From amplitudes to cross sections and events

[2024 Summer School in Particle Physics
Phenomenology after the Higgs Discovery]

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Useful references

- 0709.2092 by Nason, Frixione, Oleari: comprehensive review of **NLO calculations** (sec. 2,5,6)
- 0902.0293 lecure notes on the 2006 "Work- shop on Monte Carlo's, Physics and Simulations at the LHC", especially Sec. 2 on **Parton Showers** by P. Nason
- 1101.2599 General-purpose event generators for LHC physics by Buckley et al., especially for **matching** and **merging** (Sec. 5), and **non-perturbative physics** in MC (Secs. 6-7)
- 0407286 by Banfi, Salam, Zanderighi: comprehensive review of **NLL resummation** in direct QCD (summarised in Sec. 2 of 1412.2126 by Banfi, McAslan, Monni and Zanderighi)
- 1410.1892 by Becher, Broggio and Ferroglia, introduction to **SCET**

(*) quickly covered in these lectures, (*) not covered but good for your education

The Standard Model of Particle Physics

- The **STANDARD MODEL LAGRANGIAN** encapsulates our understanding of the **fundamental interactions** between **elementary particles**

$$\mathscr{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(iD^{\mu}\gamma_{\mu} - m)\psi + \ldots = \mathscr{L}(m, \alpha, \ldots)$$

Masses, coupling constants are free parameters, that we need to extract from **data**

- We need to calculate quantities that depends on m, α, that the experimentalists can measure: **cross sections**

