Semileptonic and hadronic *b*-decays

KSETA Plenary Workshop 2023

Manuel Egner | Durbach, March 27, 2023

INSTITUTE FOR THEORETICAL PARTICLE PHYSICS

based on work with:
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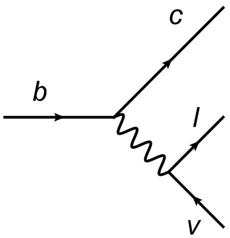
Outline

1. Introduction
2. Calculation
3. Conclusion and Outlook

Motivation



- $|V_{cb}|$ is extracted from inclusive $B \to X_c l \bar{\nu}$. Precise predictions for semileptonic b-decays are crucial input.
- Heavy Quark Expansion (HQE): Decay width of $B \to X_c l \bar{\nu}$ as sum of decay width of $b \to c l \bar{\nu}$ and corrections suppressed by the mass m_b.

$$\Gamma\left(B_c\right) = \Gamma_3 + \Gamma_5 \frac{\langle \mathcal{O}_5 \rangle}{m_b^2} + \Gamma_6 \frac{\langle \mathcal{O}_6 \rangle}{m_b^3} + \dots + 16\pi^2 \left(\Gamma_6 \frac{\langle \tilde{\mathcal{O}}_6 \rangle}{m_b^3} + \Gamma_7 \frac{\langle \tilde{\mathcal{O}}_7 \rangle}{m_b^4} + \dots \right).$$

- $\Gamma_3 \leftrightarrow$ decay of free b-quark.
- In this talk: Semileptonic decay channel $b \to c l \bar{\nu}$ at $\mathcal{O}\left(\alpha_s^2\right)$.

Motivation

- Semileptonic calculation as method test
- Next step: Hadronic decay channels $b \to c\bar{u}d$ and $b \to c\bar{c}s$ at $\mathcal{O}\left(\alpha_s^2\right)$.
- Input for the calculation of B-meson lifetimes in HQE.
- Knowing the $\mathcal{O}\left(\alpha_s^2\right)$ contributions will reduce the uncertainty induced by the renormalization scale variation.

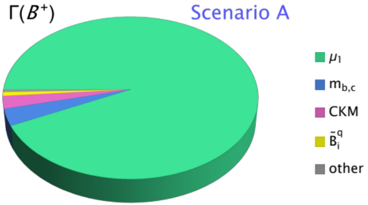


Figure: Error contributions on B-meson lifetimes [Lenz, Piscopo, Rusov (2022)]

What is known?

Previous calculations including finite charm quark mass for Γ_3:

Semileptonic decay channel:

- $\mathcal{O}\left(\alpha_s^1\right)$ [Nir (1989)]
- $\mathcal{O}\left(\alpha_s^2\right)$ [Czarnecki, Pak (2008)], [Czarnecki, Dowling, Piclum (2008)]
- $\mathcal{O}\left(\alpha_s^3\right)$ [Fael, Schönwald, Steinhauser (2021)]

$\mathcal{O}\left(\alpha_s^2\right)$ and $\mathcal{O}\left(\alpha_s^3\right)$ calculations as expansions in mass ratios, no analytic results.

Hadronic decay channel:

- $b \to c\bar{u}d$: $\mathcal{O}\left(\alpha_s^1\right)$ [Bagan, Ball, Braun, Gosdzinsky (1994)]
- $b \to c\bar{c}s$: $\mathcal{O}\left(\alpha_s^1\right)$ [Bagan, Ball, Fiol, Gosdzinsky (1995)]
- $\mathcal{O}\left(\alpha_s^2\right)$, including only massless quarks in the final state, $b \to u$, only one operator [Czarnecki, Slusarczyk, Tkachov (2006)]

