Progress on 3-loop Feynman integrals for 4-point 1-mass processes

Nikolaos Syrrakos

based on 2112.14275(JHEP), in collaboration with Dhimiter Canko

and ongoing work with Thomas Gehrmann, Petr Jakubčík, Cesare Carlo Mella, Lorenzo Tancredi.

Technical University Munich

Mathematical structures in Feynman Integrals, 16.02.2023

Table of Contents

Table of Contents

Introduction

- QCD corrections for $2 \to 2$ scattering at NNLO are well under control.

Introduction

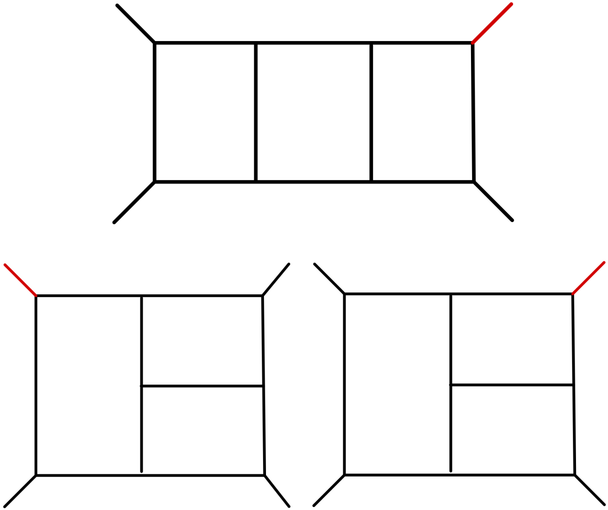
- QCD corrections for $2 \to 2$ scattering at NNLO are well under control.
- All three-loop master integrals for massless $2 \to 2$ scattering have been recently computed:
  - **Henn, Smirnov, Smirnov, JHEP07(2013)128.**
  - **Henn, Mistlberger, Smirnov, Wasser, JHEP04(2020)167.**

Introduction

- QCD corrections for $2 \to 2$ scattering at NNLO are well under control.
- All three-loop master integrals for massless $2 \to 2$ scattering have been recently computed:
  - **Henn, Smirnov, Smirnov, JHEP07(2013)128.**
  - **Henn, Mistlberger, Smirnov, Wasser, JHEP04(2020)167.**
- Three-lopp planar master integrals for $2 \to 2$ scattering with one massive leg:
  - **Di Vita, Mastrolia, Schubert, Yundin, JHEP09(2014)148.**
  - **Canko and NS, JHEP02(2021)080.**
  - **Canko and NS, JHEP04(2022)134.**

Table of Contents

Planar top sectors



Master integrals and kinematics

- 83(Ladder), 117(TennisCourt1) and 166(TennisCourt2) master integrals (KIRA2, FIRE6).
- $\sum_{i=1}^{4} p_i = 0,\ p_4^2 = m^2,\ p_i^2 = 0$ for $i = 1, 2, 3,\ s_{ij} = (p_i + p_j)^2$.
- Euclidean region: $s_{13} < 0, \quad s_{23} < 0, \quad m^2 < s_{13} + s_{23}$.
- Scattering kinematics

$$\text{s-channel}: m^2 > 0, \quad s_{12} \geq m^2, \quad s_{23} \leq 0, \quad s_{13} \leq 0 \qquad (1)$$

$$\text{t-channel}: m^2 > 0, \quad s_{12} \leq 0, \quad s_{23} \geq m^2, \quad s_{13} \leq 0 \qquad (2)$$

$$\text{u-channel}: m^2 > 0, \quad s_{12} \leq 0, \quad s_{23} \leq 0, \quad s_{13} \geq m^2. \qquad (3)$$

Main steps

- Construct a canonical basis[1] $\mathbf{g}$.
- Apply the Simplified Differential Equations approach[2].
- Compute necessary boundary terms.
- Results in terms of MPLs and analytic continuation.

$$\mathcal{G}(a_1, a_2, \ldots, a_n; x) = \int_0^x \frac{\mathrm{dt}}{t - a_1} \mathcal{G}(a_2, \ldots, a_n; t) \tag{4}$$

$$\mathcal{G}(0, \ldots, 0; x) = \frac{1}{n!} \log^n(x) \tag{5}$$

[1] Henn, Phys. Rev. Lett. **110** (2013), 251601.
[2] Papadopoulos, JHEP **07** (2014), 088

Canonical basis

- Up to seven propagators: Magnus series expansions[3] (Federico Gasparotto, Luca Mattiazzi).
- Up to nine propagators: `Mathematica` package `DlogBasis`[4].
- Top sector: Analyse leading singularities in 4D loop-by-loop and use known 1-,2-loop results as building blocks[5].

