1. Tensor Reduction & IBPs

Bozen '24

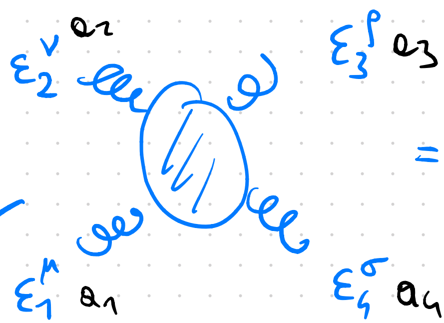
Lorenzo Tancredi

Scattering Amplitudes

(S, A) Are fundamental building blocks in QFT

$$\sigma \sim \int d\phi \, |A|^2$$

Example $gg \to gg$



$$= \underbrace{\varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\rho \varepsilon_4^\sigma}\ \overbrace{A^{a_1 \dots a_4}_{\mu\nu\rho\sigma}}^{\text{Lorentz Covariant}}$$

external states are ON-SHELL (necessary to guarantee gauge invariance of S matrix that only physical degrees of freedom are allowed to propagate)

polarization vectors "neutralize" Lorentz indices & leave an object that is Little-Group Covariant

In these lectures we will focus on $A_{\mu\nu\rho\sigma}$

[no time unfortunately for ON-SHELL methods!]

what can we say in general? <u>work by example</u>:

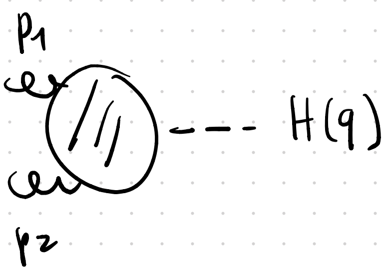
$gg \to H$ (Higgs production @ LHC)

- work in QCD
- Parity invariant theory!

$$S' = \varepsilon_1^{\mu_1} \varepsilon_2^{\mu_2} A_{\mu_1\mu_2}$$

ignore dependence on gluon color indices for simplicity

we "see" this has to be a rank-2 Lorentz scalar!



$q = p_1 + p_2 \qquad q^2 = m_H^2$

1. Lorentz Covariance implies:

$$A_{\mu_1\mu_2} = F_1\, p_{1\mu_1} p_{1\mu_2} + F_2\, p_{1\mu_1} p_{2\mu_2} + F_3\, p_{2\mu_1} p_{1\mu_2}$$
$$+ F_4\, p_{2\mu_1} p_{2\mu_2} + F_5\, g_{\mu_1\mu_2}$$

2. gluons physical $\quad \varepsilon_1 \cdot p_1 = \varepsilon_2 \cdot p_2 = 0$

only part that survives is

$$A^{phys}_{\mu_1\mu_2} = F_3 \, p_{2\mu_1} p_{1\mu_2} + F_5 \, g_{\mu_1\mu_2}$$

3. Gauge Invariance $\equiv$ Ward Identities

$$p_{1\mu_1} A_{\mu_1\mu_2} \varepsilon_2^{\mu_2} = p_{2\mu_2} A_{\mu_1\mu_2} \varepsilon_1^{\mu_1} = 0$$

$$0 = (F_3 \, p_1 \cdot p_2 + F_5) \, \varepsilon_2 \cdot p_1$$

$$= (F_3 \, p_1 \cdot p_2 + F_5) \, \varepsilon_1 \cdot p_2$$

use $(p_1+p_2)^2 = m_H^2 = 2 p_1 \cdot p_2 \quad ; \quad p_i^2 = 0$

$$F_3 = -\frac{F_5}{p_1 \cdot p_2} = -\frac{2F_5}{m_H^2}$$

So using Lorentz invariance & gauge invariance we find

$$A^{phys}_{\mu_1\mu_2} = F\left[g_{\mu_1\mu_2} - \frac{2\, p_{2\mu_1} p_{1\mu_2}}{m_H^2} \right]$$

F is called a scalar form factor, its explicit form depends on # loops etc

$$F(m_H^2, m_{q_i}^2, \varepsilon)$$

if quarks have masses

$\varepsilon \longrightarrow$ DIM-REGULATOR

This general form is valid @ any number of loops!

$$S = F(m_H^2, \dots)\left[\varepsilon_1 \cdot \varepsilon_2 - \frac{2}{m_H^2} \varepsilon_1 \cdot p_2\, \varepsilon_2 \cdot p_1 \right]$$

this procedure is called TENSOR DECOMPOSITION.

