Based on the lectures by B. Bornschein „Messmethoden und Techniken in der Experimentalphysik“, SS 2013, lectures #3 and #4

Measurement uncertainties – GUM & Co.

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KSETA Workshop, 17.10.2013

Right or wrong?

„The more often I repeat a measurement, the more accurate it gets.“

Right or wrong?

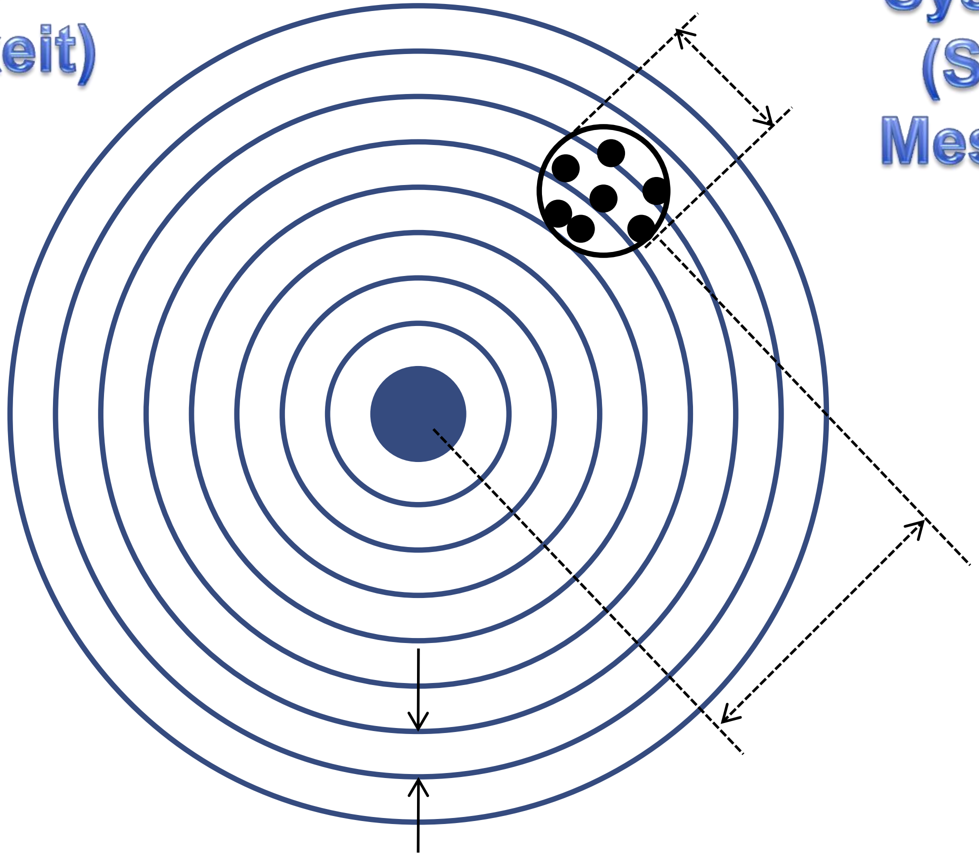
„The more often I repeat a measurement, the more accurate it gets.“

Wrong!

Some basic terms

Accuracy?
(Genauigkeit)

Systematic error?
(Systematische Messabweichung)



Random error?
(Zufällige Mess-abweichung)

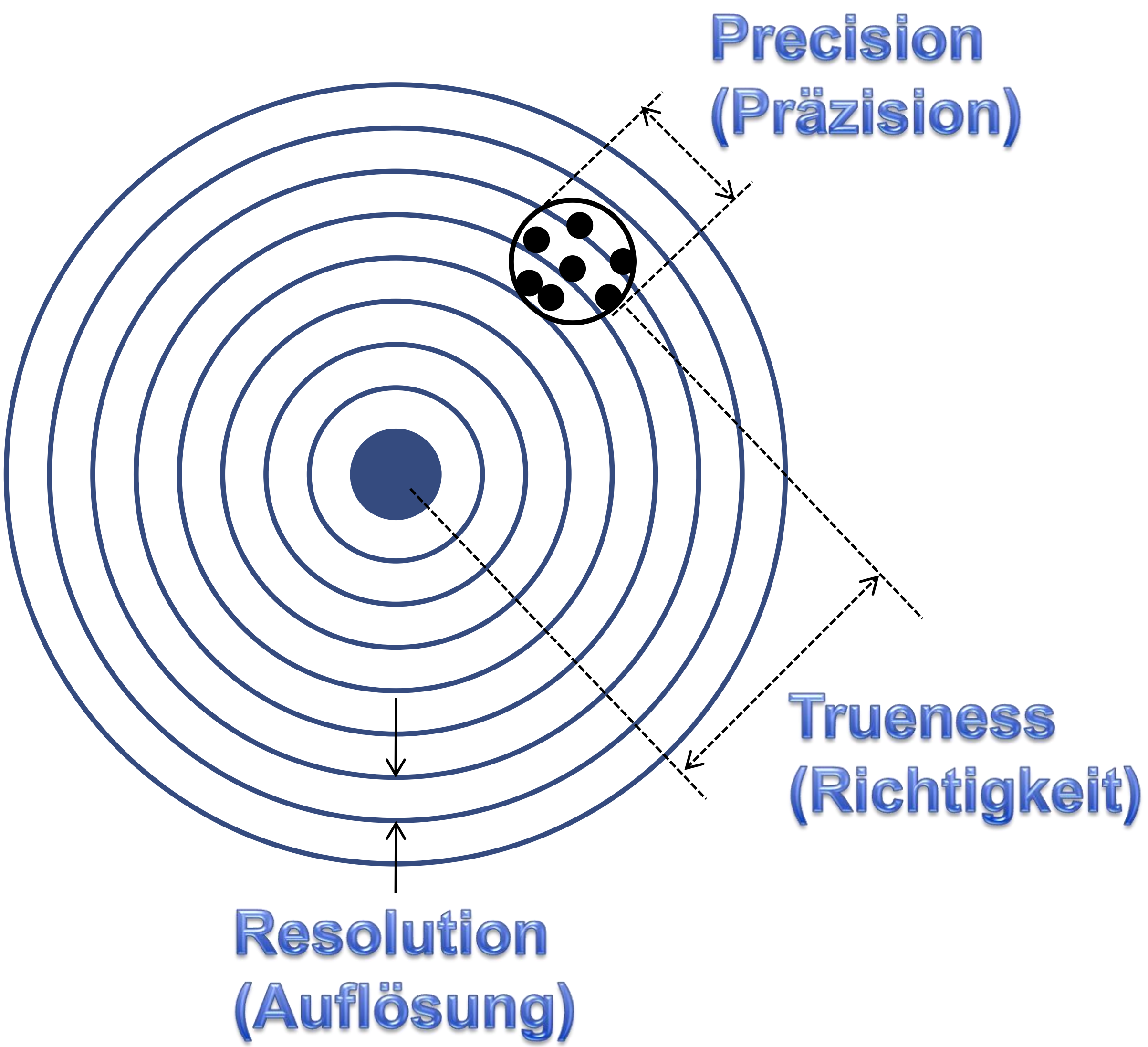
Precision?
(Präzision)

