPLs.

To **bootstrap** *Hggg* amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight, impose various constraints to get a unique answer

Hopf algebra for MPLs

Chen, Goncharov, Brown,…; Duhr, Gangl, Rhodes

- Differential definition: $\boxed{dF = \sum_{s_k \in \mathcal{L}} F^{s_k} \, d \ln s_k}$

- Hopf algebra "co-acts" on space of polylogarithms,

$$\Delta: F \to F \otimes F$$

- Derivative dF is one piece of Δ: $\boxed{\Delta_{n-1,1} F = \sum_{s_k \in \mathcal{L}} F^{s_k} \otimes \ln s_k}$

- So refer to F^{s_k} as a $\{n\text{-}1,1\}$ coproduct of F

- **s_k are letters in the symbol alphabet $\mathcal{L}$**

Iterate to get **symbol**

- $\{n\text{-}1,1\}$ coaction can be applied iterately
- Define $\{n\text{-}2,1,1\}$ double coproducts, $F^{s_k,,s_j}$,

via derivatives of $\{n\text{-}1,1\}$ single coproducts F^{s_j}:

$$\boxed{dF^{s_j} \equiv \sum_{s_k \in \mathcal{L}} F^{s_k,,s_j}\, d\ln s_k}$$

- And so on for $\{n\text{-}m,1,\ldots,1\}$ m^{th} coproducts of F.
- Maximal iteration, n times for weight n function, is

the symbol, ["ln" is implicit in s_{i_k}]

$$\mathcal{S}[F] \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in \mathcal{L}} F^{s_{i_1},\ldots,s_{i_n}}\, s_{i_1} \otimes \ldots \otimes s_{i_n}$$

where now $F^{s_{i_1},\ldots,s_{i_n}}$ are just rational numbers

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

Symbol alphabets for n-gluon amplitudes

parity-odd letters, algebraic in $\hat{u}, \hat{v}, \hat{w}$

$n = 6$ has 9 letters: $\mathcal{L}_6 = \{\hat{u}, \hat{v}, \hat{w}, 1 - \hat{u}, 1 - \hat{v}, 1 - \hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703;
LD, Drummond, Henn, 1108.4461; Caron-Huot,
LD, von Hippel, McLeod, 1609.00669

$n = 7$ has 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617,
1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

Arkani-Hamed, Lam, Spradlin, 1912.08222;
Drummond, Foster, Gürdoğan, Kalousios, 1912.08217, 2002.04624;
Henke, Papathanasiou 1912.08254, 2106.01392;
Z. Li, C. Zhang, 2110.00350

Heuristic view of function space

weight

$\int\int\int\int\int\int$

derivatives

…

$\int\int\int\int\int$

4

$\int\int\int\int$

3

$\mathrm{Li}_3(1-1/u_i)$, true 2D HPLs, …

2

$\mathrm{Li}_2(1-1/u_i)$ $\ln^2 u_i$ $\ln u_i \ln u_{i+1} - \zeta_2$

1

$\ln u$ $\ln v$ $\ln w$

0

1

$$\boxed{\frac{\partial F(u,v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}}$$

Back to 3-gluon form factor

- Motivated by 6 gluon experience, we switch to an equivalent symbol alphabet

$$\mathcal{L}' = \{\, a = \frac{u}{vw},\ b = \frac{v}{wu},\ c = \frac{w}{uv},\ d = \frac{1-u}{u},\ e = \frac{1-v}{v},\ f = \frac{1-w}{w} \}$$

- Symbols of form factor $F_3^{(L)}$ at 1 and 2 loops: just 1 and 2 terms, plus D_3 dihedral images(!!!):

$$\mathcal{S}\left[F_3^{(1)}\right] = (-1)\, b \otimes d + \text{dihedral}$$

$$\mathcal{S}\left[F_3^{(2)}\right] = 4\, b \otimes d \otimes d \otimes d + 2\, b \otimes b \otimes b \otimes d + \text{dihedral}$$

Brandhuber, Travaglini, Yang, 1201.4170

dihedral cycle: $a \rightarrow b \rightarrow c \rightarrow a, \quad d \rightarrow e \rightarrow f \rightarrow d$

dihedral flip: $a \leftrightarrow b, \quad d \leftrightarrow \mathrm{e}$

Examples of patterns

- Every term in the symbol starts with a, b, c; never d, e, f
- Physical reason related to causality, which dictates where branch cuts can appear: only for $(p_i + p_j)^2 \sim 0$
- Empirically, 12 pairs of adjacent letters are forbidden:

$$\ldots a \otimes d \ldots, \quad \ldots b \otimes e \ldots, \quad \ldots c \otimes f \ldots$$
$$\ldots d \otimes a \ldots, \quad \ldots e \otimes b \ldots, \quad \ldots f \otimes c \ldots$$
$$\ldots d \otimes e \ldots, \quad \ldots e \otimes f \ldots, \quad \ldots f \otimes d \ldots$$
$$\ldots e \otimes d \ldots, \quad \ldots f \otimes e \ldots, \quad \ldots d \otimes f \ldots$$

- Resemble constraints from causality:
  Steinmann relations Steinmann, Helv. Phys. Acta (1960)
- But not quite, which mystified us for a while…

Number of remaining parameters in form-factor ansatz after imposing constraints

weight	4	6	8	10	12	14	16
L	2	3	4	5	6	7	8
symbols in $\mathcal{C}$	48	249	1290	6654	34219	????	????
dihedral symmetry	11	51	247	1219	????	????	????
$(L-1)$ final entries	5	9	20	44	86	191	191
L^{th} discontinuity	2	5	17	38	75	171	164
collinear limit	0	1	2	8	19	70	6
OPE $T^2 \ln^{L-1} T$	0	0	0	4	12	56	0
OPE $T^2 \ln^{L-2} T$	0	0	0	0	0	36	0
OPE $T^2 \ln^{L-3} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-4} T$	0	0	0	0	0	0	0
OPE $T^2 \ln^{L-5} T$	0	0	0	0	0	0	0

Antipodal duality in full 2d

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

$$\boxed{F_3^{(L)}(u,v,w) = S\left(A_6^{(L)}(\hat{u},\hat{v},\hat{w})\right)}$$

Antipode map S, at symbol level, reverses order of all letters:

$$S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m \, x_m \otimes \cdots \otimes x_2 \otimes x_1$$

Kinematic map is

$$\hat{u} = \frac{vw}{(1-v)(1-w)}, \qquad \hat{v} = \frac{wu}{(1-w)(1-u)}, \qquad \hat{w} = \frac{uv}{(1-u)(1-v)}$$

Maps $u + v + w = 1$ to parity-preserving surface

$$\Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0$$

6-gluon alphabet and symbol map

- $\mathcal{L}_6 = \{\, \hat{u}, \hat{v}, \widehat{w}, 1-\hat{u}, 1-\hat{v}, 1-\widehat{w}, \cancel{\hat{y}_u, \hat{y}_v, \hat{y}_w}\}$ $\rightarrow$ 1 for $\Delta = 0$

$\rightarrow \mathcal{L}_6' = \{\, \hat{a} = \frac{\hat{u}}{\hat{v}\widehat{w}},\ \hat{b} = \frac{\hat{v}}{\widehat{w}u},\ \hat{c} = \frac{\widehat{w}}{\hat{u}\hat{v}},\ \hat{d} = \frac{1-\hat{u}}{\hat{u}},\ \hat{e} = \frac{1-\hat{v}}{\hat{v}},\ \hat{f} = \frac{1-\widehat{w}}{\widehat{w}}\}$

- Kinematic map on letters:

$$\sqrt{\hat{a}} = d, \quad \hat{d} = a, \qquad \textit{plus cyclic relations}$$

$$\mathcal{S}\left[A_6^{(1)}\right] = (-\tfrac{1}{2})\hat{b}\otimes\hat{d} + \text{dihedral}$$

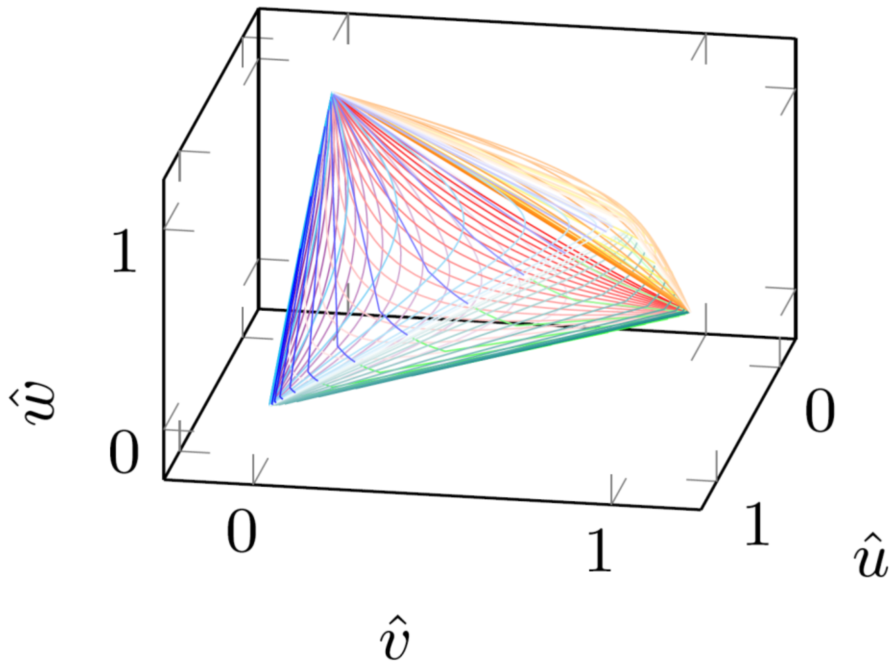
$$\mathcal{S}\left[A_6^{(2)}\right] = \hat{b}\otimes\hat{d}\otimes\hat{d}\otimes\hat{d} + \frac{1}{2}\hat{b}\otimes\hat{b}\otimes\hat{b}\otimes\hat{d} + \text{dihedral}$$