↳ Can you prove $S \neq 0$ only for equal helicity g? 3

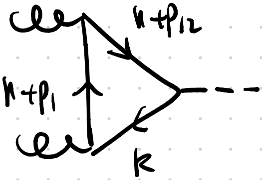
PROJECTOR METHOD

start from:

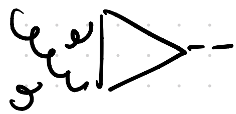
$$A^{phys}_{\mu_1\mu_2} = F\left[g_{\mu_1\mu_2} - \frac{2\, p_{2\mu_1} p_{1\mu_2}}{m_H^2} \right] \qquad (*)$$

on the other hand, Feynman Diagrams @ L-loops provide explicit representation for $A^{phys}_{\mu_1\mu_2}$: **@ 1 loop**

DIM REG



$$\propto \int \frac{d^D k}{(2\pi)^D} \, \mathrm{Tr}\left[\frac{1}{\not k - m} \gamma_{\mu_1} \frac{1}{\not k + \not p_1 - m} \gamma_{\mu_2} \frac{1}{\not k + \not p_{12} - m} \right]$$

and similar expression from 

not immediately obvious how these can be put in form $(*)$, it requires performing algebra, integration etc, non trivial for more complicated processes

Form of (*) provides seed to general solution

$\Rightarrow$ Define a Projector Operator that projects out Lorentz indices :

$$P^{\mu_1\mu_2} = c\left[g^{\mu_1\mu_2} - \frac{2p_2^{\mu_1}p_1^{\mu_2}}{m_H^2}\right]$$ such that

$$\left\{ P^{\tilde{\mu}_1\tilde{\mu}_2}\left[-g_{\mu_1\tilde{\mu}_1}\right]\left[-g_{\mu_2\tilde{\mu}_2}\right] A_{\mu_1\mu_2} \equiv F \right.$$

sum over pol $\Rightarrow$ METRIC in the vector space of "tensors"

$$c\left(g^{\mu_1\mu_2} - \frac{2p_1^{\mu_2}p_2^{\mu_1}}{m_H^2}\right)\left(g_{\mu_1\mu_2} - \frac{2p_{1\mu_2}p_{2\mu_1}}{m_H^2}\right)F =$$

$$= \left(D - \frac{2p_1\cdot p_2}{m_H^2} - \frac{2p_1\cdot p_2}{m_H^2} + \cancel{\frac{2p_1^2p_2^2}{m_H^4}}\right)cF =$$

space-time DIM

used $m_H^2 = (p_1+p_2)^2$!

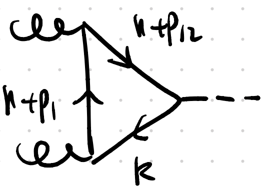
$$(D-2)\,cF \equiv F \quad \Longrightarrow \quad \boxed{c = \frac{1}{D-2}}$$

so the projector reads

$$P^{\mu_1\mu_2} = \frac{1}{D-2}\left[g^{\mu_1\mu_2} - \frac{2\, p_2^{\mu_1} p_1^{\mu_2}}{m_H^2} \right]$$

necessary to use dimensional regularisation

By Applying this projector on



DIM REG

$$\propto \int \frac{d^D k}{(2\pi)^D}\, \mathrm{Tr}\left[\frac{1}{\not{k}-m}\, \gamma_{\mu_1}\, \frac{1}{\not{k}+\not{p}_1-m}\, \gamma_{\mu_2}\, \frac{1}{\not{k}+\not{p}_{12}-m} \right]$$

all indices are contracted, and we are left with "scalar integrals"

$$\int \frac{d^D k}{(2\pi)^D}\, \frac{\{1,\ k^2,\ k\cdot p_1,\ k\cdot p_2\}}{(k^2-m^2)\,((k+p_1)^2-m^2)\,((k+p_2)^2-m^2)}$$

Numerator is a polynomial in scalar products!