Trueness?
(Richtigkeit)

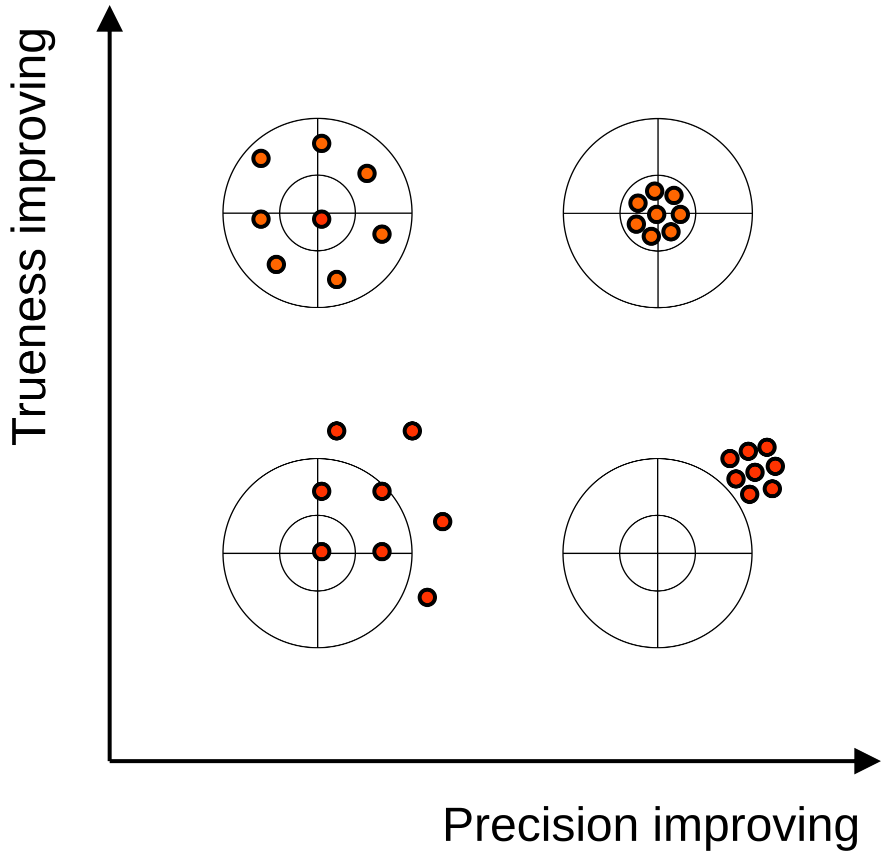
True value?
(Wahrer Wert)

Resolution?
(Auflösung)

Some basic terms



And what about „accuracy“?