Calculation setup

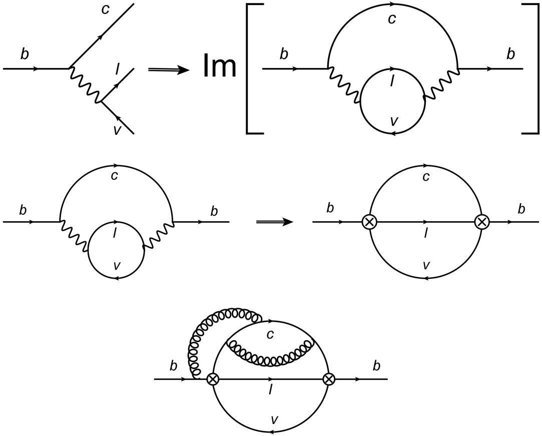
- Optical theorem:

$$\Gamma = \frac{1}{m_b} \mathrm{Im}\left[\mathcal{M}(b \to b)\right]$$

- Integrate out W-boson

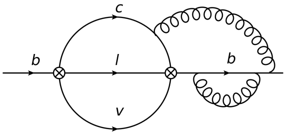
$$\frac{1}{(m_W^2 - p^2)} \to \frac{1}{m_W^2}$$

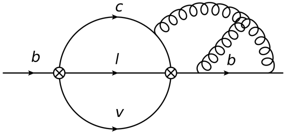
- At $\mathcal{O}\left(\alpha_s^2\right)$ calculate imaginary part of 4-loop diagrams



Calculation setup

- Generate diagrams with `QGRAF` [Nogueira (1993)].
- Mapping on integral families with `TAPIR` [Gerlach, Herren, Lang (2022)] and `EXP` [Harlander, Seidensticker, Steinhauser (1998-1999)].
- Reduction to master integrals with `Kira` [Klappert, Lange, Maierhöfer, Usovitsch (2021)].
  - Choose good basis of master integrals, where ϵ and $x = m_c/m_b$ factorize, with `ImproveMasters.m` [Smirnov, Smirnov (2020)].





$$d4l1[\vec{n}] = \int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \mathrm{d}^d k_3 \mathrm{d}^d k_4}{(-k_1^2)^{n_1}(-k_2^2)^{n_1}(m_b^2 - k_1^2)^{n_3}(-(k_3 - k_4)^2)^{n_4}\ldots}$$

$$\begin{aligned} d4l1[n_1, n_2, n_3, n_4, n_5, 0, \ldots] &= \ldots d4l1[1,1,1,1,1,0,\ldots] \\ &+ \ldots d4l1[2,1,1,1,1,0,\ldots] \end{aligned}$$

Master integrals, method 1: analytic solutions

- Calculating master integrals via differential equations:
  - Determination of ϵ-form with `Canonica` [Meyer (2018)] and `Libra` [Lee (2020)].
  - Part of the differential equations which can not be brought to ϵ-form with `OreSys` [Gerhold (2002)].
  - Handling of iterated integrals with `HarmonicSums` [Ablinger (2010)].
- Fix constants $\vec{c}$ by matching to boundary conditions
- Boundary conditions: asymptotic expansion in $\delta = 1 - m_c^2/m_b^2$ the limit $m_c \approx m_b$

$$\frac{m_c}{m_b} = \frac{1-t^2}{1+t^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{I} = A(\epsilon, t) \cdot \vec{I}$$

$$\underrightarrow{\vec{I} = T \cdot \vec{J}} \quad \frac{\mathrm{d}}{\mathrm{d}t}\vec{J} = \epsilon\tilde{A}(t) \cdot \vec{J}$$

$$\rightarrow \quad \vec{J} = \int^t \epsilon\tilde{A}(t') \cdot \vec{J}\,\mathrm{d}t' + \vec{c}$$

Master integrals, method 1: analytic solutions

- Obtain master integrals in terms of iterated integrals over the alphabet

$$\frac{1}{t}, \quad \frac{1}{1+t}, \quad \frac{1}{1-t}, \quad \frac{t}{1+t^2}, \quad \frac{t^3}{1+t^4}$$

- Analytic result for $\mathcal{O}\left(\alpha_s^2\right)$ contributions (new!)
- Analytic expression $\rightarrow$ allows for expansion around different kinematic limits:
  - Massless charm quark:

$$x = \frac{m_c}{m_b} = 0 \quad \leftrightarrow \quad t = 1$$

  - Heavy charm quark:

$$x = \frac{m_c}{m_b} = 1 \quad \leftrightarrow \quad \delta = 0 \quad \leftrightarrow \quad t = 0$$