General Formalism:

$$A = \sum_i F_i \, T_i \leftarrow \text{tensors}$$

$\uparrow$ form Factors

T_i should be thought of as elements of a vector space

we can define "dual vectors" $\Rightarrow T_i^\dagger$ (for ex build out of ε_j^*)

a "scalar product" in this vector space

using $\sum_{pol} \varepsilon_j^\mu \varepsilon_j^{*\nu} = -g^{\mu\nu} \pm \dots$

Depending on conditions we used to define T_i (restrict vector space!)

$T_i^\dagger \cdot T_j$ implies $\sum_{pol} \Leftarrow$

then

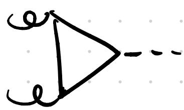
$$P_j = \sum_k C_k^{(j)} T_k^\dagger \quad \text{where} \quad C_k^{(j)} = [M]^{-1}_{jk}$$

check! Linear Algebra $\rightarrow$ $M_{ij} = [T_i^\dagger \cdot T_j]$

IRREDUCIBLE SCALAR PRODUCTS :

@ 1 loop, all scalar products can always be rewritten in terms of the propagators of the problem

EXAMPLE ABOVE :

• 

$$\begin{cases} D_1 = k^2 - m^2 \\ D_2 = (k+p_1)^2 - m^2 \\ D_3 = (k+p_1+p_2)^2 - m^2 \end{cases} \qquad \begin{cases} k \cdot k \\ k \cdot p_1 \\ k \cdot p_2 \end{cases}$$

$$\Rightarrow \quad k \cdot k = D_1 + m^2$$

$$k \cdot p_1 = \frac{1}{2}\left[D_2 - D_1 - p_1^2\right]$$

$$k \cdot p_2 = \frac{1}{2}\left[D_3 - D_2 - p_2^2 - 2 p_1 \cdot p_2\right]$$

so substituting these, all scalar ints become of the type

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{D_1^{a_1} D_2^{a_2} D_3^{a_3}} = I(a_1, a_2, a_3)$$

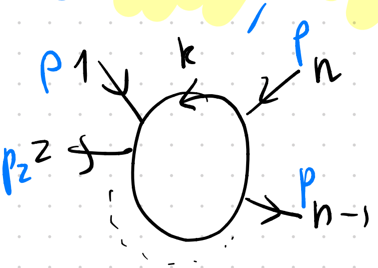
$$a_i \in \mathbb{Z}$$

This construction can be generalized for any process

$\Rightarrow$ Scattering Amplitude always decomposed into **Tensors** & **Form Factors**

1-LOOP CASE IS SPECIAL :

@ 1 loop n points $\rightarrow$ n "propagators" $D_1 \dots D_n$



$$\Rightarrow \begin{cases} D_1 = (k+p_1)^2 - m_1^2 \\ \vdots \\ D_n = k^2 - m_n^2 \end{cases}$$

n - scalar products $\begin{Bmatrix} k \cdot k \\ k \cdot p_i \end{Bmatrix}$

Scalar integrals will always be of the type

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{D_1^{a_1} \cdots D_n^{a_n}} = I(a_1 \dots a_n)$$

$a_i \in \mathbb{Z}$

SCALAR FEYNMAN INTEGRALS @ L-loops

@ L loops, $L \geq 2$, not all scalar products can be expressed in terms of D_i, we need to generalise the notation:

COMBINATORIC EXERCISE SHOWS THAT L loops, N points

$\#$ SCAL PRODS ρ is:

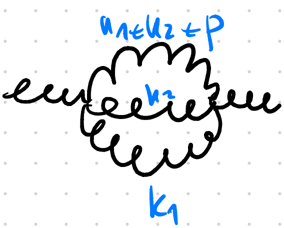
$$\rho = L\Big(\underset{\Uparrow}{N} + \frac{L}{2} - \frac{1}{2}\Big) \implies L = 1 \quad \boxed{\rho = N}$$

$\#$ of Legs, not indep momenta

@ 1 Loop $\rho = N$

(can you prove it?)