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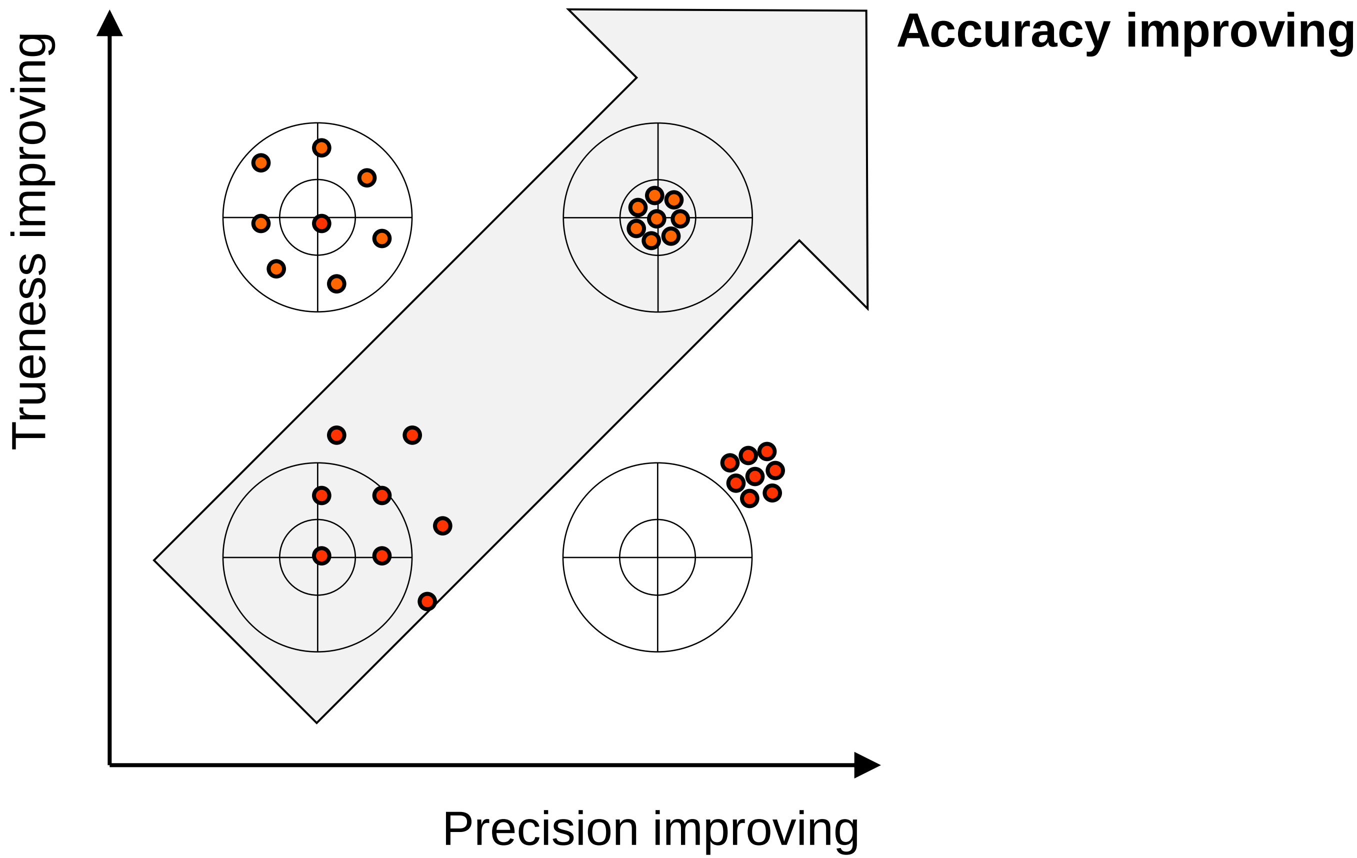


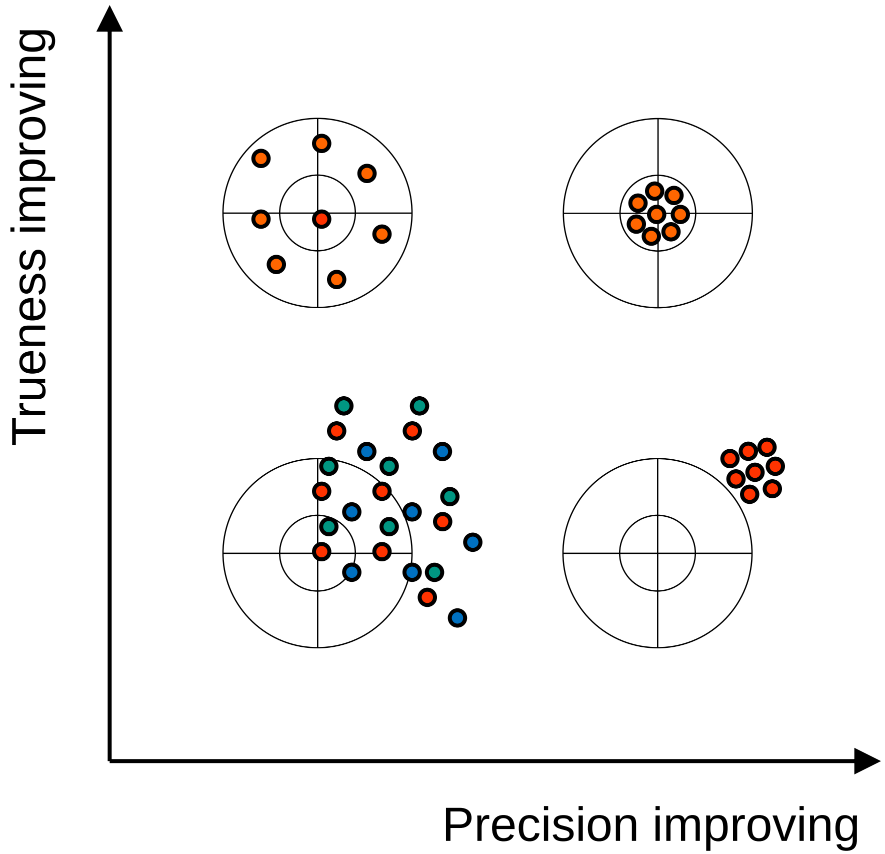
Right or wrong? Next try!

„The more often I repeat a measurement, the more precise it gets."

Right or wrong? Next try!

„The more often I repeat a measurement, the more precise it gets.“

Wrong!


Trueness improving
Precision improving

So what is it then?

„The more often I repeat a measurement, ...“

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GUM

Guide to the expression of Uncertainty in Measurement

International discussion on uncertainties

→ „Guide to the Expression of Uncertainty in Measurement“

- Published 1993 by ISO (International Organization for Standardization), corrected versions 1995 and 2008
- Involved:
  - International Bureau of Weights and Measures (BIPM)
  - International Electrotechnical Commission (IEC)
  - International Federation of Clinical Chemistry (IFCC)
  - International Organization for Standardization (ISO)
  - International Union of Pure and Applied Chemistry (IUPAC)
  - International Union of Pure and Applied Physics (IUPAP)
  - International Organization of Legal Metrology (OIML)
- *www.bipm.org/utils/common/documents/jcgm/JCGM_100_2008_E.pdf*

Method for evaluating the measurement uncertainty according to GUM

- GUM distinguishes two cases in the evaluation of measurement uncertainties:
  Type A and **Type B**
- **Type A**: Method of evaluation of uncertainty based on a series of observations (repeated measurements)
  ➡ **statistical analysis**
- **Type B**: Method of evaluation of uncertainty by means other than the statistical analysis of series of observations
  ➡ **if it‘s not type A, then it‘s type B**

Method A

Basic formulae

We have performed *n* independent repeat measurements of quantity x under the same conditions. Then:

- Best available estimate of the expected value μ = arithmetic mean/average $\overline{x}$

$$\overline{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$

- Best estimate of the variance σ^2 = empirical variance $s^2(x)$

$$s^2(x) = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \overline{x})^2$$

- Best estimate for the variance of the mean = empirical variance of the mean $s^2(\overline{x})$

$$s^2(\overline{x}) = \frac{s^2(x)}{n}$$

normalizes variance to the sample size
→ comparability

This is what we need!