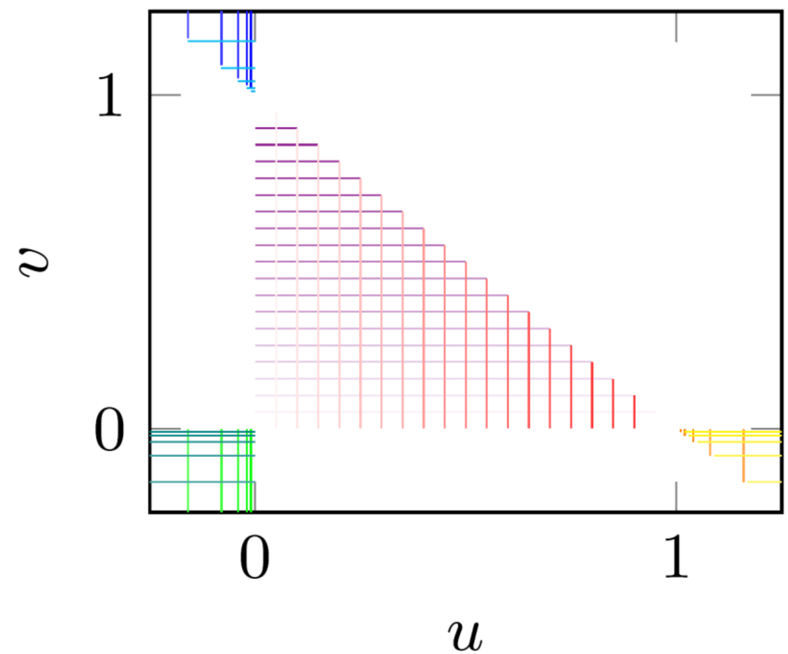
$$\ldots$$

L	number of terms
1	6
2	12
3	636
4	11,208
5	263,880
6	4,916,466
7	$\boxed{92{,}954{,}568}$
8	1,671,656,292

- Works through 7 loops!

Map covers entire phase space for 3-gluon form factor



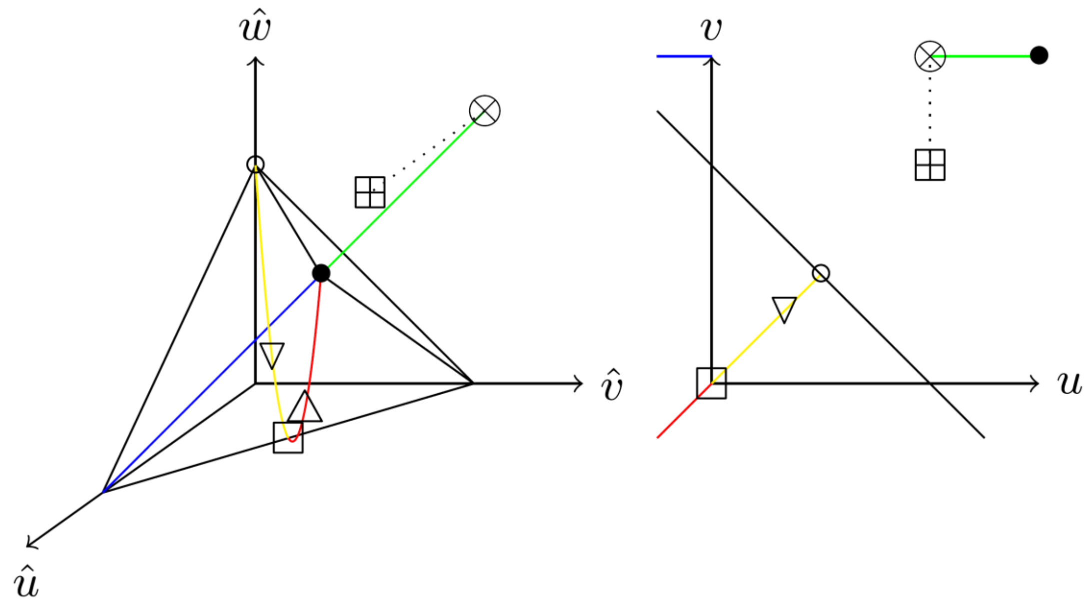


- Soft is dual to collinear; collinear is dual to soft
- White regions in (u, v) map to some of $\hat{u}, \hat{v}, \widehat{w} > 1$

Many special dual points

There is an "f" alphabet at all these points: a way of writing multiple zeta values (MZV's) so that coaction is manifest.

F. Brown, 1102.1310;
O. Schnetz, HyperlogProcedures



	$(\hat{u},\hat{v},\hat{w})$	(u,v,w)	functions
$\triangledown$	$(\frac{1}{4},\frac{1}{4},\frac{1}{4})$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$	$\sqrt[6]{1}$
$\square$	$(\frac{1}{2},\frac{1}{2},0)$	$(0,0,1)$	$\mathrm{Li}_2(\frac{1}{2})$ + logs
$\bullet$	$(1,1,1)$	$\lim_{u\to\infty}(u,u,1-2u)$	MZVs
$\circ$	$(0,0,1)$	$(\frac{1}{2},\frac{1}{2},0)$	MZVs + logs
$\triangle$	$(\frac{3}{4},\frac{3}{4},\frac{1}{4})$	$(-1,-1,3)$	$\sqrt[6]{1}$
$\boxplus$	(∞,∞,∞)	$(1,1,-1)$	alternating sums
$\otimes$	$\lim_{\hat{v}\to\infty}(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(1,v,-v)$	MZVs
	$(1,\hat{v},\hat{v})$	$\lim_{v\to\infty}(u,v,1-u-v)$	HPL$\{0,1\}$
	$(\hat{u},\hat{u},(1-2\hat{u})^2)$	$(u,u,1-2u)$	HPL$\{-1,0,1\}$

Simplest point

- $(\hat{u},\hat{v},\hat{w}) = (1,1,1) \iff u,v \to \infty$
- At this point,

$$A_6^{(1)}(\cdot) = 0 \qquad F_3^{(1)}(\cdot) = 8\zeta_2$$

$$A_6^{(2)}(\cdot) = -9\zeta_4 \qquad F_3^{(2)}(\cdot) = 31\zeta_4$$

$$A_6^{(3)}(\cdot) = 121\zeta_6 \qquad F_3^{(3)}(\cdot) = -145\zeta_6$$

$$A_6^{(4)}(\cdot) = 120 f_{3,5} - 48\zeta_2 f_{3,3} - \frac{6381}{4}\zeta_8 \qquad F_3^{(4)}(\cdot) = 120 f_{5,3} + \frac{11363}{4}\zeta_8$$

$$A_6^{(5)}(\cdot) = -2688 f_{3,7} - 1560 f_{5,5} + \mathcal{O}(\pi^2) \qquad F_3^{(5)}(\cdot) = -2688 f_{7,3} - 1560 f_{5,5} + \mathcal{O}(\pi^2)$$

$$A_6^{(6)}(\cdot) = 48528 f_{3,9} + 37296 f_{5,7} + 21120 f_{7,5} + \mathcal{O}(\pi^2) \qquad F_3^{(6)}(\cdot) = 48528 f_{9,3} + 37296 f_{7,5} + 21120 f_{5,7} + \mathcal{O}(\pi^2)$$

- Reversing ordering of letters in *f*-alphabet, blue values show that antipodal duality holds beyond symbol level, modulo $i\pi$
- modulo $i\pi$ is best we can get from mathematical antipode map

“OPE” coordinates simplify kinematic map

- Amplitude:

$$\hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})}\,,$$

$$\hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2\,, \qquad \hat{w} = \frac{\hat{T}^2}{1+\hat{T}^2}$$

$(\hat{F} = 1 \text{ for } \Delta = 0)$

- Form factor:

$$u = \frac{1}{1+S^2+T^2}\,, \qquad v = \frac{T^2}{1+T^2}\,,$$

$$w = \frac{1}{(1+T^2)(1+S^{-2}(1+T^2))}\,,$$

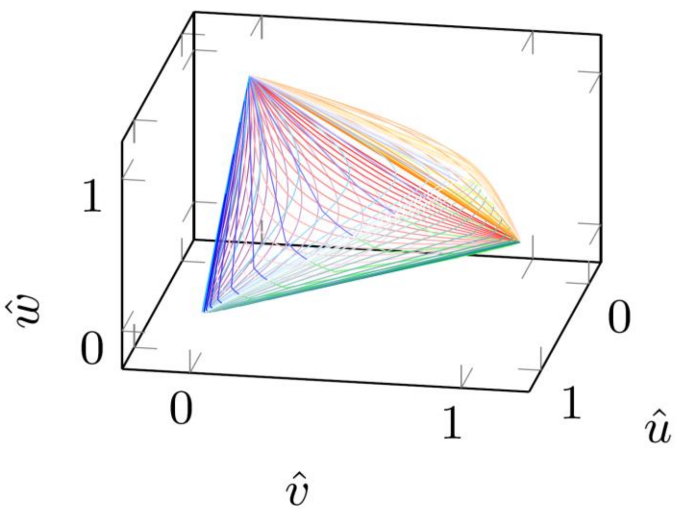
- Kinematic map →

$$\boxed{\hat{T} = \frac{T}{S}\,, \qquad \hat{S} = \frac{1}{TS} \qquad T = \sqrt{\frac{\hat{T}}{\hat{S}}}\,, \qquad S = \sqrt{\frac{1}{\hat{T}\hat{S}}}}$$