EXAMPLE: Two loop gluon propagator



$l = 2$

$N = 2$

$$\rho = 2\Big(2 + 1 - \frac{1}{2}\Big) = 5$$

3 propagators, 5 scalar products

$$S_i = \{ k_1^2, k_2^2, k_1\cdot k_2, k_1\cdot p_1, k_2\cdot p \}$$

given propagators $\left\{ \begin{array}{c} k_1^2 \\ k_2^2 \\ (k_1+k_2+p)^2 \end{array} \right\}$ a possible choice for ISPs $k_1\cdot p,\ k_2\cdot p$

$$\begin{cases} k_1\cdot k_1 = D_1 = k_1^2 \\ k_2\cdot k_2 = D_2 = k_2^2 \\ k_1\cdot k_2 = \frac{1}{2}\Big[D_3 - 2\underbrace{k_1\cdot p}_{S_1} - 2\underbrace{k_2\cdot p}_{S_2} - D_1 - D_2 - p^2\Big] \end{cases}$$

these are my two ISPs

FAMILY OF INTEGRALS : $\Rightarrow$ our object of study!

$$\int \frac{d^D k_1}{(2\pi)^D}\frac{d^D k_2}{(2\pi)^D}\ \frac{S_1^{b_1} S_2^{b_2}}{D_1^{a_1} D_2^{a_2} D_3^{a_3}} =$$

$$= I(a_1, a_2, a_3; \underbrace{-b_1, -b_2}_{\text{numerators}})$$

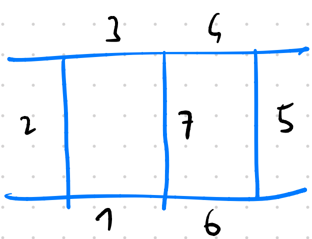
General Nomenclature:

$$I(a_1,\dots,a_\tau;\underbrace{-b_1,\dots,-b_\sigma}_{})=\int\prod_{l=1}^{L}\frac{d^Dk_l}{(2\pi)^D}\,\frac{S_1^{b_1}\cdots S_\sigma^{b_\sigma}}{D_1^{a_1}\cdots D_\tau^{a_\tau}}$$

negative indices **mean numerators!**

$I(1,\dots,1;0,\dots,0)$ = defines the **TOP SECTOR** or **TOP-TOPOLOGY**, i.e the graph we are considering

EXAMPLE: a 2-loop double box for $gg\to gg$ in QCD



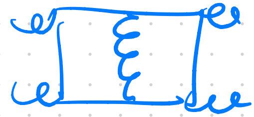
$$=\int\frac{d^Dk_1}{(2\pi)^D}\frac{d^Dk_2}{(2\pi)^D}\,\frac{1}{D_1\cdots D_7}$$

$\Uparrow$

we draw graph associated to scalar integral, we mean

— no Feynman Rules —

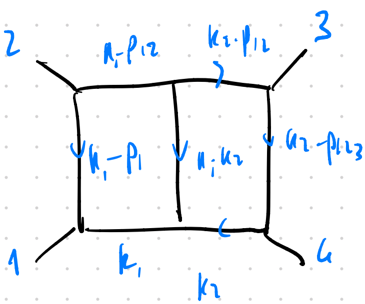
contributes to these dig



etc

SKIP

counting shows 9 scalar products, 7 propagators $\Big\}$ $\underline{2 \text{ ISPs}}$



$$= \left\{ \begin{array}{l} k_1^2 \\ k_2^2 \\ (k_1-k_2)^2 \\ (k_1-p_1)^2 \\ (k_1-p_{12})^2 \\ (k_2-p_{12})^2 \\ (k_2-p_{123})^2 \\ \text{ADD entire props} \\ (k_2-p_1)^2 \\ (k_1-p_{123})^2 \end{array} \right.$$

FAMILY

$p_{ijk} = p_i + p_j + p_k$ etc

So Integrals we are interested in will be

$$I(n_1,\dots,n_7;n_8,n_9) \qquad \left\{ \begin{array}{l} n_1,\dots,n_7 \geq 0 \\ n_8,\ n_9 \leq 0 \end{array} \right\}$$