Solution

„The more often I repeat a measurement, the smaller gets the empirical variance of the mean.“

Remarks (according to GUM)

- The **number of observations *n*** should be **large enough** to ensure that $\bar{x}$ provides a **reliable estimate** of the expectation μ of the random variable *x*, and $s^2(\bar{x})$ of the variance $\sigma^2(\bar{x}) = \sigma^2 / n$.

  The **difference** between s^2 and σ^2 has to be considered when one constructs confidence intervals!

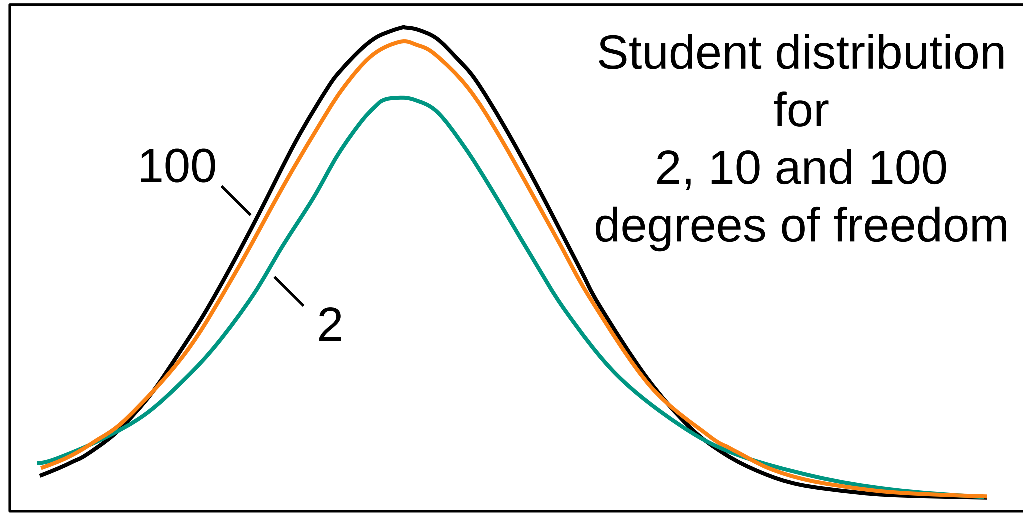
  → If the probability distribution of *x* is a normal distribution, the difference is taken into account through the *t*-distribution.

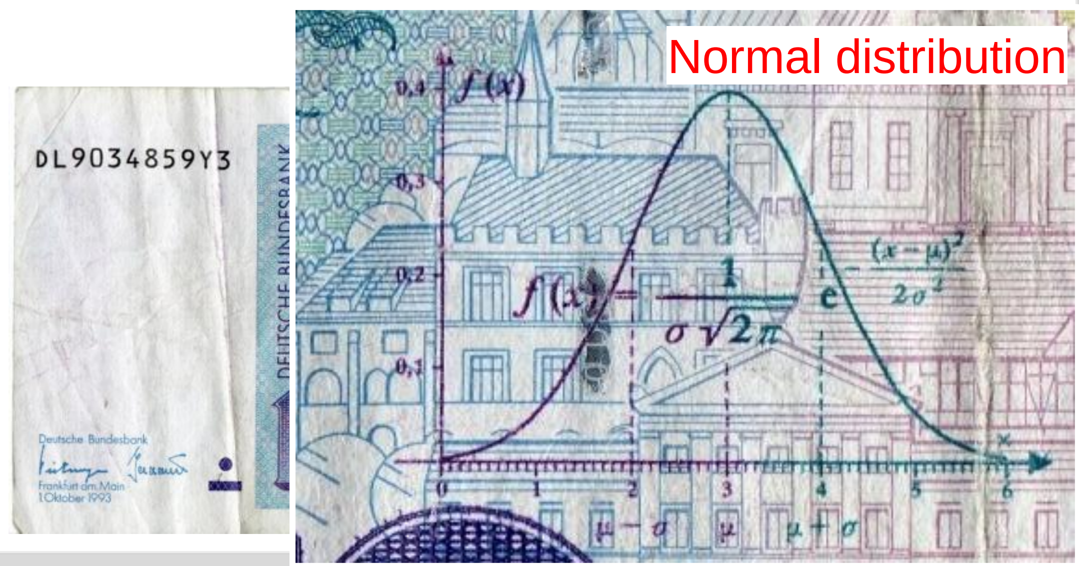
Student distribution / *t*-distribution

- t-distribution → normal distribution for ∞ degrees of freedom ν (ν = n-1)

- **Normal distribution**:
  68.3 % of measured values inside region $\pm \sigma$

  ***t*-distribution:**
  Same probability for region $t_{68,3} * s$ with t = student factor (t-factor)





Some values for *t*-factors (from GUM)

Table G.2 — Value of $t_p(\nu)$ from the t-distribution for degrees of freedom ν that defines an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses the fraction p of the distribution