$$\sigma_{\text{exp}} = \frac{N}{\mathscr{L}}$$

Number of events per unit of time

Number of colisions per unit of area per unit of time

$$\sigma_{\text{th}} = \int \frac{|\mathscr{A}|^2}{2s} d\Phi\, \Theta(\Phi)$$

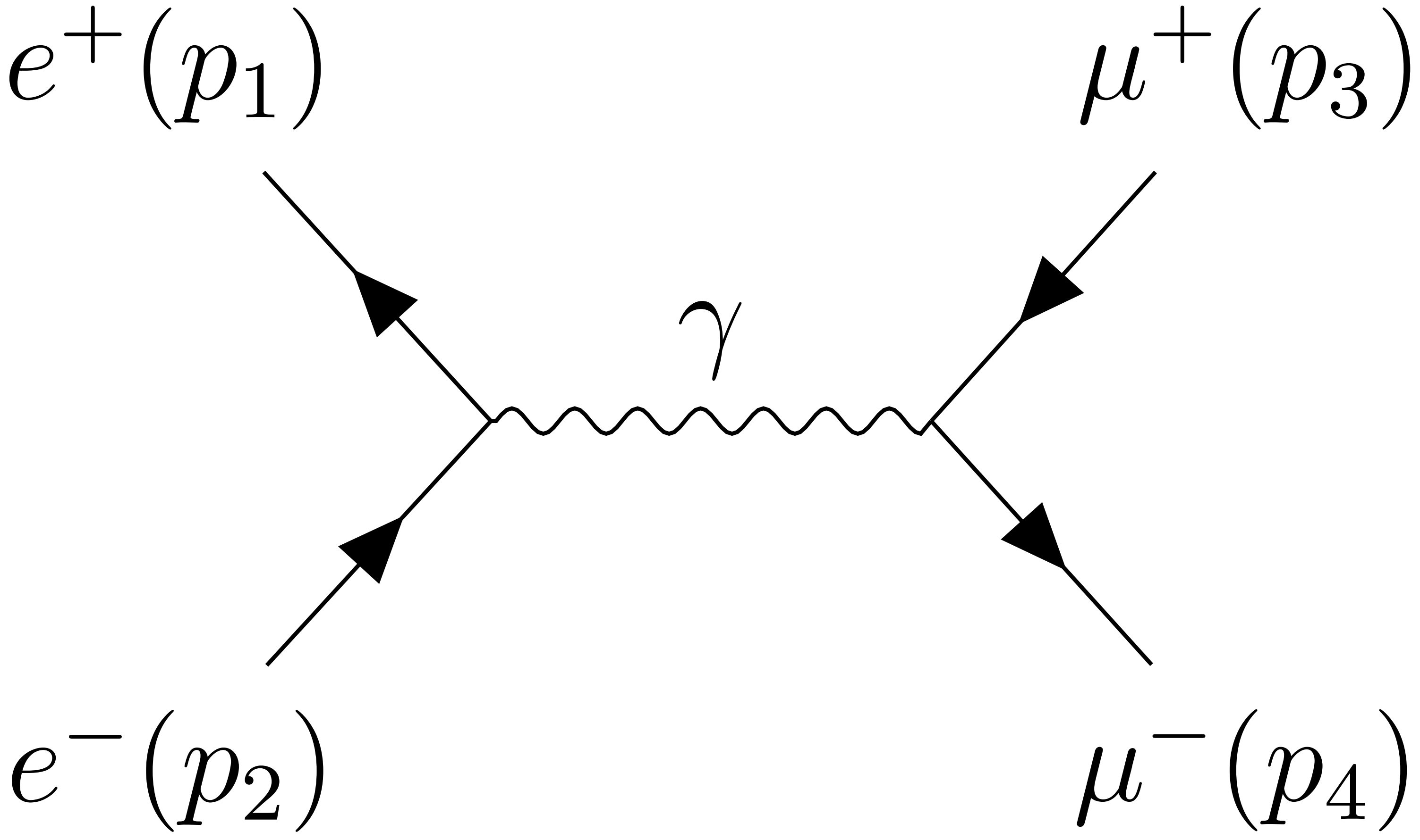
Squared amplitude

Phase space

Flux factor

Acceptance cuts

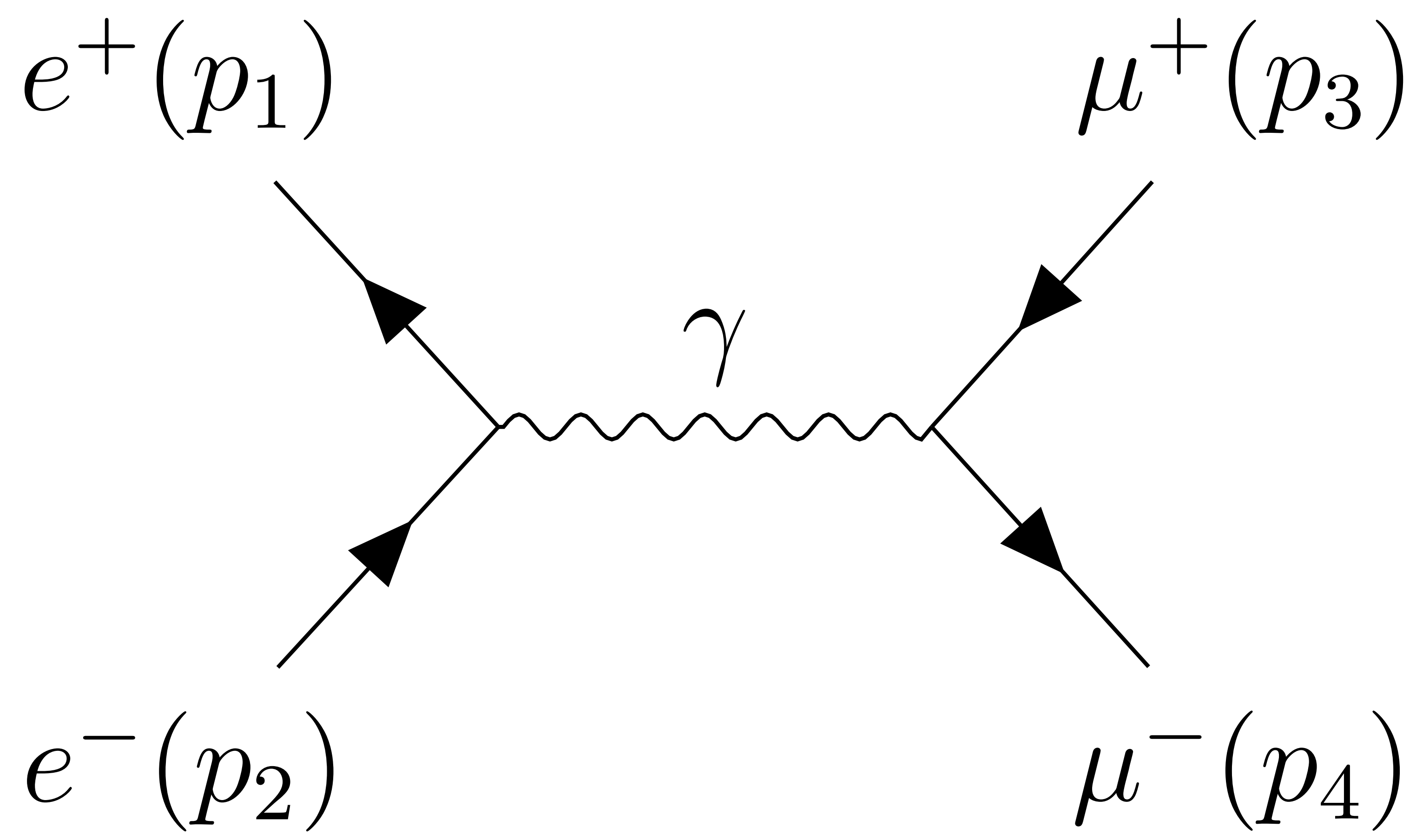
Example: $e^+e^- \to \mu^+\mu^-$



Let's calculate the cross-section for the production of a $\mu^+\mu^-$ pair in e^+e^- collisions, we need:

- **Amplitude** $\mathscr{A}$
- Two-body **phase space** $d\Phi_2$

Amplitude for $e^+e^- \to \mu^+\mu^-$



$$\mathcal{A} = [\bar{u}_e(p_1)(-ie\gamma^\mu)v_e(p_2)] \times [\bar{v}_\mu(p_3)(-ie\gamma^\nu)u_\mu(p_4)] \times \frac{-ig_{\mu\nu}}{(p_1+p_2)^2}$$

$$|\mathcal{A}|^2 = \frac{1}{4}\left(\frac{e^2}{2p_1\cdot p_2}\right)^2 \mathrm{Tr}(\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu)\mathrm{Tr}(\not{p}_3\gamma_\mu\not{p}_4\gamma_\nu)$$

Square and sum over polarisations

Average initial-state polarisations

$$= 2e^4\frac{(p_1\cdot p_3)^2+(p_1\cdot p_4)^2}{(p_1\cdot p_2)^2}$$

Two-body phase space

$$d\Phi = \left(\prod_{i=3}^{4}[dp_i]\right)(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4) \quad with \quad [dp_i] = \frac{d^4p_i}{(2\pi)^4}\delta_+(p_i^2-m_i^2) = \frac{d^3\vec{p}_i}{(2\pi)^3 2E_i}$$

➤ Lorentz invariant →can be computed in any frame, in the **center-of-mass frame**

$$p_1 = \frac{\sqrt{s}}{2}\{1,0,0,1\} \qquad p_2 = \frac{\sqrt{s}}{2}\{1,0,0,-1\}$$

→ in this frame $\vec{p}_3 + \vec{p}_4 = \vec{0}$: we use the x, y, z components of the δ to integrate over $\vec{p}_4$

$$d\Phi_2 = \frac{d^3\vec{p}_3}{((2\pi)^3 2|\vec{p}_3|)^2}(2\pi)^4\delta(\sqrt{s}-2|\vec{p}_3|) = \frac{|\vec{p}_3|^2 d|\vec{p}_3| d\Omega_2}{16\pi^2|\vec{p}_3|^2}\delta(\sqrt{s}-2|\vec{p}_3|) = \frac{d\Omega_2}{32\pi^2}$$