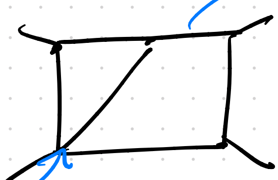
- $I(1,1,1,1,1,1,1,0,0)$ is <u>TOP SECTOR</u>

  <u>TOP TOPOLOGY</u>

- we call all ints obtained removing one or more propagators in all possible ways SUBSECTORS or SUB TOPOLOGIES

$\Rightarrow$ they give subtopology <u>tree</u>

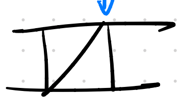
For example $I(0,1,1,1,1,1,1,0,0)$

=  it's a sub sector

pinch

pinch

$\uparrow$

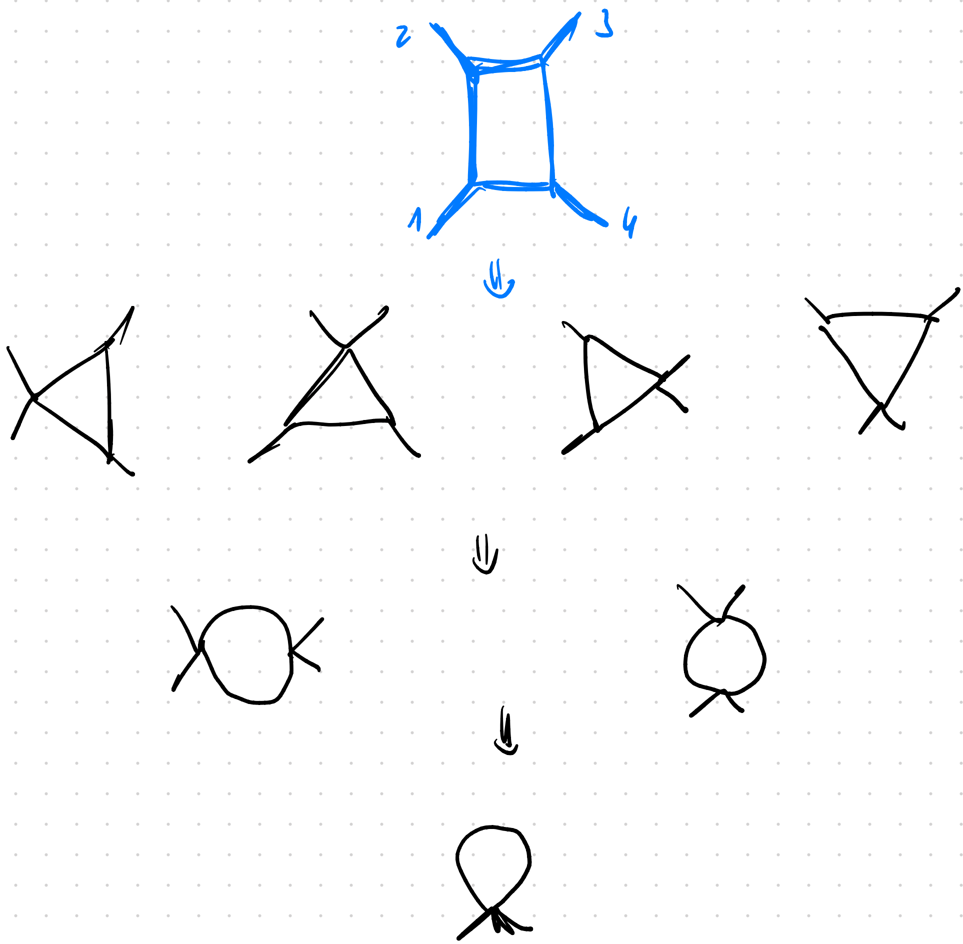
$I(0,1,1,1,1,0,1,0,0)$ =  subsector of

<u>etc</u>

→ HERE

In these cases we often say that subsectors are obtained by **PINCHING** propagators of top sector

INTEGRAL FAMILY & SUBTOPOLOGY TREE



Now, as you can imagine, if we start writing down all Feynman diagrams for a given problem, perform projection to compute scalar form factors, and finally collect all scalar integrals, we will in general find huge number of apparently different

integrals ⇒

\# $gg \to gg$ (in QCD)

@ 1 loop $\mathcal{O}(100)$ ints

@ 2 loops $\mathcal{O}(10000)$ ints

@ 3 loops $\mathcal{O}(10^7)$ ints !

simple combinatorics — clearly hopeless to compute all of them one by one —

Luckily, not all these integrals are independent!