Degrees of freedom	Fraction p in percent					
ν	68,27 1σ	90	95	95,45 2σ	99	99,73 3σ
1	1,84	6,31	12,71	13,97	63,66	235,80
2	1,32	2,92	4,30	4,53	9,92	19,21
3	1,20	2,35	3,18	3,31	5,84	9,22
4	1,14	2,13	2,78	2,87	4,60	6,62
5	1,11	2,02	2,57	2,65	4,03	5,51
6	1,09	1,94	2,45	2,52	3,71	4,90
7	1,08	1,89	2,36	2,43	3,50	4,53
8	1,07	1,86	2,31	2,37	3,36	4,28
9	1,06	1,83	2,26	2,32	3,25	4,09
10	1,05	1,81	2,23	2,28	3,17	3,96

Some values for *t*-factors (from GUM)

dof	1σ			2σ		3σ
20	1,03	1,72	2,09	2,13	2,85	3,42
25	1,02	1,71	2,06	2,11	2,79	3,33
30	1,02	1,70	2,04	2,09	2,75	3,27
35	1,01	1,70	2,03	2,07	2,72	3,23
40	1,01	1,68	2,02	2,06	2,70	3,20
45	1,01	1,68	2,01	2,06	2,69	3,18
50	1,01	1,68	2,01	2,05	2,68	3,16
100	1,005	1,660	1,984	2,025	2,626	3,077
∞	1,000	1,645	1,960	2,000	2,576	3,000

***t*-factor important also for more than 10 repeats!**

Example: Type A

- 5 measurements of a voltage

Table G.2 — Value of $t_p(\nu)$ from the t-distribution f
an interval $-t_p(\nu)$ to $+t_p(\nu)$ that encompasses t

Degrees of freedom ν	Fraction p		
	68,27 a)	90	95
1	1,84	6,31	12,71
2	1,32	2,92	4,30
3	1,20	2,35	3,18
4	1,14	2,13	2,78

i	X_i in V	$(X_i-\mu)^2$ in V^2
1	6,7	0,04
2	6,4	0,01
3	6,3	0,04
4	6,6	0,01
5	6,5	0,00
Sum	32,5	0,10

Discussion:

What is the arithmetic mean, the empirical standard deviation, the uncertainty of the mean and the total uncertainty taking into account the student factor (68,27%)?

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Example: Type A

- 5 measurements of a voltage
- Mean: 32,5 V / 5 = 6,5 V
- Empirical varicance:
  0,10 V² / (5-1) = 0,025 V²
- Empirical std. deviation:

$$s = \sqrt{s^2} = \sqrt{0{,}025}\ \mathrm{V} = 0{,}156\ \mathrm{V}$$

i	X_i in V	$(X_i-\bar{x})^2$ in V²
1	6,7	0,04
2	6,4	0,01
3	6,3	0,04
4	6,6	0,01
5	6,5	0,00
Sum	32,5	0,10

- Uncertainty of the mean:
  $$s(\bar{x}) = s/\sqrt{n} = 0{,}156\ \mathrm{V}/\sqrt{5} = 0{,}070\ \mathrm{V}$$
- Student factor for 68,3% probability: 1,14 ($\nu = 4$)
- Final result: $\mu \pm s(\bar{x}) \cdot t = 6{,}5\mathrm{V} \pm 0{,}08\mathrm{V}$

Method for evaluating the measurement uncertainty according to GUM

- GUM distinguishes two cases in the evaluation of measurement uncertainties:
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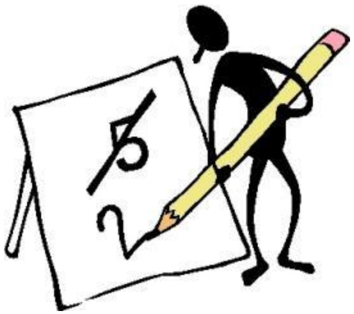
  ⇒ **if it's not type A, then it's type B**

Method B

Evaluation of the variance from other source („everything apart from statistical analysis“)

According to Hoffmann, Handbuch der Messtechnik

- Sometimes one can only perform one single measurement → variance?
  - Own experience (e.g. from former experiments)
  - Experience of others
  - literature values
- Often there are upper/lower bounds of the uncertainty given



What to do?

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Evaluation of the variance if only upper/ lower bounds of the uncertainty are known

- **First step:**
  Determination of the distribution function $f(x_\mathrm{i})$

- **Second step:**
  Calculation of the variance using these formulae:

Discrete distribution

$$u^2 = \sum_{i=1}^{N} (x_\mathrm{i} - x_0)^2 f(x_\mathrm{i})$$

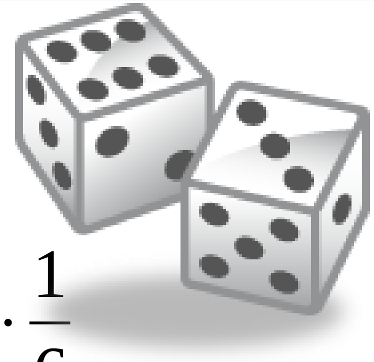
$$\text{with } x_0 = \sum_{i=1}^{N} x_\mathrm{i} f(x_\mathrm{i})$$

Continuous distribution

$$u^2 = \int_{-\infty}^{\infty} (x - x_0)^2 f(x)\, dx$$

$$\text{with } x_0 = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

Example: Dice



$$x_0 = \sum_{i=1}^{N} x_\mathrm{i} \, f(x_\mathrm{i}) = \sum_{i=1}^{6} x_i \cdot \frac{1}{6}$$

$$= (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6}$$

$$= 3{,}5$$

$$u^2 = \sum_{i=1}^{N} (x_\mathrm{i} - x_0)^2 \, f(x_\mathrm{i})$$

$$= ((-2{,}5)^2 + (-1{,}5)^2 + (-0{,}5)^2 + 0{,}5^2 + 1{,}5^2 + 2{,}5^2)\frac{1}{6}$$