Analytic results

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$$\Gamma\left(b \to c l \bar{\nu}\right) = \frac{A_{ew} G_F^2 m_b^5 |V_{cb}|^2}{192\pi^3} \left(X_0 + \frac{\alpha_s}{\pi} C_F X_1 + \left(\frac{\alpha_s}{\pi}\right)^2 C_F X_2 + \mathcal{O}\left(\alpha_s^3\right) \right),$$

with

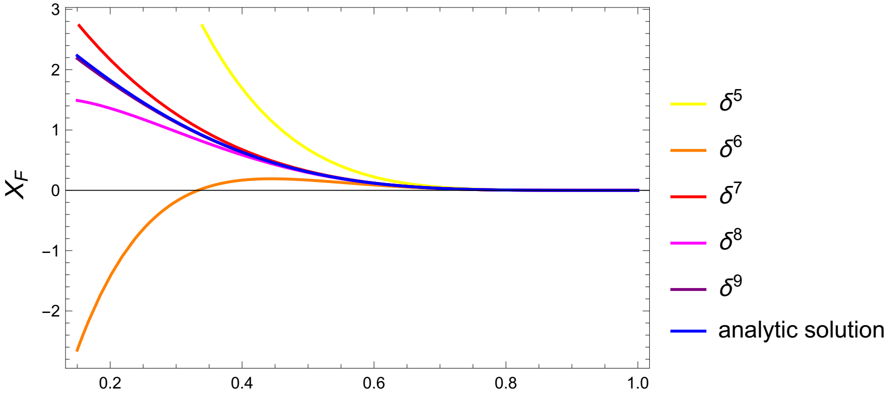
$$X_2 = C_F X_F + C_A X_A + T_F \left(n_l X_l + n_c X_c + n_h X_h\right).$$

- $n_l = 3$, $n_c = 1$, $n_h = 1$ number of light, charm and bottom quarks.
- All $\{X_F,\ X_A,\ X_l,\ X_c,\ X_h\}$ depending on the mass ratio m_c/m_b as functions of t, for example:

$$X_l = \frac{1}{(-1+t)^2(1+t)^2(1+t^2)^{12}} \left(-38 - 76t^2 + 190t^4 + \ldots\right) + \ldots$$
$$+ \frac{H_{-1,-1}(t)}{24t^6(1+t^2)^4} \left(405 + 100t^2 - 770t^4 + 3462t^6 + \ldots\right) + \ldots + H_{-1,0,0,\{4,1\}}(t)\left(\ldots\right) + \ldots$$

Comparing results

- Results as an expansion in $\delta = 1 - m_c/m_b$ are known up to $\mathcal{O}\left(\alpha_s^3\right)$ [Fael, Schönwald, Steinhauser (2021)]. Obtained by expansion in δ on diagram level.
- Expand amplitude in δ after insertion of analytic master integrals, compare $\mathcal{O}\left(\alpha_s^2\right)$ contribution $\rightarrow$ Reproduce same results for all color factors!



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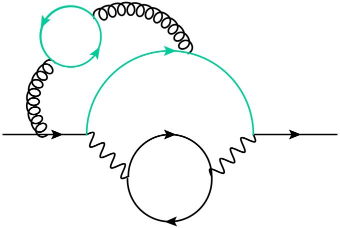
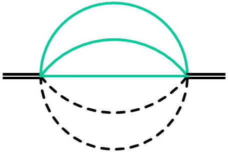
- Expand amplitude in δ after insertion of analytic master integrals, compare $\mathcal{O}\left(\alpha_s^2\right)$ contribution $\rightarrow$ Reproduce same results for all color factors!
- Results as an expansion in $x = m_c/m_b$ are known up to $\mathcal{O}\left(\alpha_s^2\right)$ [Czarnecki, Pak (2008)].
- Expand amplitude in x after insertion of analytic master integrals, compare $\mathcal{O}\left(\alpha_s^2\right)$ contribution $\rightarrow$ Reproduce same results for $T_F n_l$ and $T_F n_h$ but not for $T_F n_c$, C_F^2 and $C_F C_A$!

Cuts through 3 charm quarks





- Boundary conditions for differential equations at $m_c \approx m_b$
- Cut through 3 charm quarks is zero in this limit!
  $\rightarrow$ integral is set to zero (only imaginary part is calculated in analytic approach)
- But: additional imaginary part can arise for $m_c < m_b/3 \leftrightarrow \delta > 8/9$

$\rightarrow$ Not covered by analytic calculation

- Expect the effect to be small because of small phase space ($m_b \approx 4.5\mathrm{GeV}$, $3m_c \approx 3.9\mathrm{GeV}$)!