[3] Argeri et al., JHEP **03** (2014), 082.

[4] Henn, Mistlberger, Smirnov, Wasser, JHEP04(2020)167.

[5] P. Wasser, PhD thesis.

Simplified Differential Equations (SDE)

- Parametrise the external momenta by introducing a dimensionless parameter x in the following manner

$$p_1 = q_3, \quad p_2 = -q_1 - q_2 - q_3, \quad p_3 = xq_1, \quad p_4 = q_1 + q_2 - xq_1 \quad (6)$$

with $\sum_{i=1}^{4} q_i = 0,\ q_i^2 = 0$.

- Mapping for the kinematic invariants between the two momentum configurations

$$s_{13} = -(S_{12} + S_{23})x, \quad s_{23} = S_{23}x, \quad m^2 = S_{12}(1 - x) \qquad (7)$$

with $S_{12} = (q_1 + q_2)^2,\ S_{23} = (q_2 + q_3)^2$.

- Euclidean region: $0 < x < 1,\ S_{12} < 0,\ 0 < \tilde{y} < 1$, with $\tilde{y} = S_{23}/S_{12}$.

Canonical SDE

- The master integrals are now dependent on x through the external momenta.

Canonical SDE

- The master integrals are now dependent on x through the external momenta.

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{8}$$

- l_i: contain all kinematic dependence.
- $\mathbf{M}_i$: residue matrices corresponding to each pole l_i, consisting of rational numbers.

Canonical SDE

- The master integrals are now dependent on x through the external momenta.

$$\partial_x \mathbf{g} = \epsilon \left(\sum_{i=1}^{4} \frac{\mathbf{M}_i}{x - l_i} \right) \mathbf{g} \tag{8}$$

- l_i: contain all kinematic dependence.
- $\mathbf{M}_i$: residue matrices corresponding to each pole l_i, consisting of rational numbers.
- Pole structure:

$$\left\{ x,\ x-1,\ x - \frac{1}{1+\tilde{y}},\ x + \frac{1}{\tilde{y}} \right\}. \tag{9}$$

General solution up to weight six

$$\mathbf{g} = \epsilon^0 \mathbf{b}_0^{(0)} + \epsilon \left(\sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(0)} + \mathbf{b}_0^{(1)} \right) + \epsilon^2 \left(\sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(0)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(1)} + \mathbf{b}_0^{(2)} \right) + \dots$$
$$+ \epsilon^6 \Big(\mathbf{b}_0^{(6)} + \sum \mathcal{G}_{ijklmn} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{M}_n \mathbf{b}_0^{(0)} + \sum \mathcal{G}_{ijklm} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{M}_m \mathbf{b}_0^{(1)}$$
$$+ \sum \mathcal{G}_{ijkl} \mathbf{M}_i \mathbf{M}_j \mathbf{M}_k \mathbf{M}_l \mathbf{b}_0^{(2)} + \sum \mathcal{G}_{ijk} \mathbf{M}_i \mathbf{M}_j \mathrm{M}_k \mathbf{b}_0^{(3)} + \sum \mathcal{G}_{ij} \mathbf{M}_i \mathbf{M}_j \mathbf{b}_0^{(4)} + \sum \mathcal{G}_i \mathbf{M}_i \mathbf{b}_0^{(5)} \Big) \tag{10}$$

- $\mathcal{G}_{ab\dots} := \mathcal{G}(l_a, l_b, \dots; x)$.
- $\mathbf{b}_0^{(i)}$: boundary terms involving rational numbers and $\{\zeta(i),\ \log(-S_{12}),\ \log(\tilde{y})\}$.

Boundary terms

- Residue matrix for $l_1 = 0 \to \mathbf{M}_1 = \mathbf{SDS}^{-1}$.
- $\mathbf{R} = \mathbf{S}\mathrm{e}^{\epsilon \mathbf{D}\log(x)}\mathbf{S}^{-1}$.
- $\mathbf{b} = \sum_{i=0}^{6} \ \epsilon^i \ b_0^{(i)}$.
- IBP reduction: $\mathbf{g} = \mathbf{TI}$.
- Expansion-by-regions using `asy`: $I_i \underset{x\to 0}{=} \sum_j x^{b_j + a_j\epsilon} I_i^{(b_j + a_j\epsilon)}$.
- Master equation:

$$\mathbf{Rb} = \lim_{x\to 0} \mathbf{TI}\Big|_{\mathcal{O}\left(x^{0+a_j\epsilon}\right)} \tag{11}$$

Analytic continuation

- Tools: `HyperInt`, `PolyLogTools`, `GiNaC`.