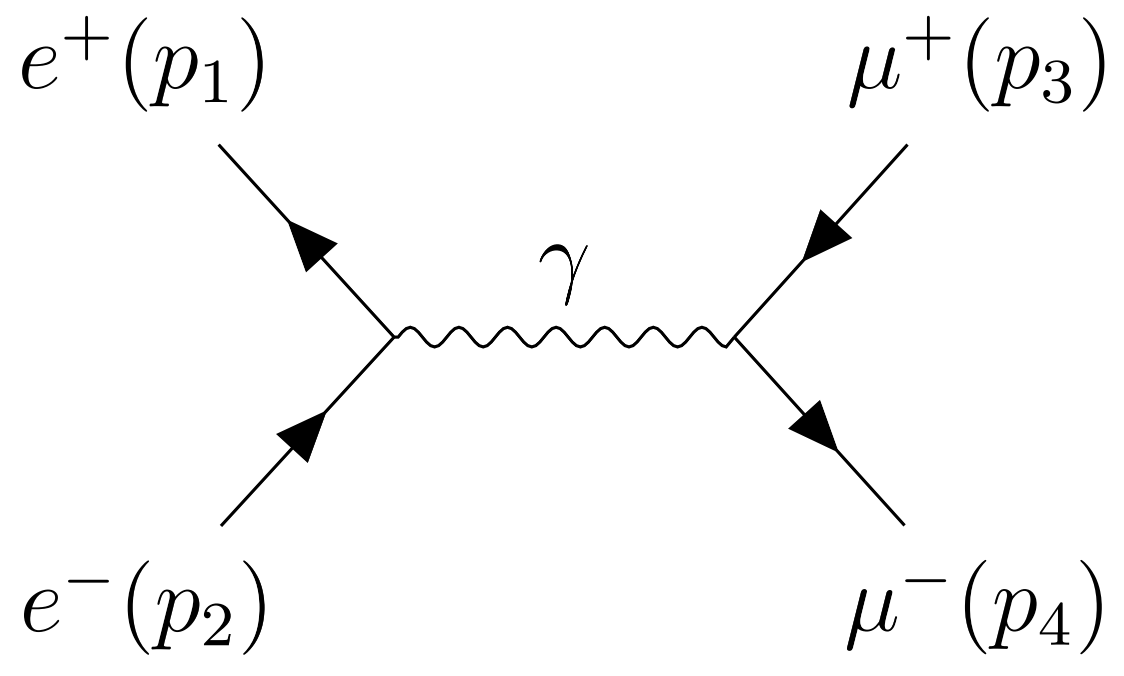
Spherical coordinates; $d\Omega_2 = d\cos\theta d\phi$

$|\vec{p}_3| = |\vec{p}_4| = \frac{\sqrt{s}}{2}$

$$p_3 = \frac{\sqrt{s}}{2}\{1, \sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta\}$$

$$p_4 = \frac{\sqrt{s}}{2}\{1, -\sin\theta\sin\phi, -\sin\theta\cos\phi, -\cos\theta\}$$

Example: $e^+e^- \to \mu^+\mu^-$



Let's calculate the cross-section for the production of a $\mu^+\mu^-$ pair in e^+e^- collisions, we need:

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- ➤ Two-body **phase space** $d\Phi_2$

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$$p_4 = \frac{\sqrt{s}}{2}\{1, -\sin\theta\sin\phi, -\sin\theta\cos\phi, -\cos\theta\}$$

$$d\Phi = \frac{d\cos\theta d\phi}{32\pi^2} \qquad |\mathscr{A}|^2 = 2e^4\frac{(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2}{(p_1\cdot p_2)^2}$$

$$d\sigma = \frac{|\mathscr{A}|^2}{2s}d\Phi = \frac{\alpha_{em}^2\pi}{2s}(1+\cos\theta^2)\, d\cos\theta\, \frac{d\phi}{2\pi}$$

$$\sigma = \frac{\alpha_{em}^2\pi}{2s}\int_{-1}^{1}(1+\cos\theta^2)\, d\cos\theta \int_0^{2\pi}\frac{d\phi}{2\pi} = \frac{4\alpha_{em}^2\pi}{3s}$$