Master integrals, method 2: Expand and Match

- Make general expansion ansatz in $\delta = 1 - m_c^2/m_b^2$ around certain point δ_0 for integral

$$I_i(\delta, \delta_0) = \sum_{j=\epsilon_{\min}}^{\epsilon_{\max}} \sum_{m=0}^{j+4} \sum_{n=0}^{n_{\max}} c[i,j,m,n]\, \epsilon^j (\delta_0 - \delta)^n \log^m(\delta_0 - \delta)$$

$$\delta_0 = \frac{8}{9} \to \log\left(\frac{8}{9} - \delta\right) = \log\left(\left|\frac{8}{9} - \delta\right|\right) - i\pi,$$

- Insert ansatz in DEQ

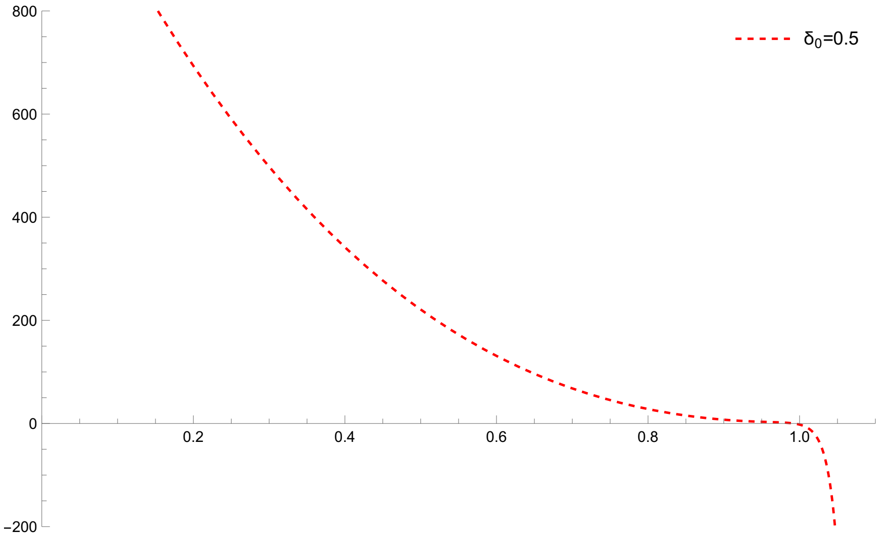
$$\ldots c[i,j,m,n] \frac{\mathrm{d}}{\mathrm{d}\delta}(\delta_0 - \delta)^n + \cdots = \ldots c[i,j,m,n-1](\delta_0 - \delta)^{n-1} + \ldots$$

$\rightarrow$ Linear equations for $c[i,j,m,n]$ for every power in δ

- Determine remaining coefficients by matching to numerical results (`AMFlow` [Liu, Ma (2022)])
- Expansions around several expansion points, match in between $\rightarrow$ cover $\delta \in [0,1]$.

Master integrals, method 2: Expand and Match

Real part of sample integral calculated by "Expand and Match" method at $\mathcal{O}\left(\epsilon^2\right)$:



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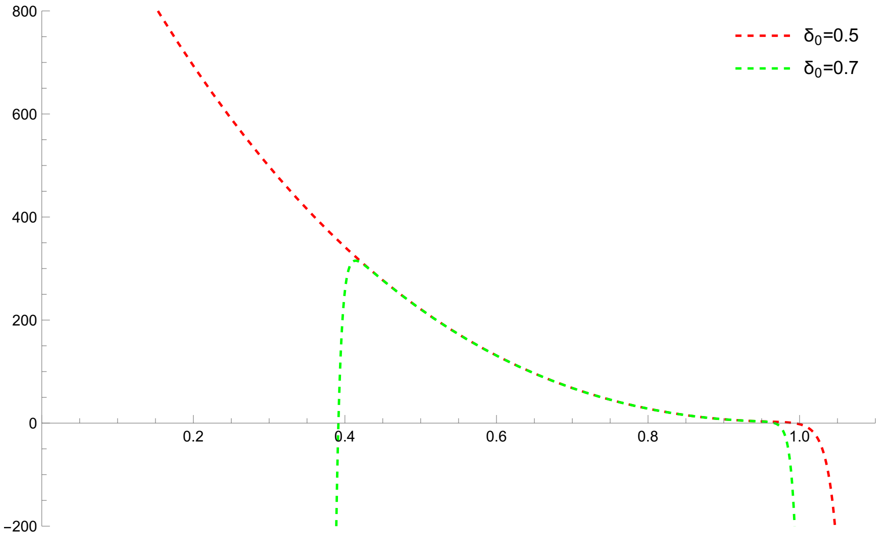