Regions	Indices	Argument	Indices	Argument
Euclidean	$\{0, 1, -1/\tilde{y}, 1/(1+\tilde{y})\}$	x	$-$	$-$
s-channel	$\{0, 1, -1/\tilde{y}, 1/(1+\tilde{y})\}$	x	$-$	$-$
t-channel	$\{0, 1, -\tilde{y}, 1+\tilde{y}\}$	$1/x$	$\{0, 1\}$	$-1/\tilde{y}$
u-channel	$\{0, 1, -\tilde{y}, 1+\tilde{y}\}$	$1/x$	$\{0, -1\}$	$\tilde{y}$

Table: Structure of MPLs appearing in each of the 4 kinematic regions.

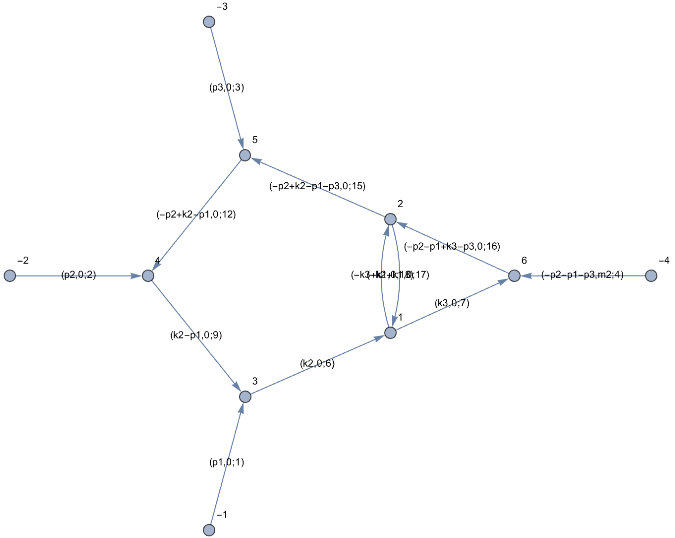
R	$W=1$	$W=2$	$W=3$	$W=4$	$W=5$	$W=6$	**Total**	**Timings (sec)**
E	4	14	50	124	367	734	1293	39.0225769
s	4	14	50	124	367	734	1293	39.2172529
t	6	18	58	155	419	603	1259	62.0567800
u	5	16	54	147	403	572	1197	55.1049640

Table: Number of MPLs per weight and region, and timings for the numerical evaluation of the total MPLs.

Are these enough for an amplitude calculation?

- E.g. $q\bar{q} \to Z + g$ at 3 loops (leading colour).
- 20 planar top sectors, 3 irreducible.
- Reducible top sectors have subsectors that contribute additional master integrals.
- Define a single family for all top sectors$\to$ $291(= 235 + 56)$ master integrals.
- Only one genuinely new master integral.

New master integral



Progress on planar master integrals

- Canonical basis for all 291 masters.
- Differential equations in $y = s_{13}/m^2$, $z = s_{23}/m^2$.
- Alphabet

$$\{y, z, 1-y, 1-z, 1-y-z, y+z\}. \tag{12}$$

Table of Contents

Progress on non-planar master integrals

- 15 irreducible top sectors!
- Sectors with many master integrals.