Exploit/test antipodal duality at 8 loops

LD, Y.-T. Liu, 2202.nnnnn

- Given form factor, antipodal duality determines symbol of MHV 6 gluon amplitude at 8 loops on $\Delta = 0$ surface.
- Lift symbol into bulk. Only 3 free parameters!
- 2 killed at origin, $(\hat{u}, \hat{v}, \hat{w}) \to (0,0,0)$
- last killed in process of lifting to full function level
- Need one OPE data point to kill one beyond-symbol ambiguity $\propto \zeta_8$



8 loop MHV 6-gluon amplitude at $(\hat{u},\hat{v},\hat{w}) = (1,1,1)$

LD, Y.-T. Liu, 2202.nnnnn

$$
\begin{aligned}
A_6^{(8)}(1,1,1) = {} & 9122624\, f_{9,7} + 11543472\, f_{7,9} + 5153280\, f_{11,5} + 19603536\, f_{5,11} + 23915376\, f_{3,13} \\
& + 371520\, f_{5,3,3,5} + 400320\, f_{3,3,5,5} + 400320\, f_{3,5,3,5} + 825216\, f_{3,3,3,7} \\
& - \zeta_2 \,(701856\, f_{7,7} + 1303232\, f_{9,5} + 430656\, f_{5,9} + 2061312\, f_{11,3} - 309696\, f_{3,11} \\
& \qquad + 160128\, f_{3,5,3,3} + 160128\, f_{3,3,5,3} + 117888\, f_{3,3,3,5} + 148608\, f_{5,3,3,3}) \\
& - \zeta_4 \,(3243888\, f_{5,7} + 3475296\, f_{7,5} + 3909696\, f_{9,3} + 3215472\, f_{3,9} + 353664\, f_{3,3,3,3}) \\
& - \zeta_6 \,(3612804\, f_{5,5} + 3791520\, f_{7,3} + 3409152\, f_{3,7}) - \zeta_8 \,(3720664\, f_{5,3} + 3456614\, f_{3,5}) \\
& - \frac{19560489}{5} \zeta_{10}\, f_{3,3} - \frac{512193667550809}{7639104} \zeta_{16}
\end{aligned}
$$

- Blue values successfully predicted by antipodal duality
- Result consistent with coaction principle at weight 16.

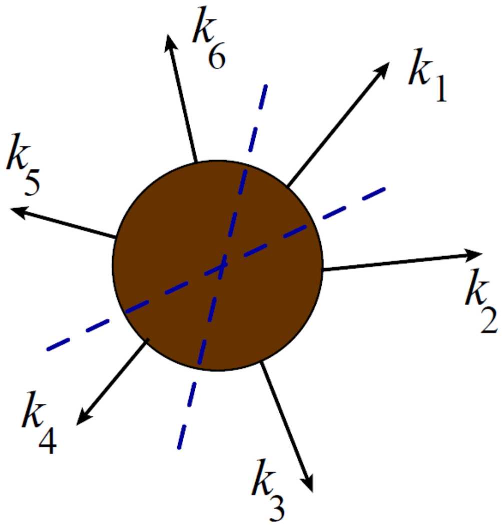
Not clear yet why antipodal duality holds

- But it does explain the mystery of the "Steinmann-like" adjacency constraints:
    - They are **actual** Steinmann constraints for the 6 gluon amplitude!

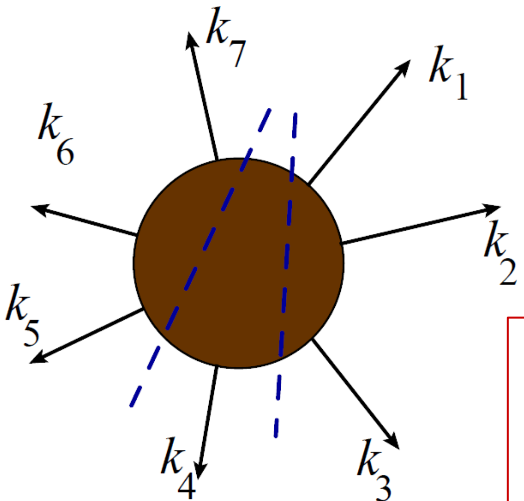
Steinmann relations

Steinmann, Helv. Phys. Acta (1960)   Bartels, Lipatov, Sabio Vera, 0802.2065
McLeod talk; Hannesdottir, McLeod, Schwartz, Vergu, 2211.07633

- Amplitudes should not have overlapping branch cuts:



Not Allowed



Allowed

can't apply to 2 particle cuts in massless case because they are not independent

$$\boxed{\mathrm{Disc}_{s_{234}} \mathrm{Disc}_{s_{123}} A_6(\hat{u}, \hat{v}, \widehat{w}) = 0}$$

if you remove IR divergences properly

Steinmann + DCI consequences

$$\hat{u} = \frac{s_{12}s_{45}}{s_{123}s_{345}}, \quad \hat{v} = \frac{s_{23}s_{56}}{s_{234}s_{123}}, \quad \widehat{w} = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$ are not ideal,

so switch to $\hat{a} \equiv \frac{\hat{u}}{\hat{v}\widehat{w}} = (s_{234}^2)^2 \times \left[s_{i,i+1}\, stuff\right]$

$$\hat{b} \equiv \frac{\hat{v}}{\widehat{w}\hat{u}}, \qquad \hat{c} \equiv \frac{\widehat{w}}{\hat{u}\hat{v}}$$

$$\boxed{\mathrm{Disc}_{\hat{b}}\ \mathrm{Disc}_{\hat{a}}\ A_6(\hat{u},\hat{v},\widehat{w}) = 0}$$

Should hold on any Riemann sheet (?)

Discontinuities via symbol

- Discontinuities commute with derivatives; discontinuities act on left entry of symbol, while derivatives act on right

$$\mathcal{S}[\mathrm{Disc}_{\hat{a}} F] = 2\pi i\, \cancel{\hat{a}} \otimes \ldots$$

- $\mathrm{Disc}_{\hat{b}} \mathrm{Disc}_{\hat{a}} A_6(\hat{u}, \hat{v}, \widehat{w}) = 0$ (+ dihedral images)

means $\mathcal{S}[A_6]$ cannot contain any terms of the form $\hat{a} \otimes \hat{b} \otimes \ldots$

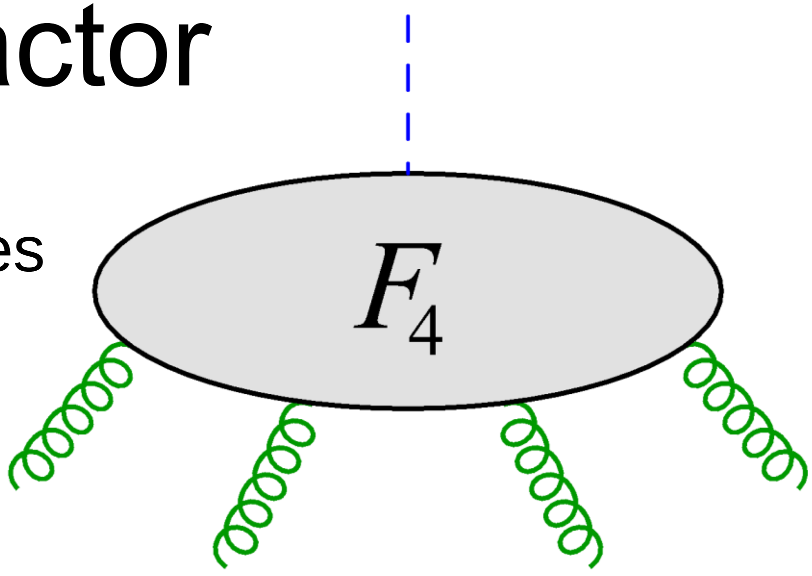
- But we actually find more generally, for any adjacent slots,

$$\boxed{\ldots \otimes \cancel{\hat{a} \otimes \hat{b}} \otimes \ldots}$$

Caron-Huot, LD, McLeod, von Hippel, Papathanasiou, 1806.01361, 1906.07116

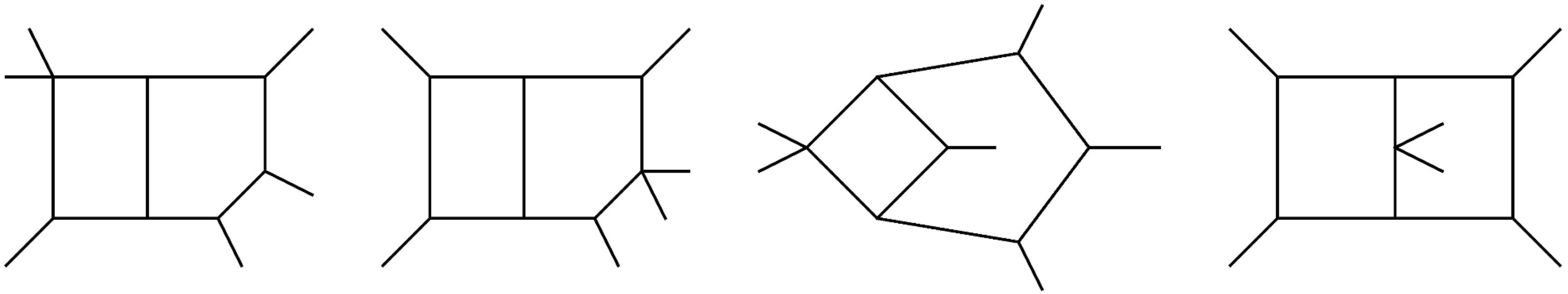
- "Extended Steinmann relations".
- With first entry condition, also find $\ldots \otimes \cancel{\hat{a} \otimes \hat{d}} \otimes \ldots$
- equivalent to "cluster adjacency" for $A_3 = \mathrm{Gr}(4,6)$ cluster algebra
  Drummond, Foster, Gürdoğan, 1710.10953

Four-gluon form factor



Depends on 5 kinematical variables instead of 2.