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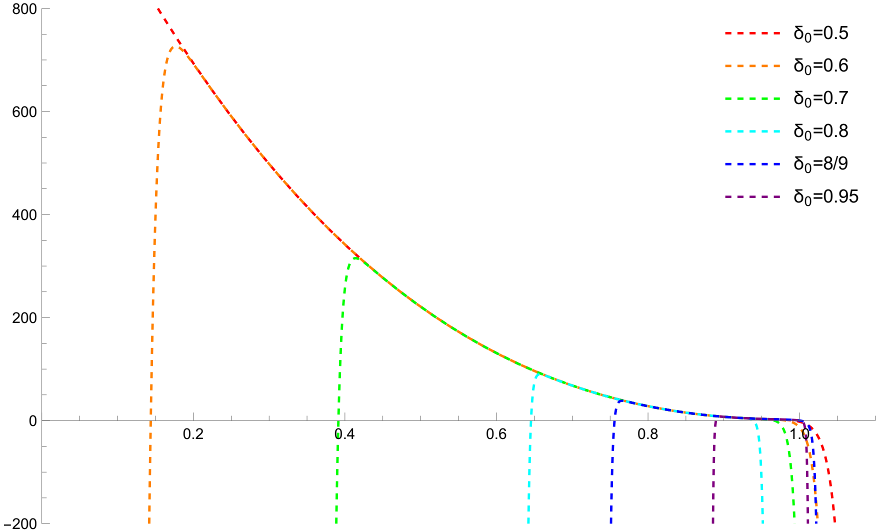

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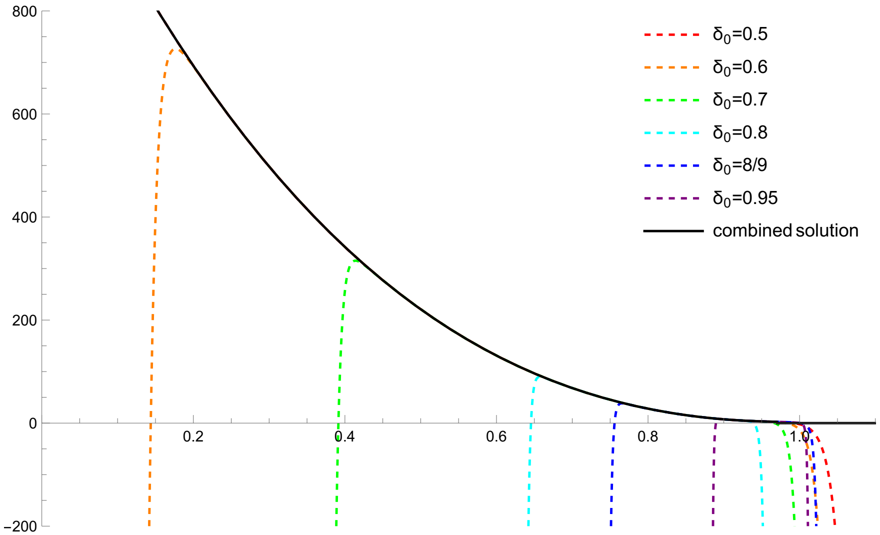

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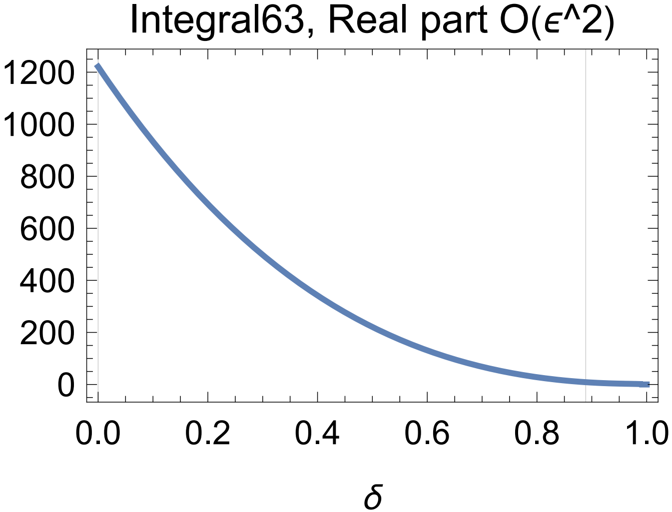