INTEGRATION BY PARTS & MASTER INTEGRALS

We work in Dim Regularization to regularize UV & IR singularities.
The action of dim reg imply that we can perform a generic transformation on loop momenta

INFINITESIMALLY $\quad k_i^\mu \to k_i^\mu + \alpha v_j^\mu \qquad v_j^\mu = \{k_j^\mu, p_j^\mu\}$

complete set of momenta

$$\Rightarrow \quad f(\vec{k},\vec{q}) \to f(\vec{k},\vec{q}) + \alpha v_j^\mu \frac{\partial}{\partial k_i^\mu} f(\vec{k},\vec{q})$$

plus:

$$d^D k_i \to (1+\alpha D)\, d^D k_i \quad \text{if } v_j^\mu = k_i^\mu$$

invariance of integral implies that

$$\int \prod_{\ell=1}^{L} \frac{d^D k_\ell}{(2\pi)^D}\, O_{ij}\, f(\vec{k},\vec{q}) = 0$$

where $\quad O_{ij} = \partial_i \cdot v_j \qquad \begin{matrix} 1 \le L \\ j \ge i \end{matrix}$

used

$\partial_i k_i = D + k_i \partial_i$

to reabsorb jacobian!

O_{ij} generate a Lie Algebra —

Let's write these identities in a human friendly form — given a FAMILY OF INTEGRALS:

$$\int \prod_{l=1}^{L} \frac{d^D k_l}{(2\pi)^D} \left[\frac{\partial}{\partial k_i^\mu} v_j^\mu \frac{S_1^{b_1} \cdots S_\sigma^{b_\sigma}}{D_1^{a_1} \cdots D_\tau^{b_\tau}} \right] = 0$$

it's nothing but generalization of 1-dimensional

$$\int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x} f(x) = 0 \quad \text{if} \quad \int_{-\infty}^{+\infty} f(x)\, dx < \infty$$

$\Rightarrow$ usually referred to as INTEGRATION BY PARTS identities (IBPs)

[Chetyrkin, Tkachov '81]

- by inspection, it is clear that by differentiating we generate integrals in the <u>same FAMILY</u> so we expect that IBPs relate apparently different integrals in same family

- Using Lie Group property, one can prove formally that all integrals can be expressed in terms of a <u>FINITE</u> NUMBER OF <u>MASTER INTS.</u>

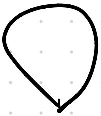
  ⇒ they are a BASIS of all integrals

  [A.V. Smirnov, A.V. Petukhov 2010]

Proof above is <u>NOT CONSTRUCTIVE</u> —

Let's see how this works in practice:

TADPOLE

 $= \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m^2 + i\varepsilon}$

In lectures I will work in Euclidean km. for convenience!

$$\xrightarrow[\text{Rotation}]{\text{Wick}} \propto \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2}$$

Family $\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+m^2)^n} = I(n)$

1 IBP $\int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left[k^\mu \frac{1}{(k^2+m^2)^n} \right] = 0$

$$\frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2+m^2)^n} = \frac{D}{(k^2+m^2)^n} - \frac{n\, k^\mu}{(k^2+m^2)^{n+1}} 2k_\mu$$

$$= \frac{D}{(k^2+m^2)^n} - 2n \frac{k^2}{(k^2+m^2)^{n+1}} = \frac{D-2n}{(k^2+m^2)^n} + \frac{2n\, m^2}{(k^2+m^2)^{n+1}}$$

which implies

$$(D-2n)\, I(n) + 2n\, m^2\, I(n+1) = 0$$

$$I(n+1) = -\frac{(D-2n)}{2n m^2} I(n)$$

$$I(2) = -\frac{D-2}{2m^2} \underline{\underline{I(1)}}$$

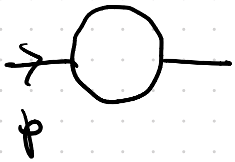
$$I(3) = -\frac{D-4}{4m^2} I(2) = \frac{(D-4)(D-2)}{8m^4} \underline{\underline{I(1)}}$$

we say that tadpole family has <u>ONE</u> <u>master integral</u>, can be chosen as $I(1)$