$e^+e^- \to$ jets

If instead of muons we want to produce quarks…

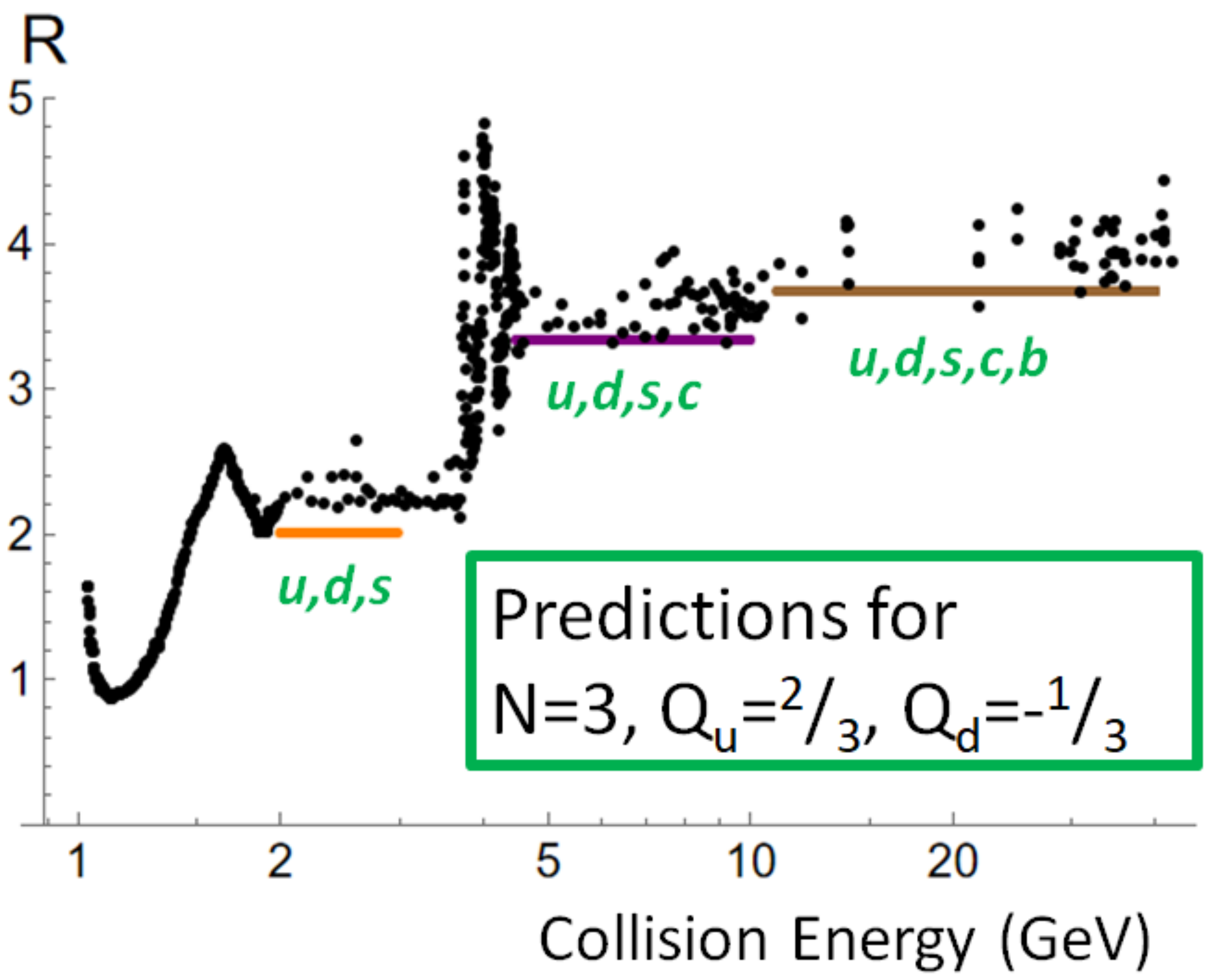


Fig. by Matt Strassler

Number of active quarks

$$d\sigma^{\mathrm{LO}}_{e^+e^-\to q\bar{q}} = N_c \sum_{i=1}^{n_f} Q_i^2 \, d\sigma^{\mathrm{LO}}_{e^+e^-\to\mu^-\mu^+}$$

Number of colours ($N_c = 3$)

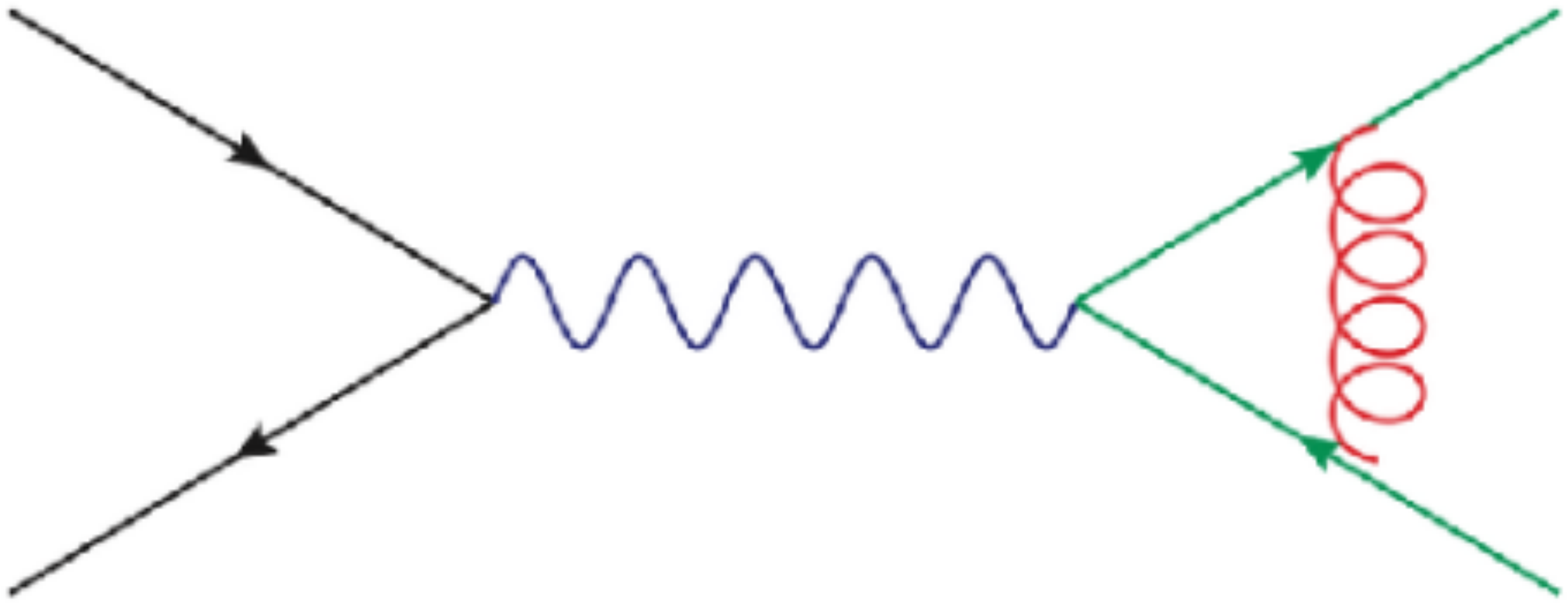
Quark charges

This is the **LEADING ORDER** prediction, can we do better?

NLO virtual corrections

The **QCD** coupling constant is quite sizeable $\alpha_s \sim 0.118$: let's include $\mathcal{O}(\alpha_s)$ **corrections**!

- At this order we have the interference between the tree level (=LO) amplitude, and the one containing a **virtual** gluon



$$\mathcal{A}_V \sim \int \frac{d^4\ell}{\ell^2(\ell+p_3)^2(\ell-p_4)^2} \xrightarrow[\ell \text{ soft}]{} \int \frac{d^4\ell}{\ell^4}$$

Infrared divergenge in 4 dimensions! Use **dimensional regularisation:** $D = 4 - 2\epsilon$

$$d\sigma_V = \frac{2\mathrm{Re}(\mathcal{A}_V \mathcal{A}_B^*)}{2s} d\Phi_2 = d\sigma_B \frac{\alpha_s}{2\pi} C_F C_\Gamma \left(\frac{\mu^2}{s}\right)^\epsilon \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2\right]$$

$$C_\Gamma = (4\pi)^\epsilon \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}$$

This regularisation introduces a non-phyisical scale..

Infrared poles

NLO corrections

The **QCD** coupling constant is quite sizeable $\alpha_s \sim 0.118$: let's include $\mathcal{O}(\alpha_s)$ **corrections**!