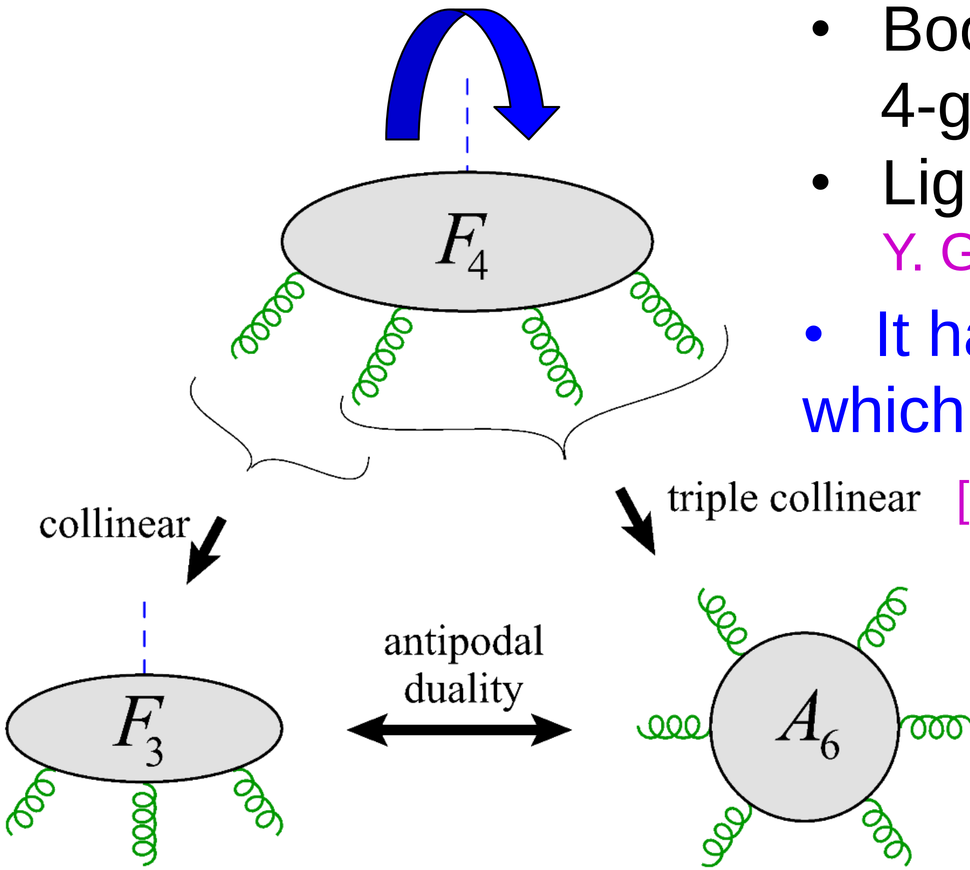
Even just at two loops, contains state-of-the art loop integrals → 113 possible symbol letters!



Abreu, Ita, Moriello, Page, Tschernow, 2005.04195;
Abreu, Ita, Page, Tschernow, 2107.14180;
Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, to appear

Antipodal Self Duality

LD, Ö. Gürdoğan, Y.-T. Liu, A. McLeod, M. Wilhelm, 2212.02410



- Bootstrapped symbol of 4-gluon form factor F_4 at 2 loops
- Light-like limit matches
  Y. Guo, L. Wang, G. Yang, 2209.06816
- It has an antipodal self-duality which subsumes the $F_3 - A_6$ duality!

[Z. B

$$T_2 = \frac{T}{S}, \qquad S_2 = \frac{1}{TS}$$

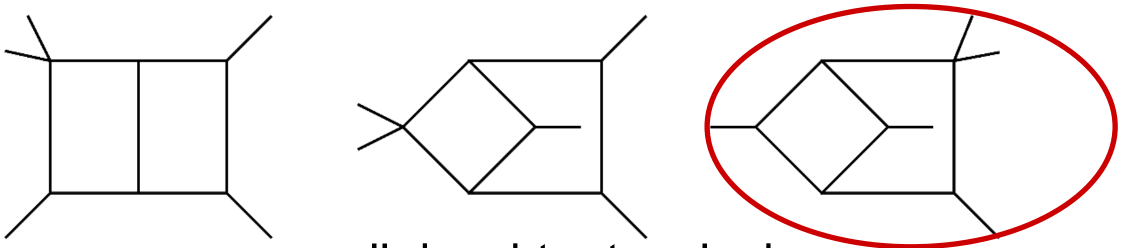
$$T = \sqrt{\frac{T_2}{S_2}}, \qquad S = \sqrt{\frac{1}{T_2 S_2}}$$

$$F_2 = 1$$

Meaning for integrals?

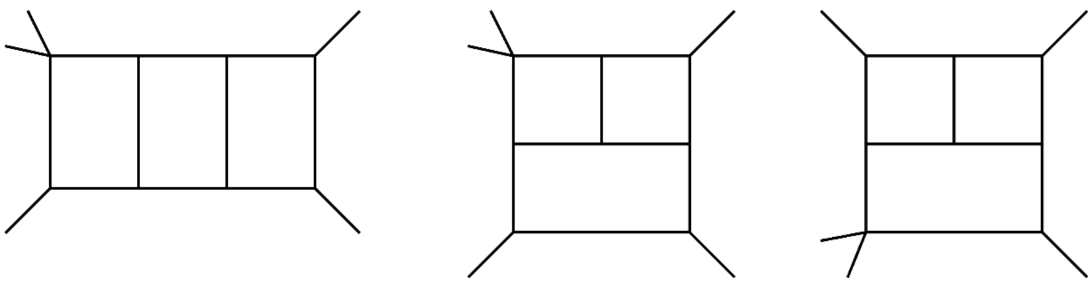
LD, McLeod, Wilhelm, 2012.12286; Chicherin, Henn, Papathanasiou, 2012.12285

Gehrmann, Remiddi, hep-ph/0008287, hep-ph/0101124



doesn't contribute to planar N=4 SYM form factor

+ all daughter topologies



DiVita, Mastrolia, Schubert, Yundin, 1408.3107

Canko, Syrrakos, 2112.14275

all have

$\ldots \cancel{d \otimes e} \ldots$

+ dihedral

half of the adjacency constraints seen in planar N=4 SYM

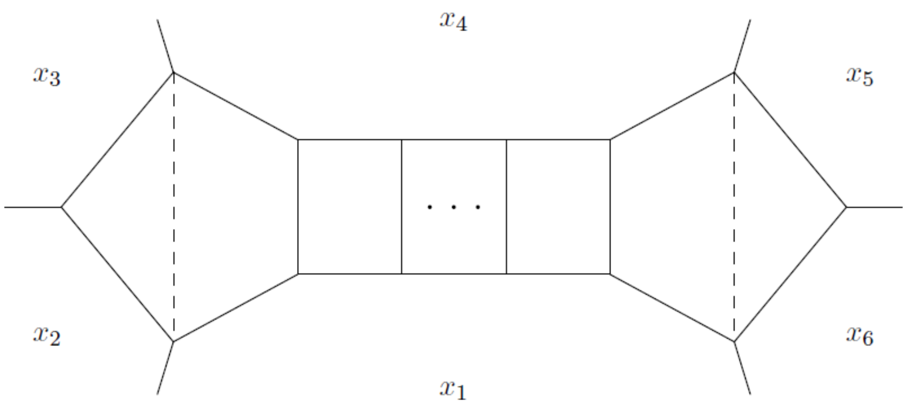
Why?