In this case, easy to solve IBP for generic "n"

in general this will not be possible $\rightarrow$ we can instead "generate" and "solve" IBPs for specific choices of indices $a_1, \dots, a_n$

. ONE LOOP BUBBLE (Euclidean signature)



$$= \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2+m^2)^a ((k+p)^2+m^2)^b}$$

$$= I(a, b)$$ family

I can derive 2 IBPs now :

① $$\int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left[k^\mu \frac{1}{D_1^a D_2^b} \right] = 0$$

② $$\int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left[p^\mu \frac{1}{D_1^a D_2^b} \right] = 0$$

Derive them for specific values of $(a,b) = \{(1,1); \cdots\}$

Prove that

$$I(1,2) = I(2,1) = -\frac{(D-2)}{2m^2(p^2+4m^2)} I(1,0) - \frac{D-3}{p^2+4m^2} I(1,1)$$

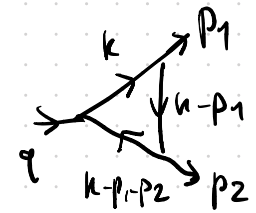
$\Rightarrow$ this problem has 2 master integrals

$$\left[\begin{array}{lll} I(1,0) & = & \text{the Tadpole} \\ I(1,1) & = & \text{the one loop bubble} \end{array}\right]$$

REDUCIBLE INTEGRALS $\Rightarrow$ QUOTE RESULT

Look up + LAPORTA

Consider the master triangle



$$= \int \frac{d^D k}{(2\pi)^D} \frac{1}{\underbrace{(k^2)^a}_{D_1^a}\underbrace{((k-p_1)^2)^b}_{D_2^b}\underbrace{((k-p_1-p_2)^2)^c}_{D_3^c}}$$

with $p_1^2 = p_2^2 = 0$; $q^2 = (p_1+p_2)^2 = s$

Three IBPs

$$\int \frac{d^D k}{(2\pi)^D} \frac{\partial}{\partial k^\mu} \left\{ \begin{array}{c} p_1^\mu \\ p_2^\mu \\ k^\mu \end{array} \right\} \frac{1}{D_1^a D_2^b D_3^c} = 0$$

EXERCISE

Prove

$$\begin{cases} s\,I(1,1,2) + (D-4)\,I(1,1,1) = 0 \\ s\,I(1,1,2) + I(1,0,2) + I(2,0,1) = 0 \\ s\,I(2,1,1) + I(1,0,2) + I(2,0,1) = 0 \end{cases}$$

NOTICE

- not all IBPs are independent! $2 \equiv 3$

- solving 1)

$$I(1,1,2) = -\frac{D-4}{s}\, I(1,1,1)$$

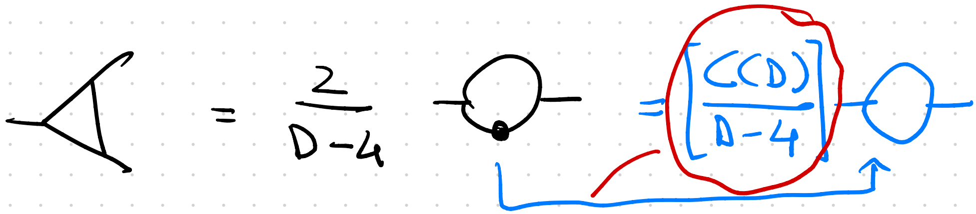
putting it into 2)

$$(D-4)\,I(1,1,1) = I(1,0,2) + I(2,0,1)$$

triangle gets "reduced" to bubbles !

noting $I(1,0,2) = I(2,0,1)$ we find

$$I(1,1,1) = \frac{2}{D-4}\, I(2,0,1)$$



reducing also bubble

integral with IR divergences

integral with UV divergences

two poles get "mixed up" by IBPs

In general, we solve IBPs in this way:

generate all of them starting from "seed" integrals

take big linear system, solve it

$\Longrightarrow$ [LAPORTA ALGORITHM '00]