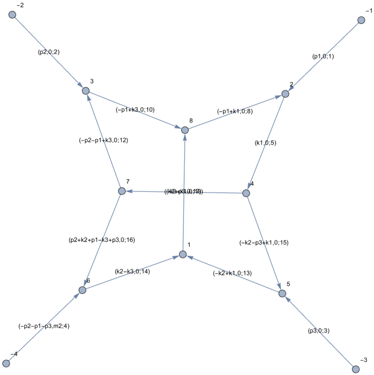
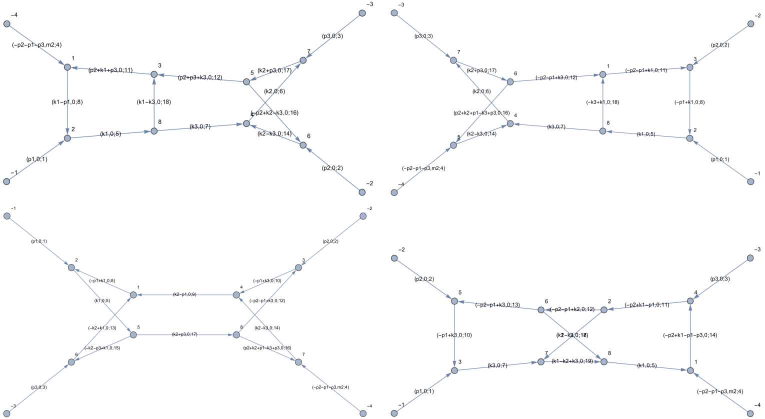
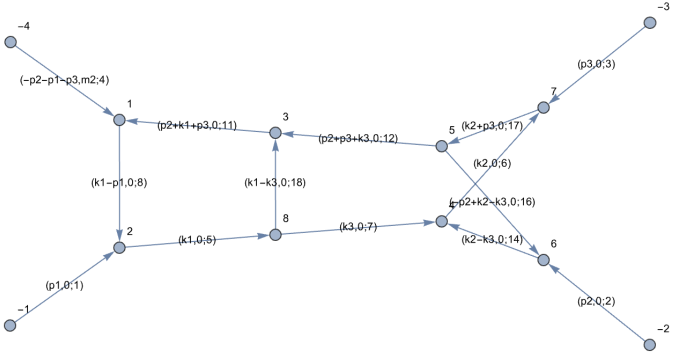


Figure: Top sector with 19 master integrals.

Ladder-like top sectors



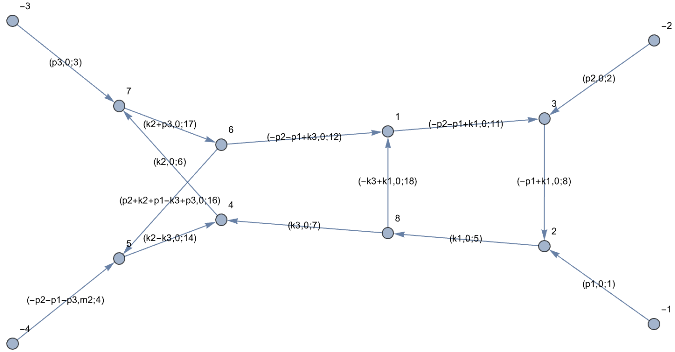
Preliminary results



- Canonical basis of 114 master integrals.
- Pole structure (SDE):

$$\left\{ x,\ x-1,\ x-\frac{1}{1+\tilde{y}},\ x+\frac{1}{\tilde{y}} \right\}. \tag{13}$$

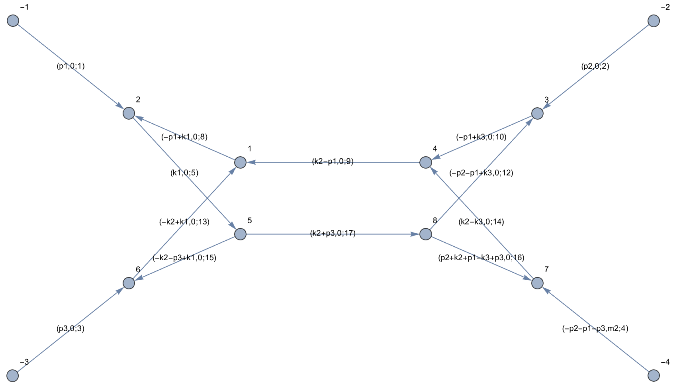
Preliminary results



- Canonical basis of 150 master integrals.
- Pole structure (SDE) $\rightarrow$ 2 new entries:

$$\left\{x,\ x-1,\ x-\frac{1}{1+\tilde{y}},\ x+\frac{1}{\tilde{y}}, x-\frac{1+\tilde{y}}{\tilde{y}}, x-\frac{\tilde{y}}{1+\tilde{y}}\right\}. \qquad (14)$$

Preliminary results



- 121 master integrals.
- Ongoing work to obtain a canonical basis.
- At least one square root $\sqrt{m^2 s_{12} s_{23} s_{13}}$.

Table of Contents

Conclusions

- Analytic results for all planar master integrals (max. W=6).

Conclusions

- Analytic results for all planar master integrals (max. W=6).
- Preliminary studies of non-planar topologies indicate that planar alphabet will be extended.

Conclusions

- Analytic results for all planar master integrals (max. W=6).
- Preliminary studies of non-planar topologies indicate that planar alphabet will be extended.
- IBP reduction main bottleneck for amplitude calculations.

Conclusions

- Analytic results for all planar master integrals (max. W=6).
- Preliminary studies of non-planar topologies indicate that planar alphabet will be extended.
- IBP reduction main bottleneck for amplitude calculations.

Thank you for your attention!