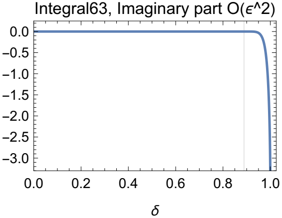

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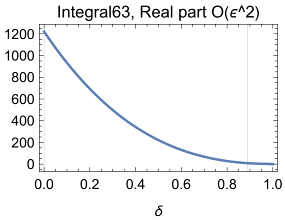
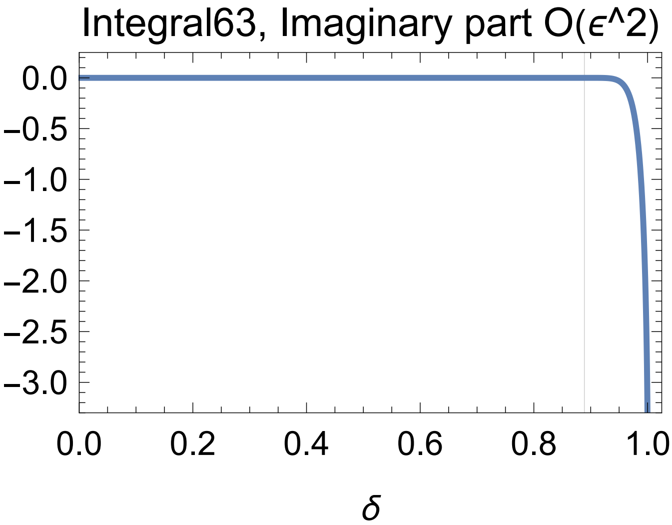
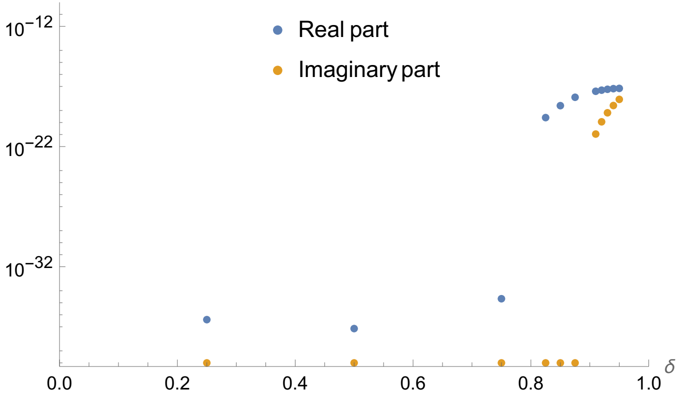




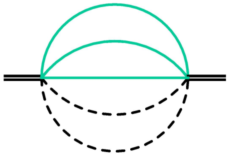
Master integrals, method 2: Expand and Match



Difference between numerical calculations with `AMFlow` [Liu, Ma (2022)] and ”Expand and Match” results at $\mathcal{O}\left(\epsilon^2\right)$

Master integrals, Work in progress

1. Calculate all master integrals for $\delta \in [0, 1]$ with method 2
2. Calculate 3 charm cut effect in the analytic result:
   - Evaluate both methods at physical point and compare
   - Assume effect to be small (phase space suppression)
3. Compare to [Czarnecki, Pak (2008)] at $m_c \approx 0$

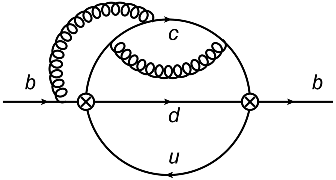


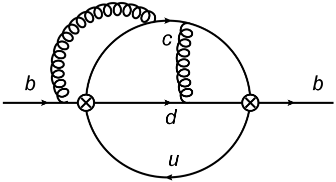
Conclusion

- We obtained an analytic expression for the semileptonic decay width $b \to cl\bar{\nu}$ at $\mathcal{O}\left(\alpha_s^2\right)$.
- We compared our results with previous calculations [Fael, Schönwald, Steinhauser (2021)] and found agreement.
- We know about the 3 charm contribution and have control over it!