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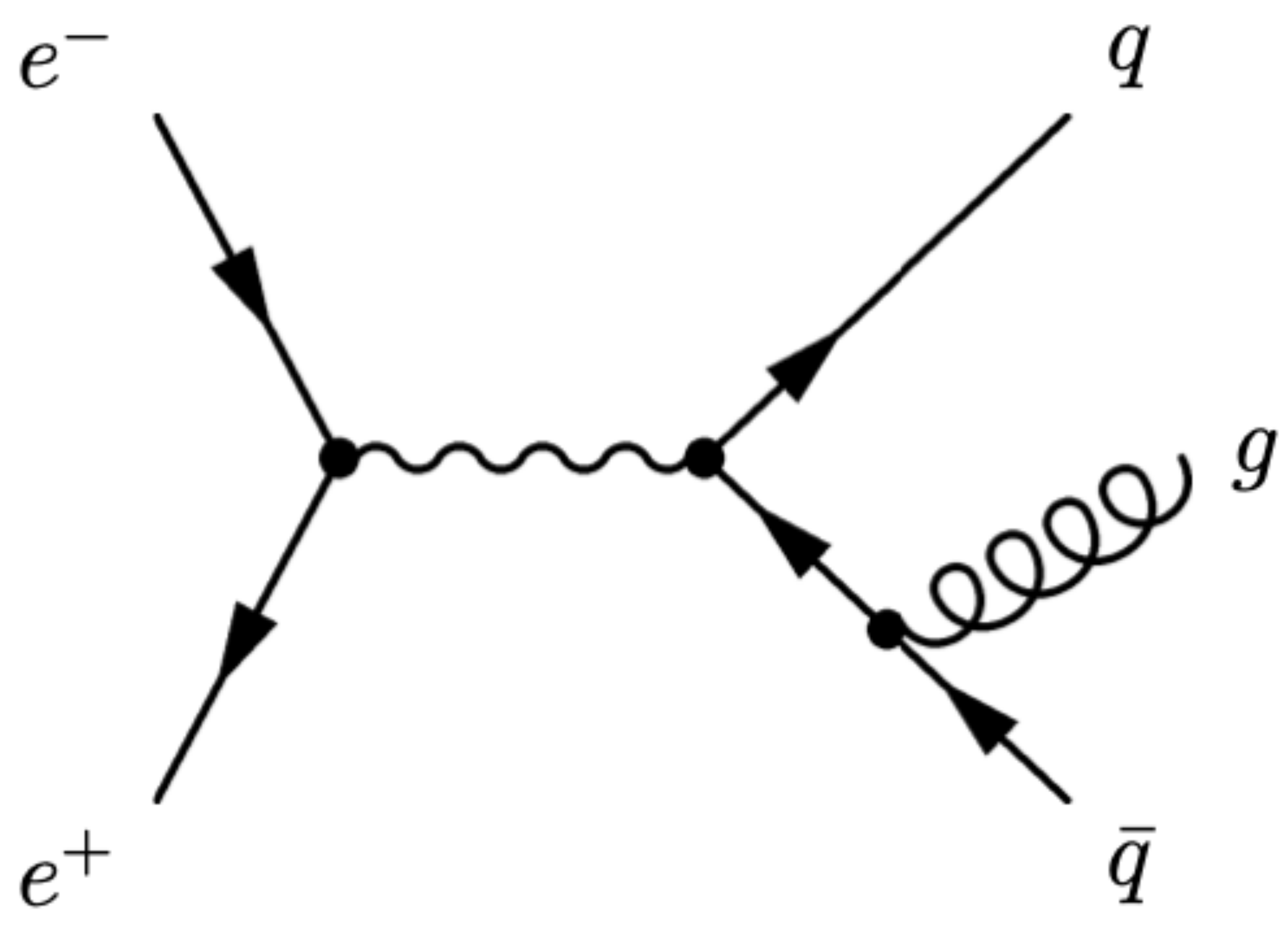
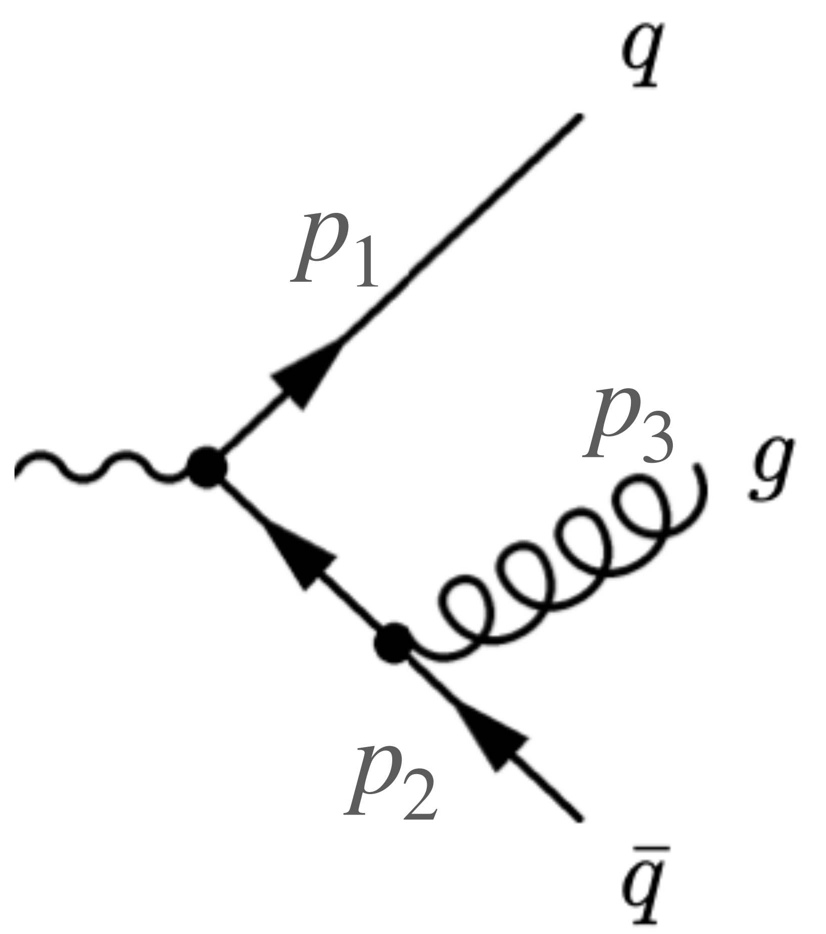
For $\epsilon \to 0$, $\sigma_V \to -\infty$: the result is meaningless, why? Because we cannot ask for the cross section for producing exactly two quarks! We cannot distinguish the case where there is also a soft (=low energy) or collinear (parallel to q or $\bar{q}$) gluon.