Remember, no Steinmann relations for massless 2-particle cuts

Some Integrals inside Hexagon Space

Drummond, Henn, Trnka, 1010.3679 ; Caron-Huot et al. 1806.01361

double pentaladders $\Omega^{(L)}(\hat{u}, \hat{v}, \hat{w})$



LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2204.11901

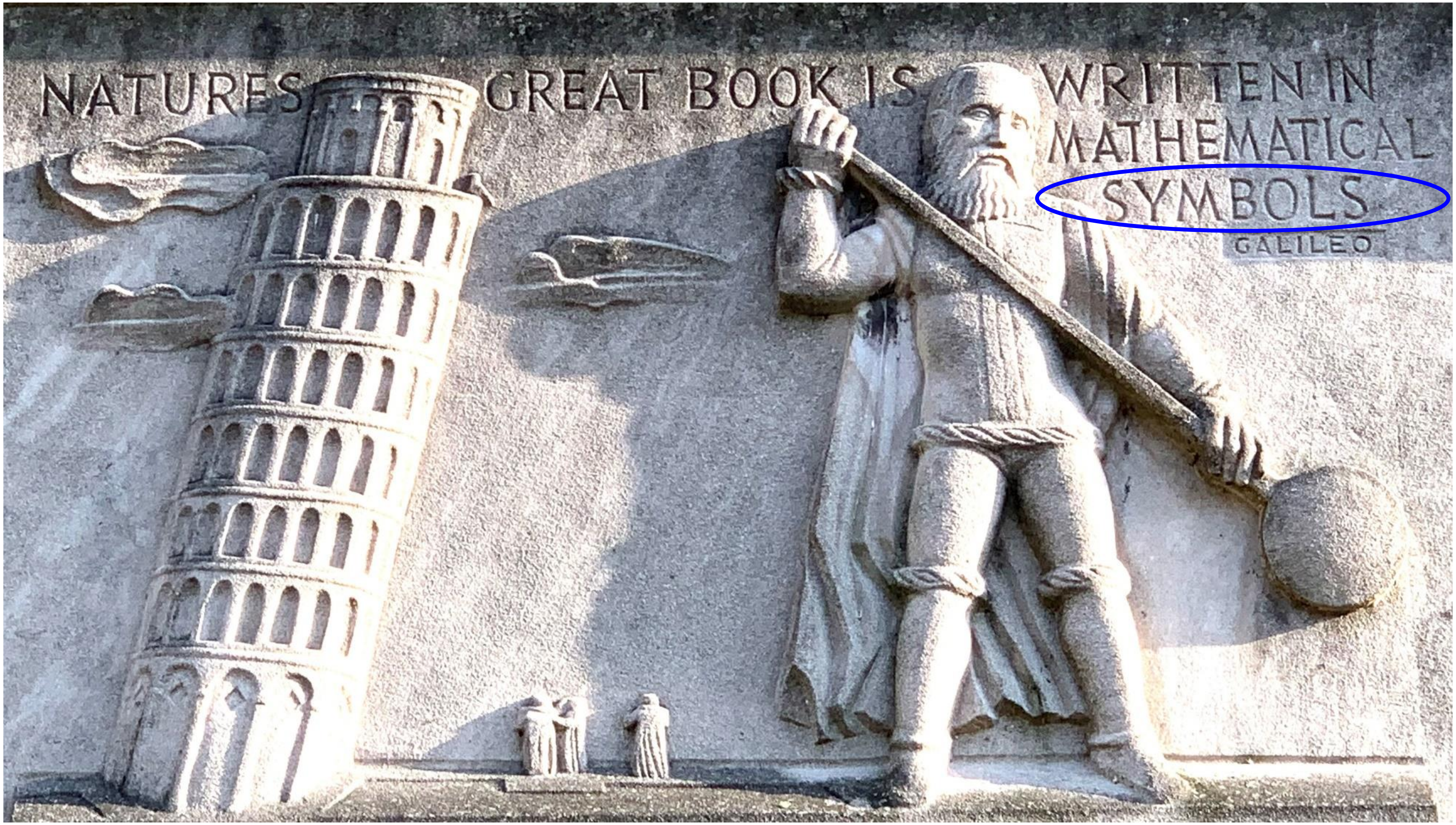
Multiple-final-entry conditions for $\Omega^{(L)}(\hat{u}, \hat{v}, \hat{w})$ don't precisely correspond to those of six-point MHV amplitude. Hence, after applying the duality, the initial (multiple) entries of the integral are not in the form factor space. Dual of $\Omega^{(1)}$ doesnt satisfy the first entry condition; dual of $\Omega^{(2)}$ satisfies that condition, but not first two entry conditions, etc. Lesson seems to be that full amplitude behaves better than individual integrals under the duality.

Summary & Open Questions

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be **bootstrapped** to high loop order
- 6-gluon amplitude and 3-gluon form factor are related by a strange new antipodal duality, swapping the role of branch cuts and derivatives
- Embedded in a 4-gluon form factor self-duality!

- Underlying physical reason for this duality?
- (How) does it hold at strong coupling?
- What other theories might it hold in?
- Are integrals providing a clue?
- How much more can we exploit it to learn more about both amplitudes and form factors?

Thank you!