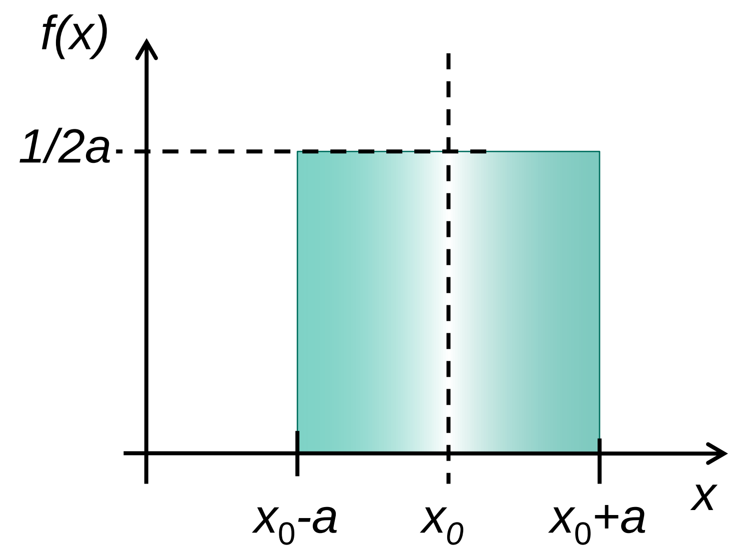
$$= 2{,}92$$

$$u^2 = \sum_{i=1}^{N} (x_\mathrm{i} - x_0)^2 \, f(x_\mathrm{i})$$

$$\text{with} \quad x_0 = \sum_{i=1}^{N} x_\mathrm{i} \, f(x_\mathrm{i})$$

Rectangular distribution

- Taken if it is only known that the uncertainty lies between certain bounds
- Example: next slide
- Variance:



$$u^2 = \int_{-\infty}^{\infty} (x - x_0)^2 f(x)\, dx$$

$$= \int_{x_0 - a}^{x_0 + a} \frac{1}{2a} (x - x_0)^2 dx = \frac{1}{2a} \frac{(x - x_0)^3}{3} \Big|_{x_0 - a}^{x_0 + a} = \frac{a^2}{3}$$

Example for rectangular distribution

- Given: Precision resistance of 100 Ω with a measurement uncertainty of ± **2 ppm** according to the calibration certificate
- Wanted: Variance and standard deviation

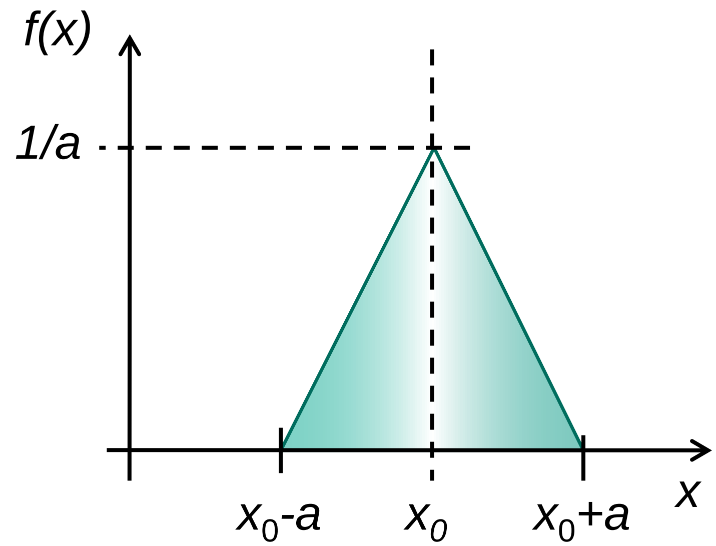
$$u^2 = \frac{1}{3}(2 \cdot 10^{-6})^2 = 1{,}33 \cdot 10^{-12} \quad \text{(relative)}$$

$$\Rightarrow u = \sqrt{1{,}33 \cdot 10^{-12}} \cdot 100\Omega = 0{,}115\,\text{m}\Omega$$

→ The resistance has a value of (100,000000 ± 0,000115) Ω.

Triangular distribution

- Taken if it is known (e.g. from experience) that there is a preference for value x_0 between certain bounds



- Variance:

$$u^2 = \int_{-\infty}^{\infty} (x - x_0)^2 f(x)\, dx$$

$$= \frac{a^2}{6} \quad \text{with} \quad f(x) = \frac{1}{6}\left(1 - \frac{1}{a}\,|x - x_0|\right)$$

- Example: According to the manufacturer, the temperature coefficient can lie between -10 ppm and +10 ppm. A preference for small deviations is assumed from experience.

→ $u^2 = \frac{1}{6}(10 \cdot 10^{-6})^2 = 1{,}67 \cdot 10^{-11}$ (relative)

Example for rectangular distribution

- Given: Precision resistance of 100 Ω with a measurement uncertainty of ± **2 ppm** according to the calibration certificate

…nd standard deviation

And why is this important?

$$u^2 = \frac{1}{3}(2\cdot 10 \ldots) \ldots = 1{,}3\ldots$$

$$\Rightarrow u = \sqrt{1{,}33\cdot 10^{-12}}\cdot 100\Omega = 0{,}115\ \mathrm{m}\Omega$$

→ The resistance has a value of (100,000000 ± 0,000115) Ω.

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Example for rectangular distribution

- Given: Precision resistance of 100 Ω with a measurement uncertainty of ± **2 ppm** according to the calibration certificate

„naive“

- Wanted: Variance and standard deviation

$$u^2 = \frac{1}{3}(2 \cdot 10^{-6})^2 = 1{,}33 \cdot 10^{-12} \quad \text{(relative)}$$

$$\Rightarrow u = \sqrt{1{,}33 \cdot 10^{-12}} \cdot 100\Omega = 0{,}115\ \text{m}\Omega$$

→ The resistance has a value of (100,000000 ± 0,000115) Ω.

better!