Kinoshita, Lee, Naumeberg theorem: **we need to sum over degenerate states to get a physical cross section!**



NLO real corrections



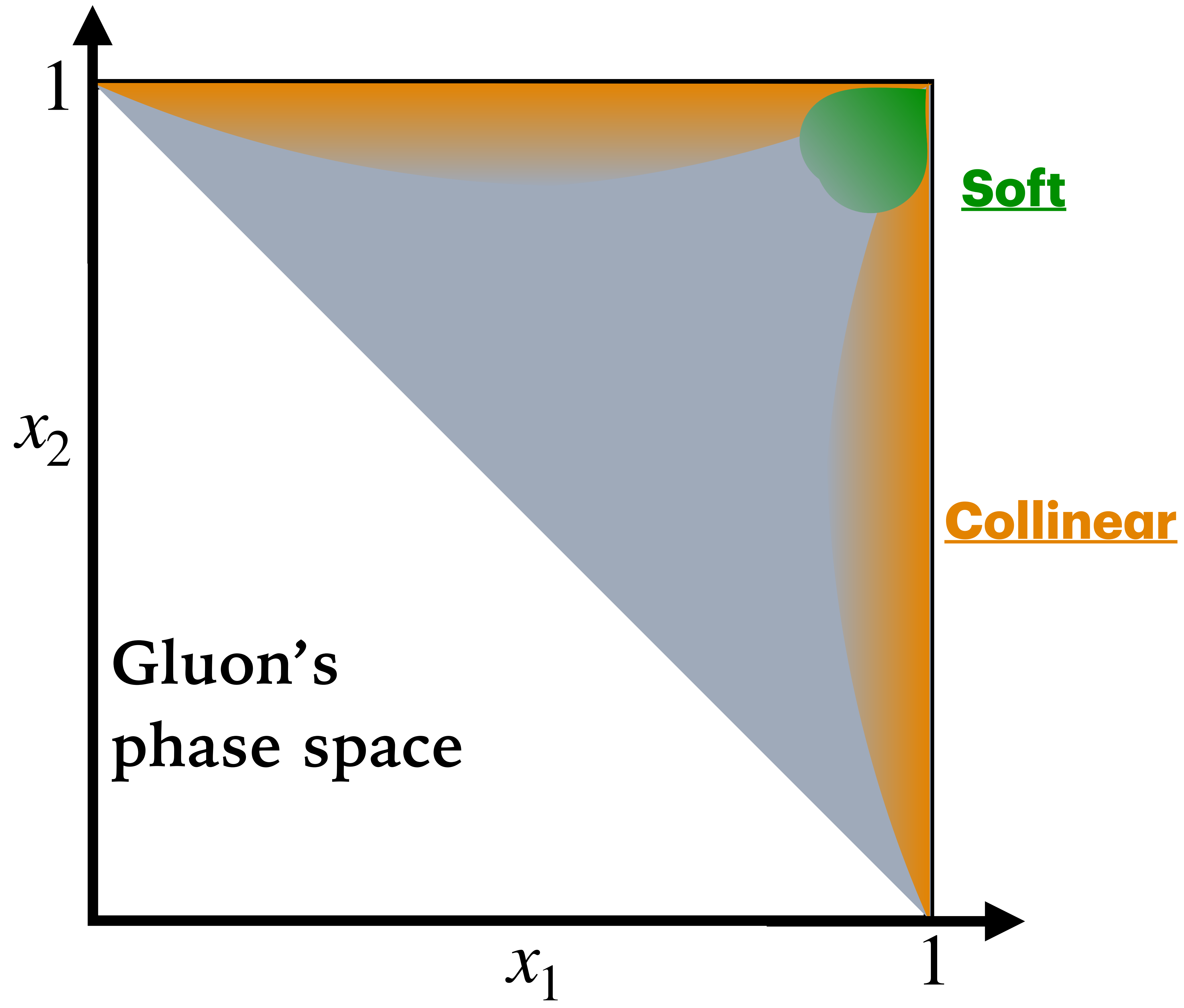
$$\frac{d\Phi_3}{\Phi_2} = \frac{dx_1 dx_2}{16\pi^2}(4\pi)^{\epsilon} s^{1-\epsilon} \frac{[(1-x_1)(1-x_2)(x_1+x_2-1)]^{\epsilon}}{\Gamma(1-\epsilon)}$$

$$x_i = \frac{2p_i \cdot p_{\text{tot}}}{p_{\text{tot}}^2} = \frac{2E_i}{E_{\text{tot}}}$$

$$\frac{|\mathscr{A}_R|^2}{|\mathscr{A}_B|^2} = \frac{2g_s^2 C_F}{(1-x_1)(1-x_2)s}\left[x_1^2 + x_2^2 - \epsilon(2-x_1-x_2)^2\right]$$

Divergence for $x_{1,2} \to 0$

$$\sigma_R = \sigma_B \frac{\alpha_s}{2\pi} C_F C_\Gamma \left(\frac{\mu^2}{s}\right)^{\epsilon} \left[+\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2\right]$$