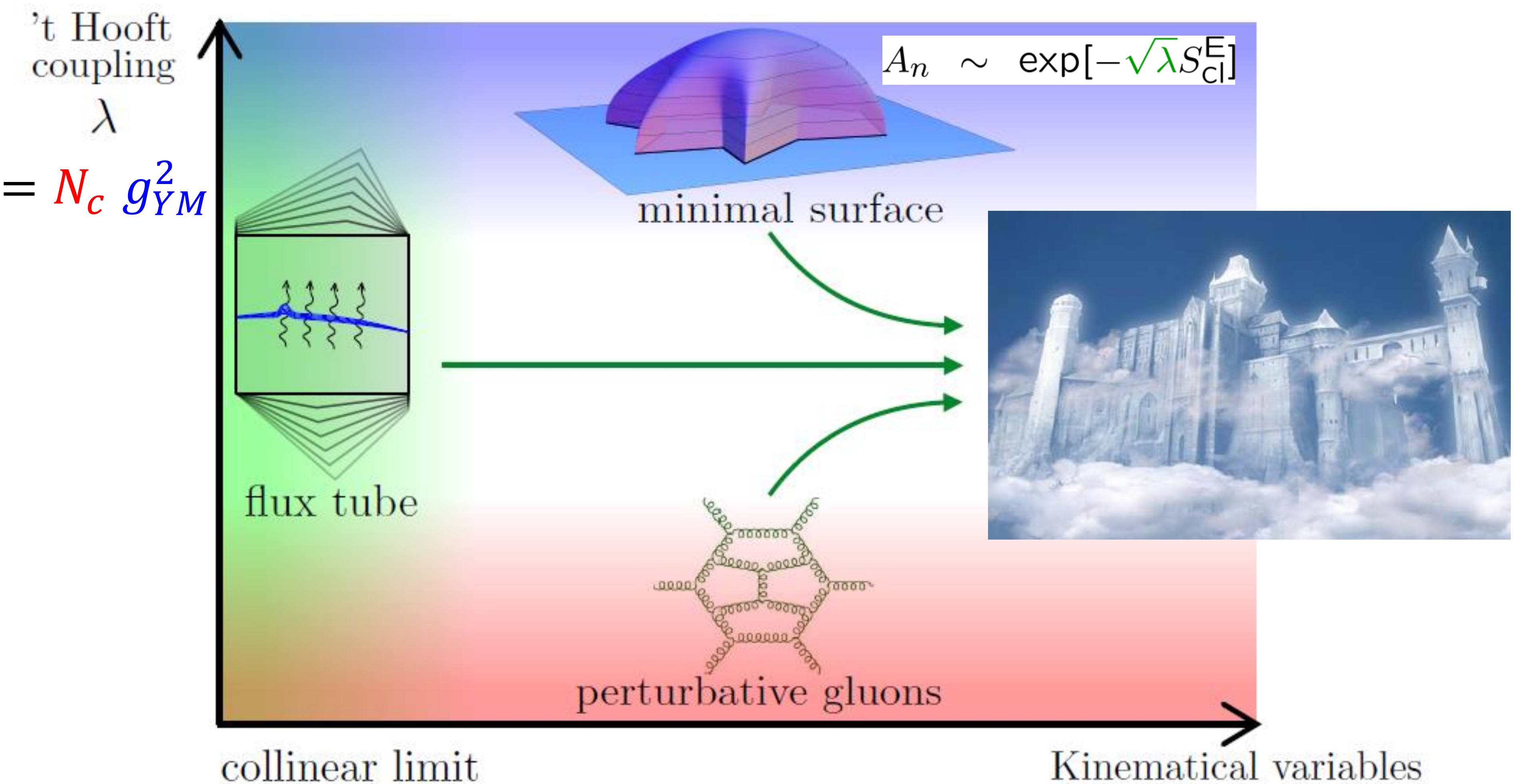


Entrance to Northwestern Physics Department

Extra Slides

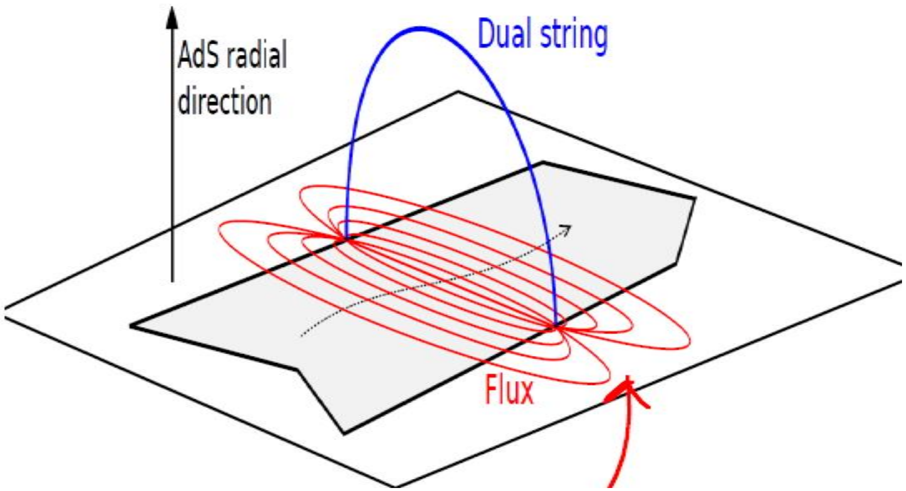
Solving Planar N=4 SYM Scattering

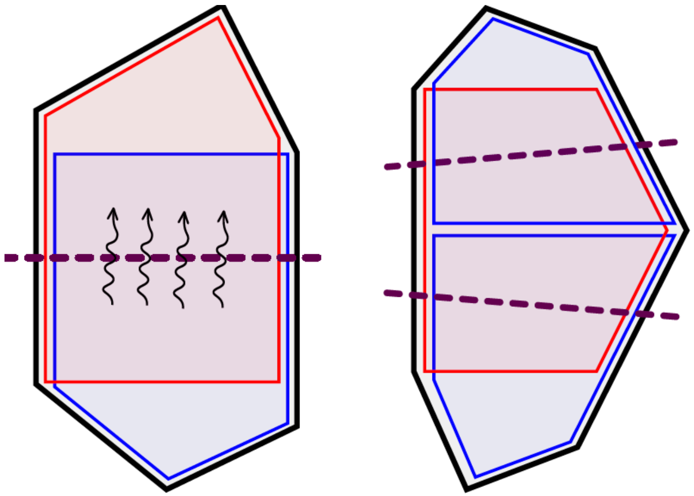
Images: A. Sever, N. Arkani-Hamed



Bootstrap boundary data:
Flux tubes at finite coupling

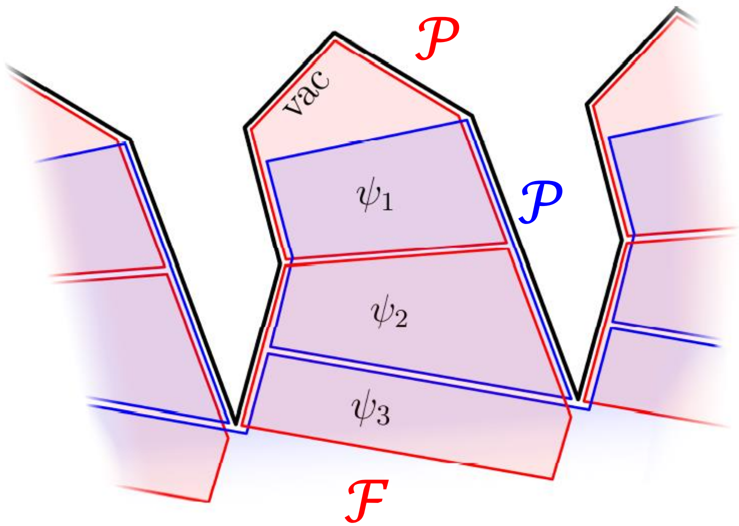
Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788;
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045
BSV+Caetano+Cordova, 1412.1132, 1508.02987





- Tile n-gon with pentagon transitions.
- Quantum integrability → compute pentagons exactly in 't Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit

A New Form Factor OPE



- Form factors are Wilson loops in a periodic space, due to injection of operator momentum

Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$

Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569

OPE representation

- 6-gluon amplitude:

$$\mathcal{W}_{\rm hex} = \sum_{\mathbf{a}} \int d\mathbf{u}\, P_{\mathbf{a}}(0|\mathbf{u}) P_{\mathbf{a}}(\bar{\mathbf{u}}|0)\, e^{-E(\mathbf{u})\tau + ip(\mathbf{u})\sigma + im\phi}$$

$T = e^{-\tau}, S = e^{\sigma}, F = e^{i\phi}$. $\quad v = \frac{T^2}{1+T^2} \to 0$,

weak-coupling, $E = k + \mathcal{O}(g^2)$ → expansion in T^k

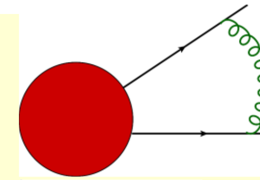
- 3-gluon form factor: $\quad \psi = helicity\ 0\ pairs\ of\ states$

$$\mathcal{W}_3 = \sum\!\!\!\!\!\!\int_{\psi} e^{-E_\psi \tau + ip_\psi \sigma} \mathcal{P}(0|\psi) \mathcal{F}(\psi)$$

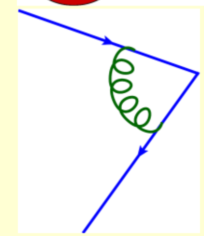
weak-coupling → expansion in T^{2k} (no azimuthal angle ϕ)

Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes IR divergent due to long-range gluons



- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\rm cusp}$



– known to all orders in planar N=4 SYM:
Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708
- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\boxed{\mathcal{E}_6(u_i) = \lim_{\epsilon\to 0} \frac{\mathcal{A}_6(s_{i,i+1},\epsilon)}{\mathcal{A}_6^{\rm BDS-like}(s_{i,i+1},\epsilon)} = \exp[\frac{\Gamma_{\rm cusp}}{4}\mathcal{E}_6^{(1)} + R_6]}$$

remainder function

BDS & BDS-like normalization for $\mathcal{F}_3$

$$\frac{\mathcal{F}_3}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{4} + \mathcal{O}(\epsilon)\right)M^{1-\mathrm{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u,v,w)\right]\right\}$$

BDS ansatz

remainder function only a function of u, v, w;
vanishes in all collinear limits,
but no adiac... y constraints

split 1-loop amplitude judiciously:

$$\frac{\mathcal{F}_3^{1-\mathrm{loop}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} \equiv M^{1-\mathrm{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u,v,w)$$

$$M(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{3}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{\epsilon} - \frac{7}{2}\zeta_2 + \sum^{3} \ldots$$

$$\mathcal{E}^{(1)}(u,v \ldots \left(1-\frac{1}{v}\right) + \mathrm{Li}_2\left(1-\frac{1}{w}\right)\Big] \qquad \mathcal{E}^{(1),u} + \mathcal{E}^{(1),1-u} = 0$$

$\mathcal{E}$ obeys "adjacency constraints"

Now div... by:

$$\frac{\mathcal{F}_3^{\mathrm{BDS-like}}}{\mathcal{F}_3^{\mathrm{MHV,\,tree}}} = \exp\left\{\sum_{L=1}^{\infty} g^{2L}\left[\left(\frac{\Gamma_{\mathrm{cusp}}}{4} + \mathcal{O}(\epsilon)\right)M(L\epsilon) + C^{(L)}\right]\right\} \Rightarrow \boxed{\mathcal{E} = \exp\left[\frac{\Gamma_{\mathrm{cusp}}}{4}\mathcal{E}^{(1)} + R\right]}$$

Finite radius of convergence

- Planar N=4 SYM has no renormalons ($\beta(g) = 0$) and no instantons ($e^{-1/g_{\mathrm{YM}}^2} = e^{-N_c/\lambda}$)
- Perturbative expansion can have finite radius of convergence, unlike QCD, QED, whose perturbative series are asymptotic.
- For cusp anomalous dimension, using coupling

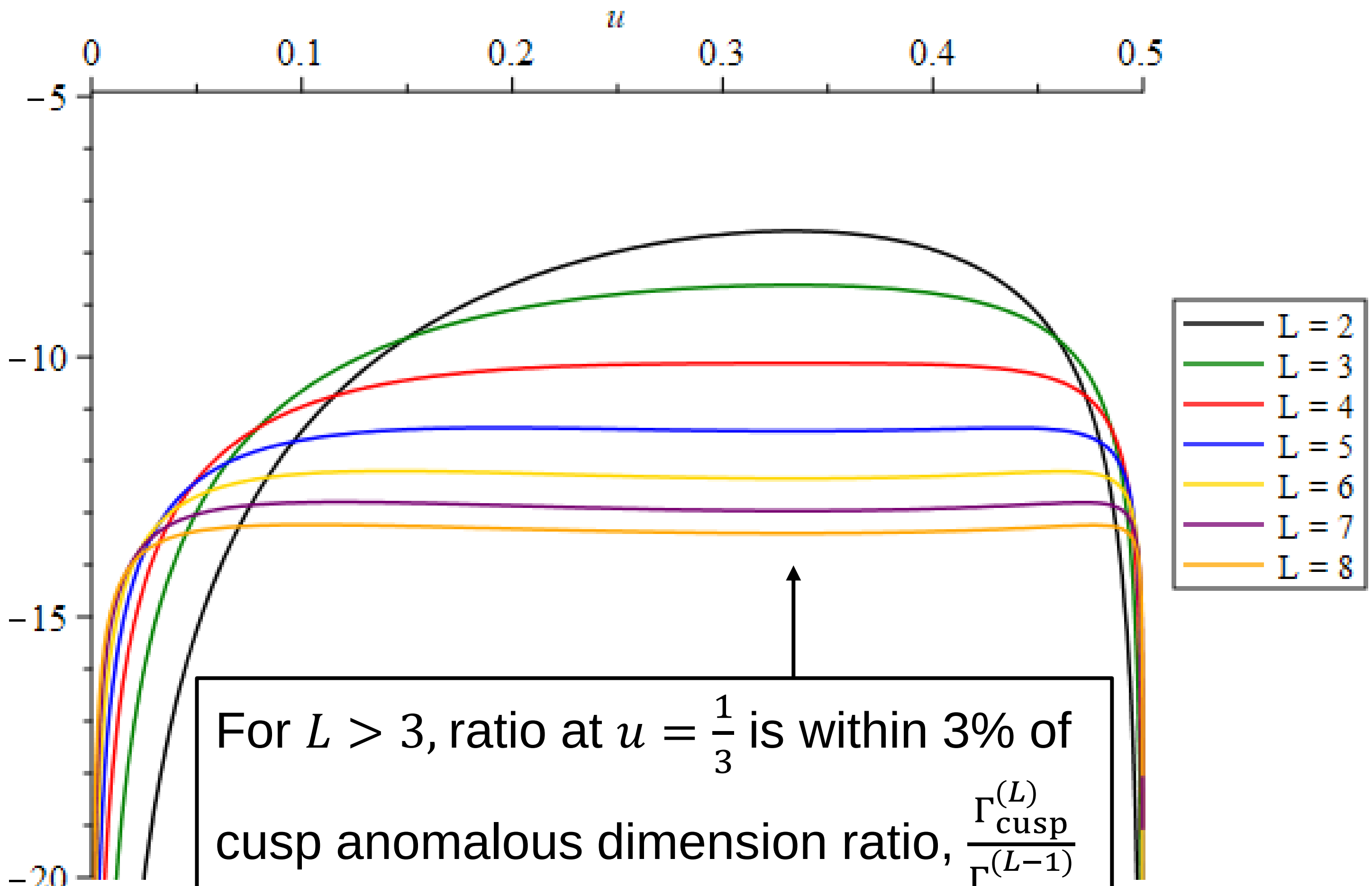
$$g^2 \equiv \frac{N_c g_{\mathrm{YM}}^2}{16\pi^2} = \frac{\lambda}{16\,\pi^2}, \quad \text{radius is } \frac{1}{16}$$

Beisert, Eden, Staudacher (BES), 0610251

- Ratio of successive loop orders $\frac{\Gamma_{\mathrm{cusp}}^{(L)}}{\Gamma_{\mathrm{cusp}}^{(L-1)}} \to -16$
- Find same radius of convergence in high-loop-order behavior of amplitudes and form factors, in most kinematic regions.