Mathematical models w/o covariance

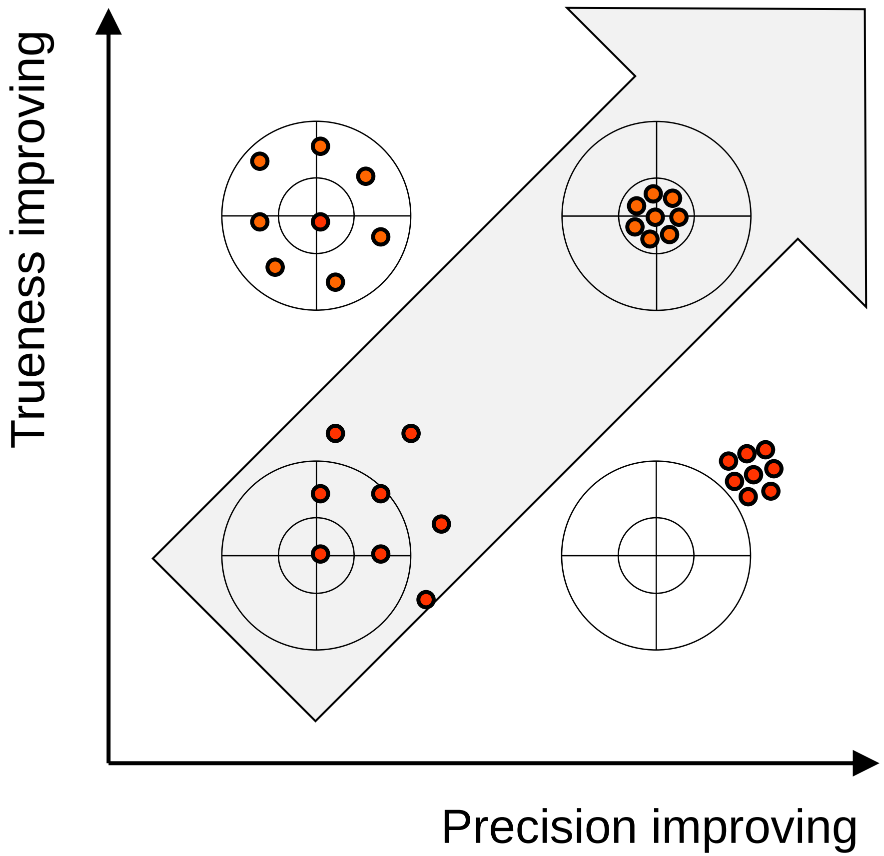
According to F. Adunka, Messunsicherheiten

- Measurement result constituted of several quantities
→ mathematical model $f(x_1, x_2, x_3, \ldots, x_N)$ needed
- Example: $R=U/I$ (U, I = measured)
- The result of the measurement is then a function of the arithmetic means $x_{0,i}$: $y_0 = f(x_{0,1}, x_{0,2}, x_{0,3}, \ldots, x_{0,N})$
- The (combined) variance is calculated via Gaussion propagation of uncertainties (here given for variances):

$$u^2(y) = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x_i} \right)^2 u^2(x_i)$$

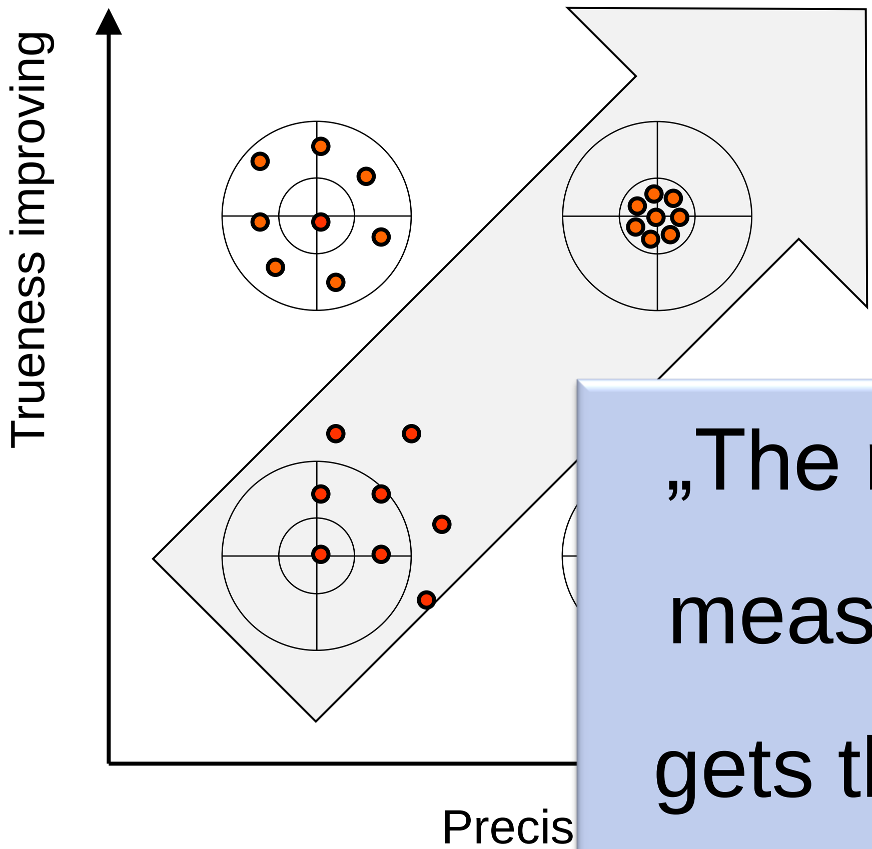
Note: Formula derived via Taylor series (to n = 1); just valid for small u_i