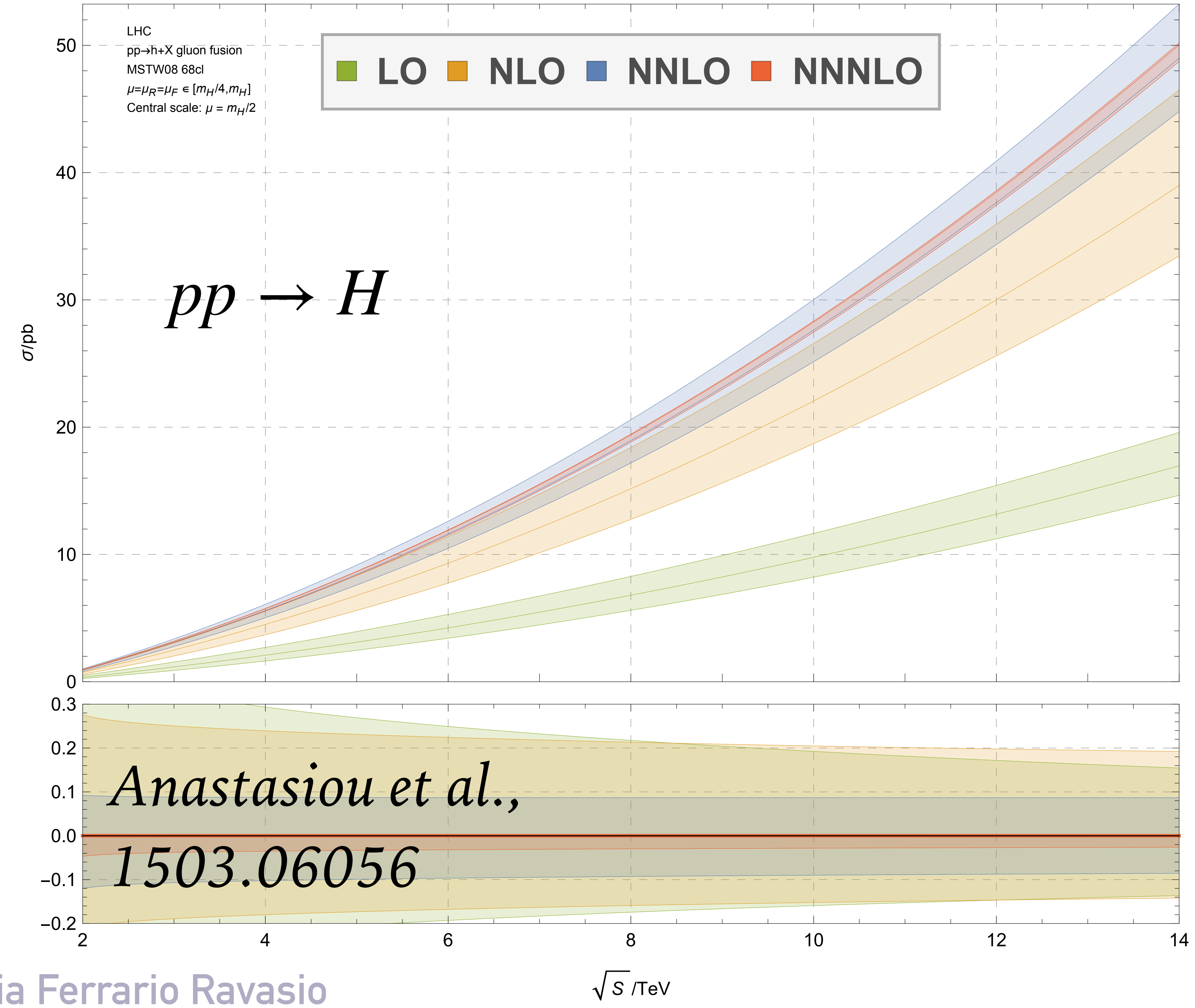
Cancels exactly the virtual divergence (KLN)!

Higer order corrections and uncertainties

- In general, virtual corrections exibit **ultraviolet** (although not for $e^+e^- \to j_1 j_2$) **and infrared singularities**: the former are reabsorbed introducing renormalised coupling constant (and masses), the latter cancel summing over degenerate states (KLN). The renormalisation procedure introduces a dependence on an unphysical scale

$$\sigma_{e^+e^- \to \text{jets}} = \sigma_{\text{LO}} \left(1 + \frac{\alpha_s(\mu_R)}{\pi} \right)$$

- When hadron colliders are involved, it is necessary to “renormalise” also the parton distributions, leading to another unphysical scale, dubber factorisation scale μ_F



- (μ_R, μ_F) variations by a factor 2 up and down conventional way to assess uncertainties from missing h.o.

- LO = gross features of an obs
  NLO = $\mathcal{O}(10-30\%)$ accuracy
  NNLO = necessary for percent-level precision
  N^{3}LO = available for $pp \to H, V$

Automating NLO calculations

Can we simplify the calculation of real corrections, so to tackle complex processes?

YES

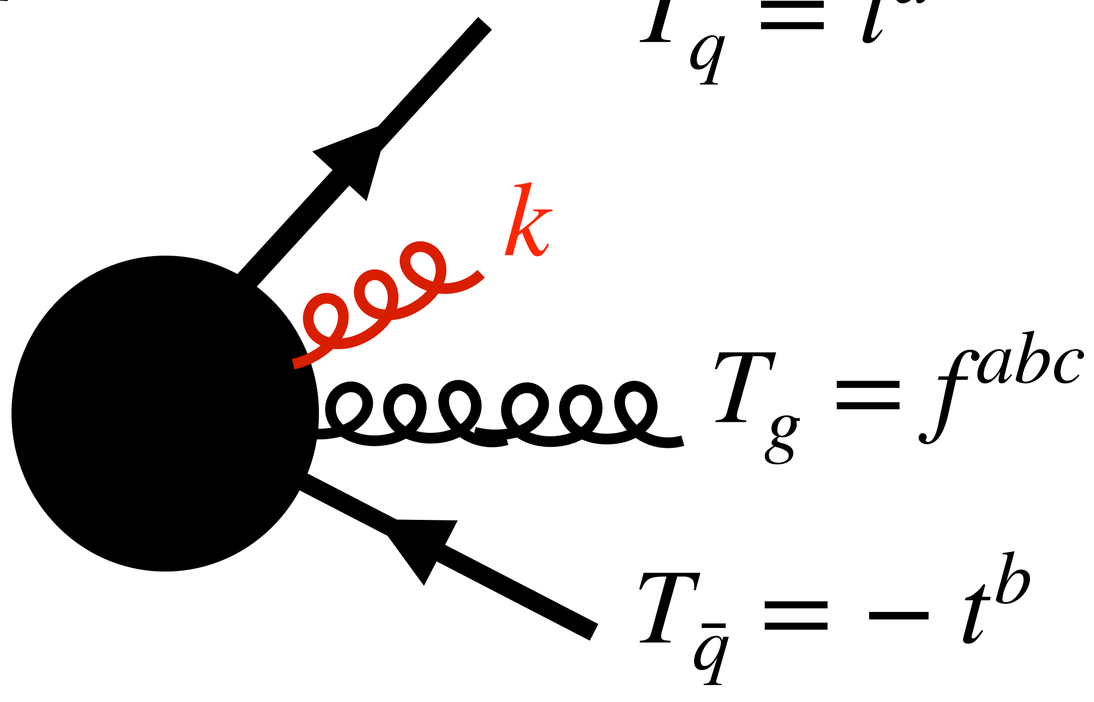
- Singular regions are associated with soft or collinear emissions