Euclidean Region numerics

$$\frac{\mathcal{E}^{(L)}(u,u,1-2u)}{\mathcal{E}^{(L-1)}(u,u,1-2u)}$$



For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma_{\rm cusp}^{(L)}}{\Gamma_{\rm cusp}^{(L-1)}}$

→ same finite radius of convergence…

$\mathcal{E} \equiv F_3$

Values of HPLs {0,1} at u = 1

- Classical polylogs

evaluate to Riemann zeta values

$$\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t}\mathrm{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}$$

$$\mathrm{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n$$

- HPL's evaluate to nested sums called multiple zeta values (MZVs):

$$\zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1>k_2>\cdots>k_m>0}^{\infty} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}$$

Weight $n = n_1 + n_1 + \ldots + n_m$

- MZV's obey many identities, e.g. stuffle

$$\zeta_{n_1}\zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}$$

- All reducible to Riemann zeta values until weight 8.

  Irreducible MZVs: $\zeta_{5,3},\ \zeta_{7,3},\ \zeta_{5,3,3},\ \zeta_{9,3},\ \zeta_{6,4,1,1},\ \ldots$

Number of (symbol-level) linearly independent $\{n, 1, \ldots, 1\}$ coproducts $(2L - n$ derivatives)

weight n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$L=1$	1	3	1														
$L=2$	1	3	6	3	1												
$L=3$	1	3	9	12	6	3	1										
$L=4$	1	3	9	21	24	12	6	3	1								
$L=5$	1	3	9	21	46	45	24	12	6	3	1						
$L=6$	1	3	9	21	48	99	85	45	24	12	6	3	1				
$L=7$	1	3	9	21	48	108	236	155	85	45	24	12	6	3	1		
$L=8$	1	3	9	21	48	108	242	466	279	155	85	45	24	12	6	3	1

- Properly normalized L loop N=4 form factors $\mathcal{E}^{(L)}$ belong to a small space $\mathcal{C}$, dimension saturates on left
- $\mathcal{E}^{(L)}$ also obeys multiple-final-entry relations, saturation on right

Multi-final entry relations

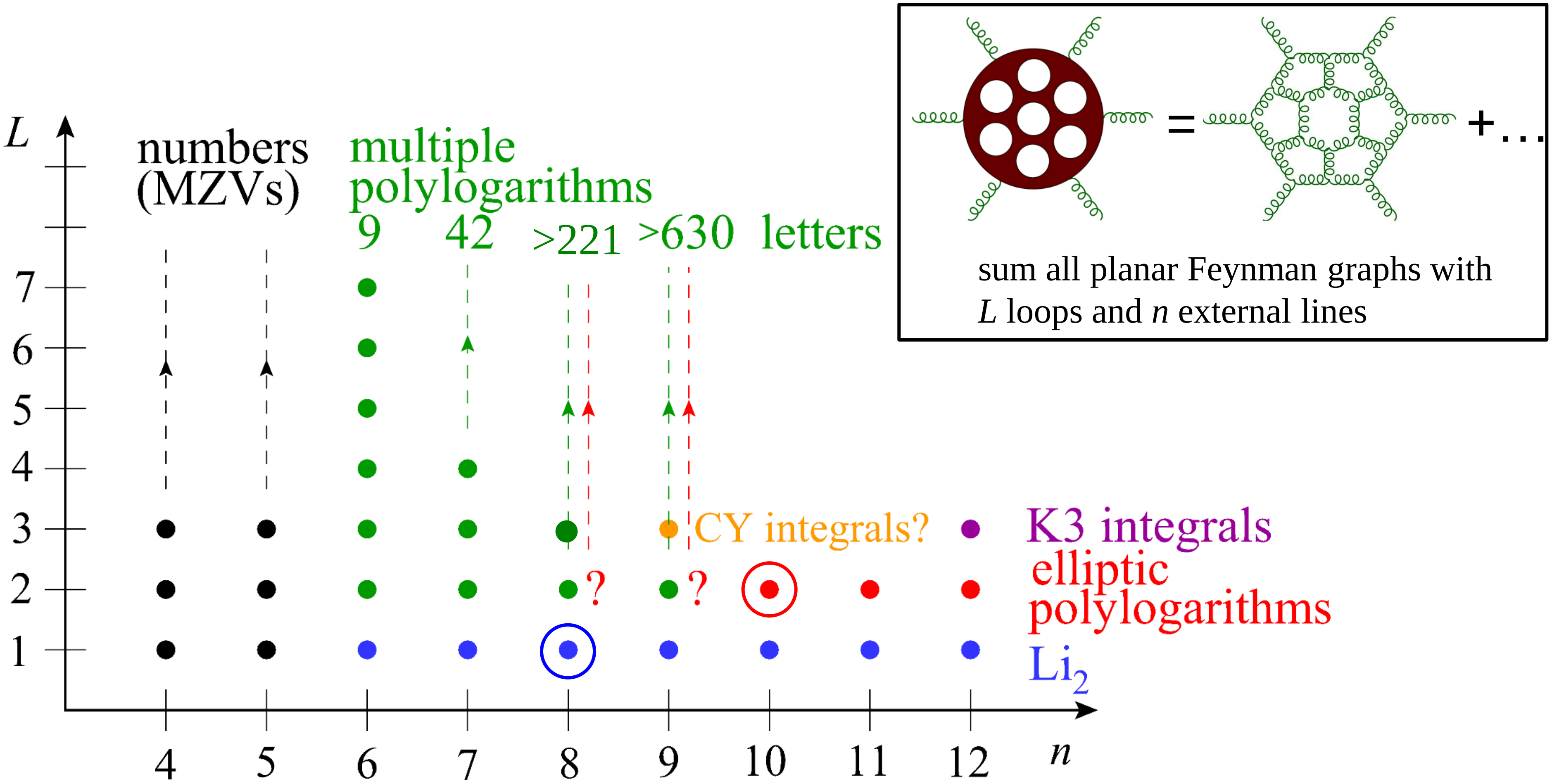
1. $\mathcal{E}^{a} = 0$ (plus dihedral images)
2. $\mathcal{E}^{a,e} = \mathcal{E}^{a,f}$ (plus …)
3. $\mathcal{E}^{a,b,d} = 0, \qquad \mathcal{E}^{a,e,e} = -\,\mathcal{E}^{a,f,f},$

$$\mathcal{E}^{e,a,f} = \mathcal{E}^{f,a,f} - \mathcal{E}^{a,f,f}$$

4. …

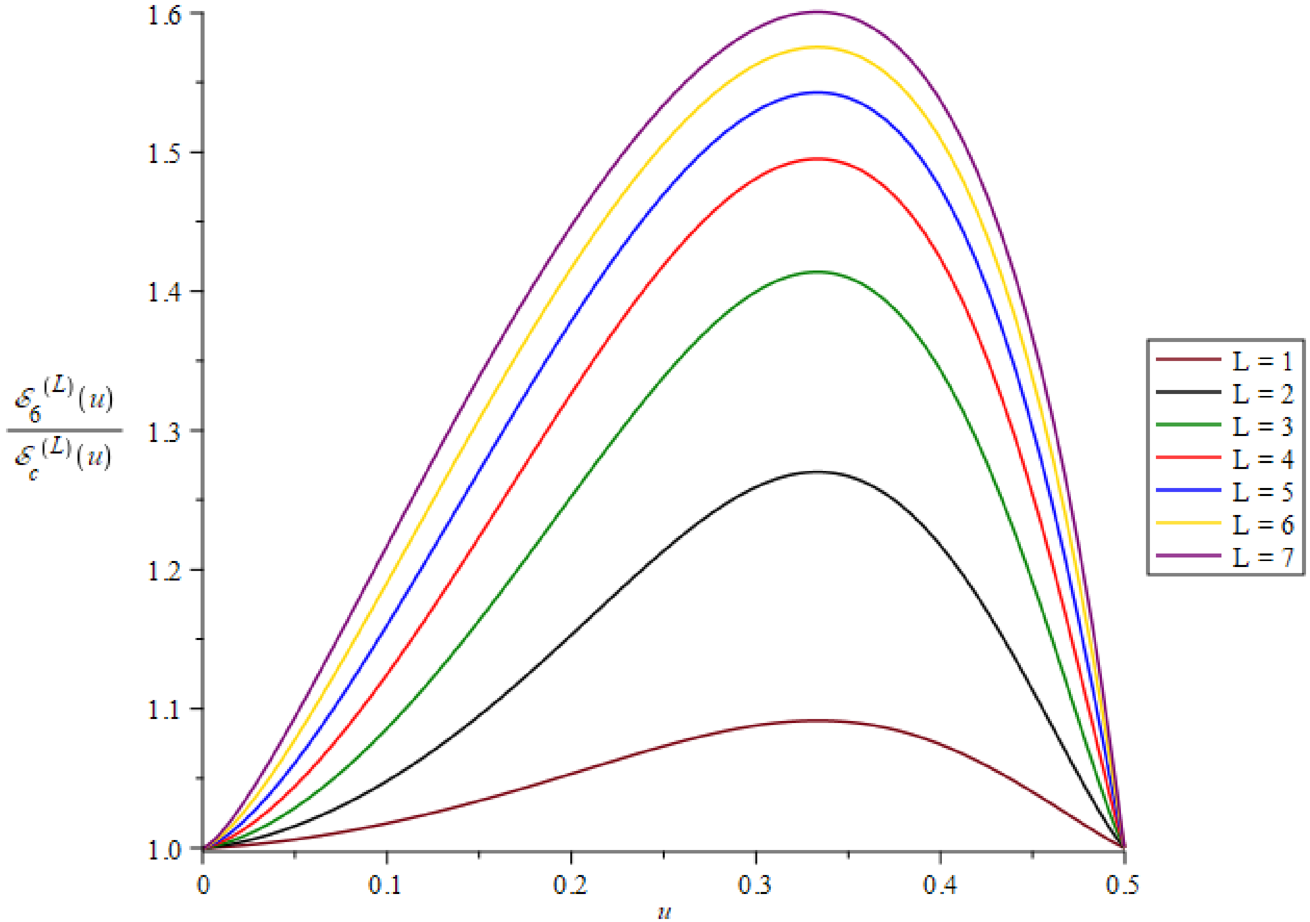
Originally empirical, but all follow from causal, initial-entry and Steinmann relations for the 6-gluon hexagon space!