Take-home-messages



Accuracy improving

Take-home-messages



„The more often I repeat a measurement, the smaller gets the empirical variance of the mean.“

Take-home-messages

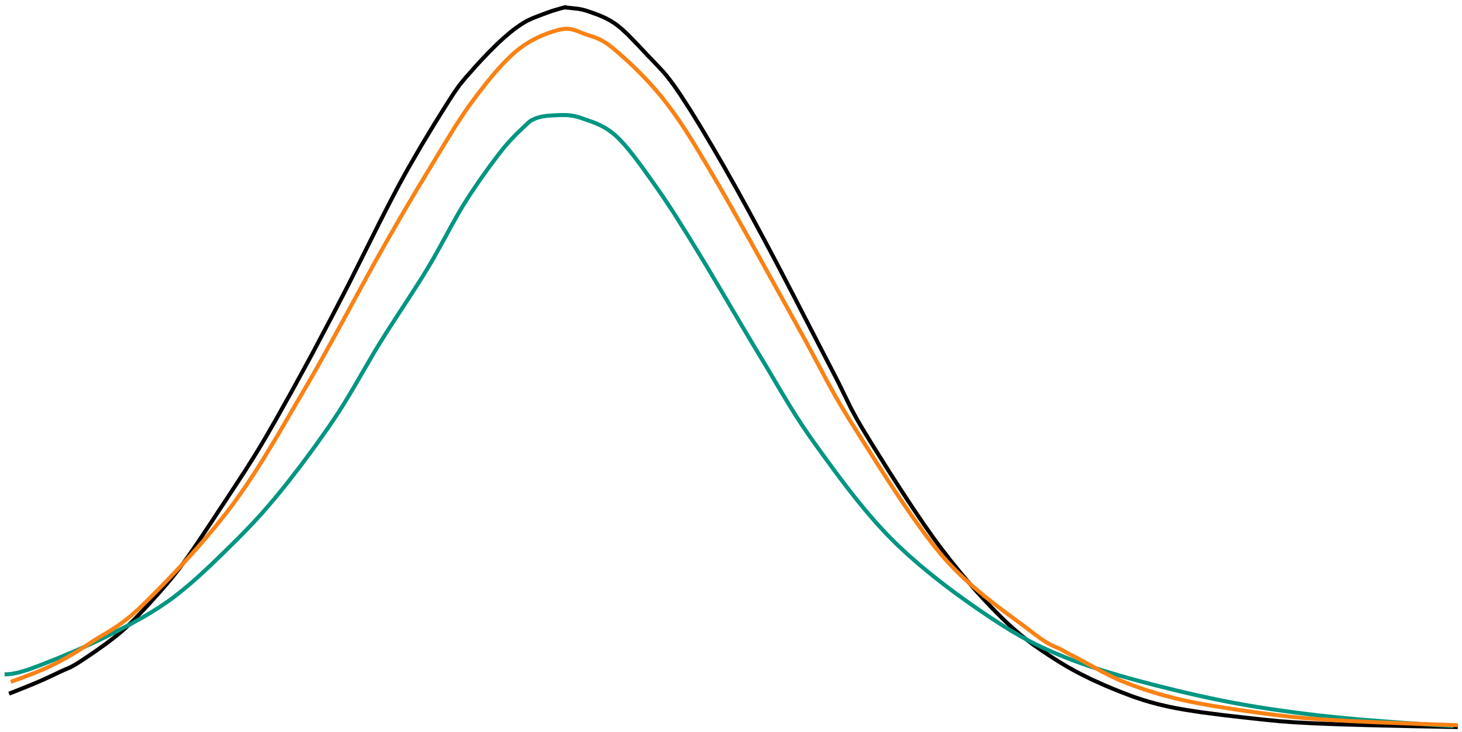
GUM

Guide to the expression of Uncertainty in Measurement

www.bipm.org/utils/common/documents/jcgm/JCGM_100_2008_E.pdf

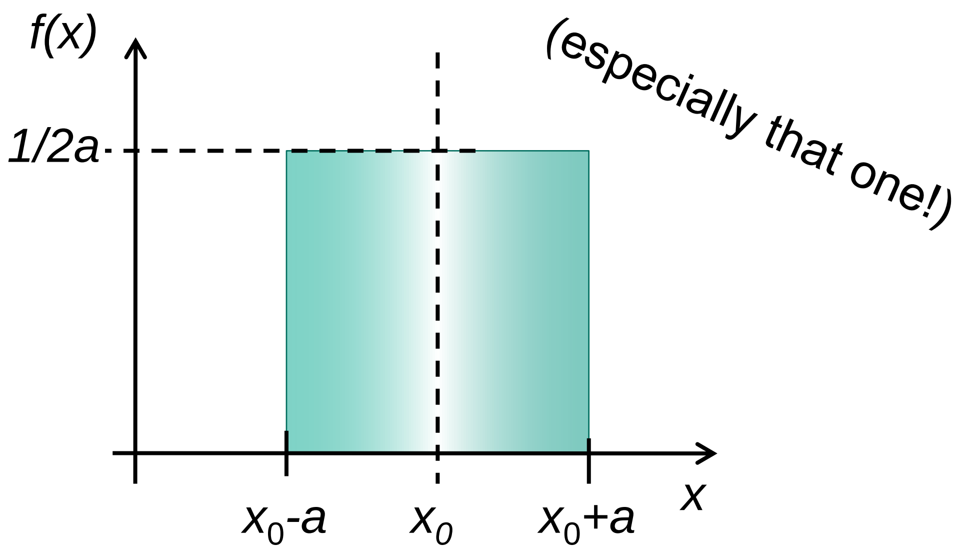
Take-home-messages

Student distribution for small sample sizes (< 30)



Take-home-messages

Think of distribution functions!



Thank you!