$$\frac{1}{(p+k)^2} = \frac{1}{2E_p E_k (1-\cos\theta_{pk})}$$

- In these limits, the amplitudes takes a **simple factorised form**

$$\frac{\mathcal{A}^2_{\text{soft}}}{\mathcal{A}^2_{\text{b}}} = -4\pi\alpha_s\mu^{2\epsilon} \times \sum_{i,j} \mathbf{T}_i \cdot \mathbf{T}_j \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}$$

Colour charges of the born partons



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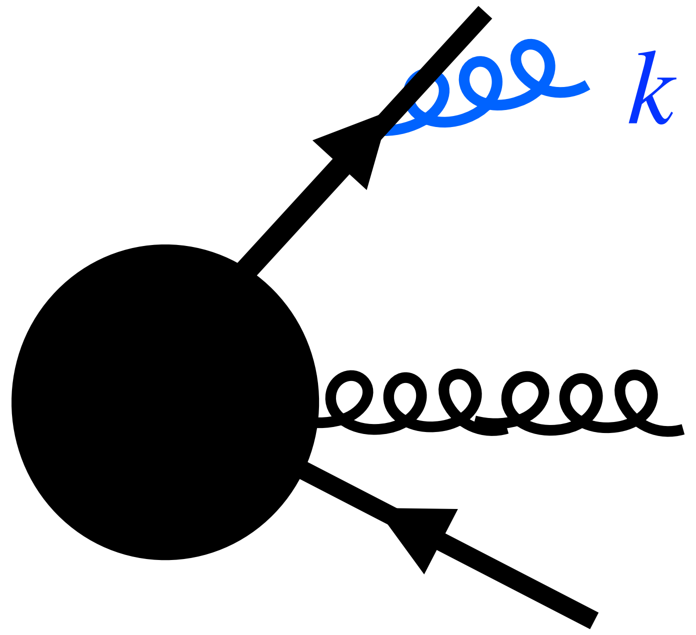

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Colour charges of the born partons

$$\frac{\mathscr{A}^2_{\text{coll}}(k \parallel p)}{\mathscr{A}^2_{\text{b}}} = \frac{4\pi\alpha_s\mu^{\epsilon}}{p \cdot k} \hat{P}_{pk}(z,\epsilon)$$

Altarelli-Parisi splitting functions

$$< \hat{P}_{gg}(z;\epsilon) >= 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z)\right]$$

$$< \hat{P}_{qq}(z;\epsilon) >= C_F \left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right]$$

$$< \hat{P}_{gq}(z;\epsilon) >= T_R \left[1 - \frac{2z(1-z)}{1-\epsilon}\right]$$

Automating NLO calculations

Can we simplify the calculation of real corrections, so to tackle complex processes?

YES

- In this limit, we know how to build an **approximation** of $d\sigma_R$ which captures the singular behaviour

$$d\sigma_R(\Phi_{n+1}) \sim d\sigma_B(\tilde{\Phi}_n)\, C(\tilde{\Phi}_n, \Phi_{\text{rad}}; \epsilon)$$

We need a mapping between $\Phi_{n+1} \leftrightarrow \Phi_n$

- $C(\tilde{\Phi}_n, \Phi_{\text{rad}})$ is usually simple enough to be integrated analytically over the radiated parton phase space Φ_{rad} in $D = 4 - 2\epsilon$

$$\bar{C}(\tilde{\Phi}_n; \epsilon) = \int d\Phi_{\text{rad}}^{(D=4-2\epsilon)} C(\Phi_{\text{rad}}, \tilde{\Phi}_b; \epsilon)$$

$$\sigma_{\text{NLO}} = \int d\Phi_n\, B(\Phi_n) \left(1 + \frac{V(\Phi_n, \epsilon)}{B(\Phi_n, \epsilon)} + \bar{C}(\Phi_n, \epsilon)\right) + \int d\Phi_{n+1} \left[R(\Phi_{n+1}) - C(\Phi_{n+1}) B(\tilde{\Phi}_n)\right]$$

D=4   The limit $\epsilon \to 0$ is finite   Can be integrated in D=4

Automating NLO calculations

$$\sigma_{\mathrm{NLO}} = \int d\Phi_n\, B(\Phi_n)\left(1 + \frac{V(\Phi_n, \epsilon)}{B(\Phi_n, \epsilon)} + \bar{C}(\Phi_n, \epsilon)\right) \quad + \quad \int d\Phi_{n+1}\left[R(\Phi_{n+1}) - C(\Phi_{n+1}) B(\tilde{\Phi}_n)\right]$$

Can we do more than inclusive cross sections?

YES

➤ We can calculate **infrared** (soft and collinear) safe observables:

$$\hat{O}(p_1, \ldots, p_n, k) \to \hat{O}(p_1, \ldots, p_n) \qquad \text{if } k \text{ is soft}$$
$$\hat{O}(p_1, \ldots, p_n, k) \to \hat{O}(p_1, \ldots, p_n + k) \qquad \text{if } k \text{ is collinear to } p_n$$

$$\frac{d\sigma}{dO} = \int d\Phi_n\, B(\Phi_n)\left(1 + \frac{V(\Phi_n, \epsilon)}{B(\Phi_n, \epsilon)} + C(\Phi_n, \epsilon)\right) \delta(\hat{O}(\Phi_n) - O)$$
$$+ \int d\Phi_{n+1}\left[R(\Phi_{n+1})\delta(\hat{O}(\Phi_{n+1}) - O) - C(\Phi_{n+1}) B(\tilde{\Phi}_n)\delta(\hat{O}(\tilde{\Phi}_n) - O)\right]$$