Beyond $n = 8$



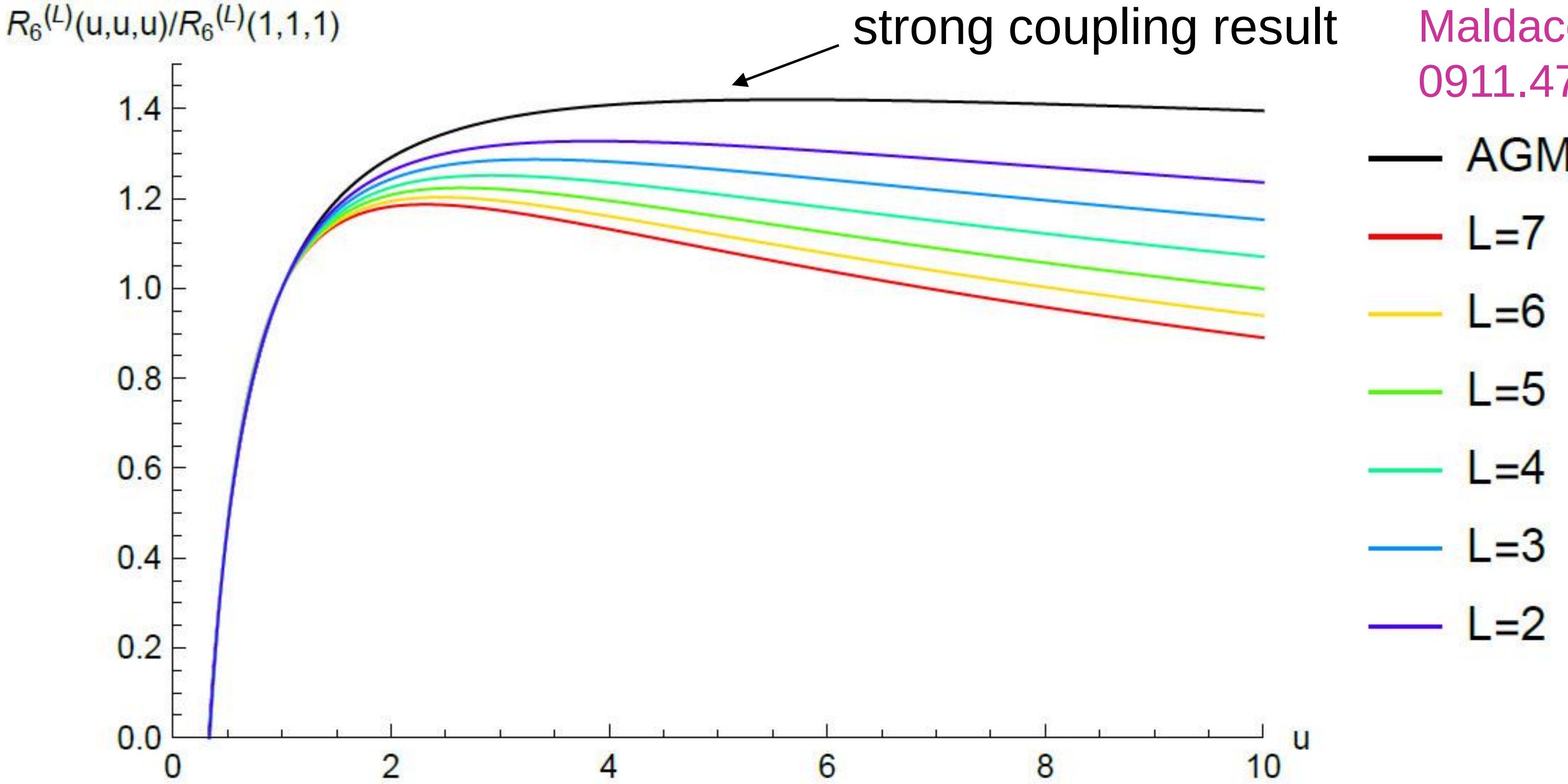
L
numbers
(MZVs)
multiple
polylogarithms
9
42
>221
>630
letters
CY integrals?
K3 integrals
elliptic
polylogarithms
Li2
?
?
1
2
3
4
5
6
7
4
5
6
7
8
9
10
11
12
n
=
+…
sum all planar Feynman graphs with
L loops and n external lines

Numerical implications of antipodal duality?



Example: MHV finite remainder $R_6^{(L)}$ on (u,u,u)

Alday, Gaiotto, Maldacena, 0911.4708



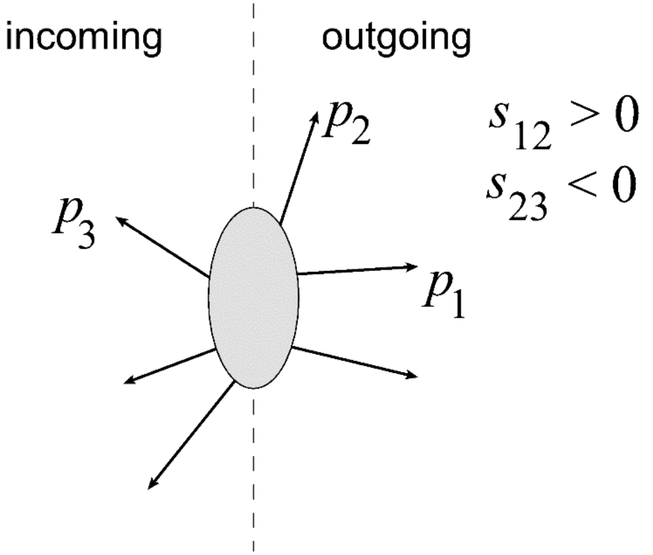
- **Amazing proportionality** of **each** perturbative coefficient at small u, and also with the strong coupling result

Steinmann for amplitudes for massless external states

Steinmann was an axiomatic quantum field theorist, and he would absolutely forbid us from applying his relations to any theory without a mass gap

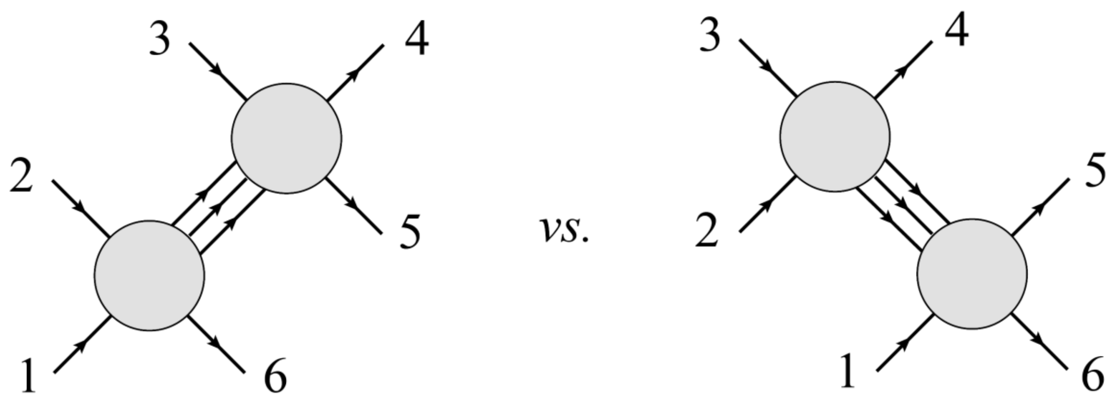
But we will do it anyway

However, we can't do it for discontinuities in 2-particle channels, because they can't be varied independently



Steinmann for 3 particle channels

3-particle channels in amplitudes with $n \geq 6$ particles can cross threshold independently of any other invariants.
Most transparent in 3 → 3 scattering:



Can move s_{345} across 0 with all other invariants generic, and similarly for $s_{561} = s_{234}$. Furthermore, there is a region where both s_{345} and s_{561} can cross 0, and this is the key to Steinmann's argument

$$\boxed{\mathrm{Disc}_{s_{234}}\ \mathrm{Disc}_{s_{345}} \mathcal{A}_6(s_{ij}, s_{ijk}, \epsilon) = 0} \qquad D = 4 - 2\epsilon$$