What's next?

- Hadronic decay channels $b \to c\bar{u}d$ and $b \to c\bar{c}s$ with finite charm mass at $\mathcal{O}\left(\alpha_s^2\right)$.





Outlook: Hadronic decay channel

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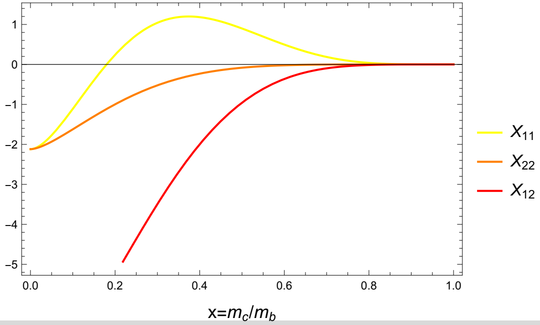
- Hadronic decay channels are more involved than semileptonic decays:
  - Different effective operators in HQE:

$$\mathcal{H}_{eff} = C_1 \cdot \left(\bar{d}_\alpha u_\beta\right)_{V-A} \left(\bar{c}_\alpha b_\beta\right)_{V-A} + C_2 \cdot \left(\bar{d}_\alpha u_\alpha\right)_{V-A} \left(\bar{c}_\beta b_\beta\right)_{V-A}$$

  - MI's of semileptonic decays are subset of MI's of hadronic decays $\rightarrow$ Use the obtained results here again
- NLO calcultaion for $b \rightarrow c\bar{u}d$ is done:

$$\Gamma\left(b \rightarrow c\bar{u}d\right) = \frac{A_{ew} G_F^2 m_b^5 |V_{cb}|^2 |V_{ud}|^2}{192\pi^3} \left(X_0 + \frac{\alpha_s}{\pi} C_F \left(C_1^2 X_{11} + C_2 X_{22} + C_1 C_2 X_{22}\right) + \mathcal{O}\left(\alpha_s^2\right)\right).$$

- Next step: $b \rightarrow c\bar{u}d$ at $\mathcal{O}\left(\alpha_s^2\right)$
- Apply analytic approach if possible. Compute all integrals with "Expand and Match"-method.



Thank you for your attention!

Backup

Boundary conditions

- Fix boundary condition in the limit $m_c \approx m_b$.
- Asymptotic expansion [Beneke, Smirnov (1997)] of master integrals in $\delta = 1 - m_c^2/m_b^2$.
- Two relevant momentum scalings:
  - hard: $|k^\mu| \sim m_b$
  - ultrasoft: $|k^\mu| \sim \delta \cdot m_b$
- Find momentum routing and scaling of the propagators for different momentum regions (check with ASY [Pak, Smirnov (2010)]), for example:

$$\frac{1}{(m_c^2 - (k + p_b)^2)} \sim \begin{cases} 1/\delta^1, & \text{if k is ultrasoft} \\ 1/\delta^0, & \text{if k is hard} \end{cases}$$

- For every region: $\{\mathrm{us}, \mathrm{us}, \mathrm{us}, \mathrm{us}\}$, $\{\mathrm{us}, \mathrm{us}, \mathrm{us}, \mathrm{h}\}, \dots$
  - Expand integral in δ.
  - Expanded integrals can be calculated in terms of Gamma functions.
  - Imaginary part only from ultrasoft integrals of the form

$$\int \frac{\mathrm{d}^d k}{(-k^2)^{n_1} (-\delta - 2k \cdot p_b)^{n_2}} \sim (-\delta)^{-2n_1 - n_2 + d},$$

  with ultrasoft loop momentum k and hard external momentum p_b.
- Sum contributions from all regions.

Master integrals with no ϵ-form

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- Differential equation of the form

$$\frac{\mathrm{d}\vec{I}}{\mathrm{d}t} = A\,(t,\epsilon)\cdot\vec{I} + \vec{B}\,(t,\epsilon)\,.$$

- Insert ansatz for master integrals $\vec{I}$

$$I_i = \sum_{j=-3} \epsilon^j \cdot f[i,j]\,(t)$$

- For every order in ϵ:
  - Decoupling of differential equations with `OreSys`
  - Solve decoupled differential equations for $f[i,j]\,(t) \rightarrow$ homogeneous and inhomogeneous solution!
  - Determine coefficients in full solution by matching to boundary conditions.