The gradient flow extended to the Standard Model

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Why gradient flow?

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Energy-momentum tensor

$$T_{\mu\nu}^{\mathrm{YM}} = G_{\mu\rho}^{a} G_{\nu\rho}^{a} - \frac{1}{4}\delta_{\mu\nu} G_{\rho\sigma}^{a} G_{\rho\sigma}^{a}$$

- Energy density: $\epsilon \sim \langle T_{00} \rangle$, Pressure: $\mathrm{P} \sim \langle T_{ii} \rangle$

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… for hadrons? → hadronic energies → lattice

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... for hadrons? → hadronic energies → lattice

- EMT = Conserved current

$$x_\mu \to x_\mu + \epsilon_\mu$$

$$\Rightarrow \partial_\mu T_{\mu\nu} = 0 + \ldots$$

$$\Rightarrow T_{\mu\nu} = \text{fin.}$$

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$$\Rightarrow \partial_\mu T_{\mu\nu} = 0 + \ldots$$

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- Lattice breaks translational invariance

⇒ EMT not finite on lattice

[Suzuki 2013]

Why gradient flow?

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- **Solution?** *Remove UV divergence of composite operators…*

- UV divergence → high fluctuations / momenta

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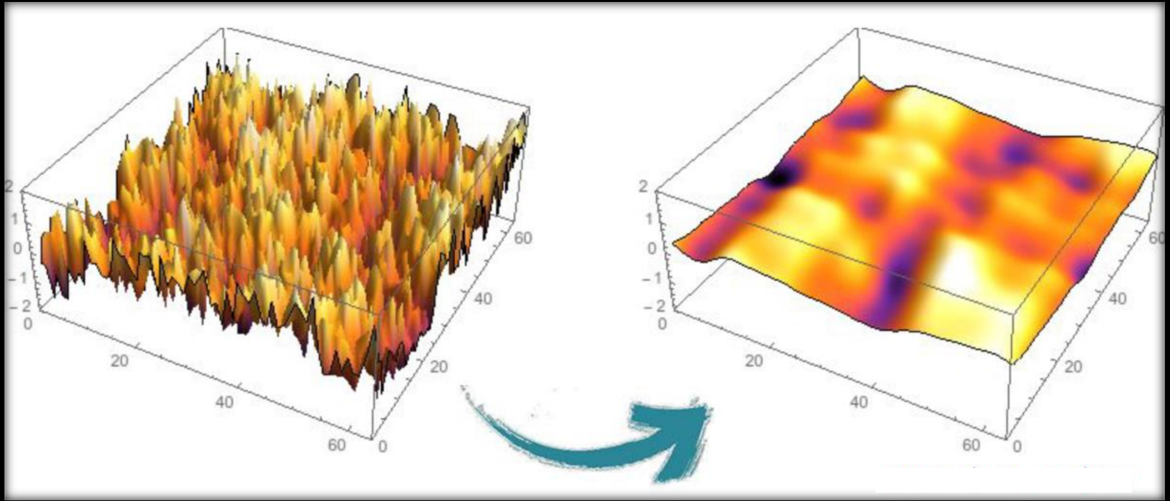
❖ **Solution?** *Remove UV divergence of composite operators…*

❖ UV divergence → high fluctuations / momenta

❖ Need low-pass filter for QCD

→ cutoff?

→ breaks Lorentz invariance



[Monahan 2015]

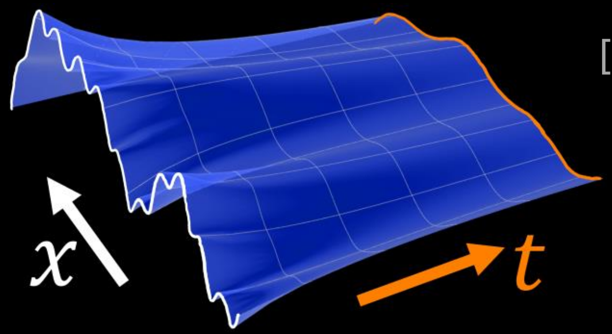
Gradient flow: Definition

- Flow equation:
  (Heat equation)

$$\partial_t \chi(t,x) = \mathcal{D}^2(t)\chi(t,x)\,, \qquad \chi(0,x) = \psi(x)$$

$$\partial_t B_\mu(t,x) = \mathcal{D}_\nu(t) G_{\nu\mu}(t)\,, \qquad B_\mu(0,x) = A_\mu(x)$$

[Narayanan, Neuberger 2006; Lüscher 2010; Lüscher, Weisz 2011; Lüscher 2013]



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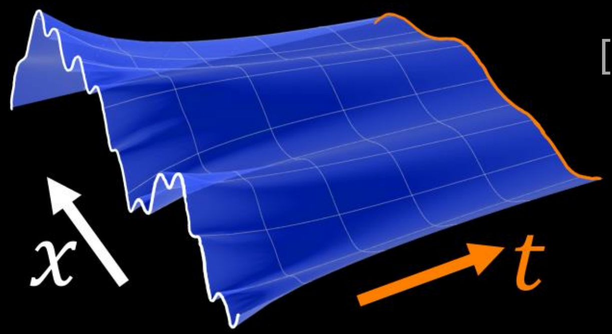

- LO solution: $\partial_t \chi(t,x) = \partial^2 \chi(t,x) \quad \Rightarrow \quad \hat{\chi}(t,p) = \hat{\psi}(p) e^{-tp^2}$

Gradient flow: Definition

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a + \overline{\psi}(\not{D} + m)\psi$$

[Lüscher, Weisz 2011]

$$\mathcal{L}_{\mathrm{flowed}} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\chi} + \mathcal{L}_{\mathrm{B}}$$

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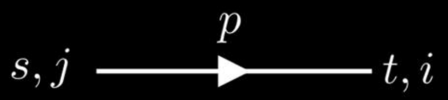
$$\mathcal{L}_{\mathrm{flowed}} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_{\chi} + \mathcal{L}_{\mathrm{B}}$$

$$\int_0^\infty \mathrm{d}t\, \bar{\lambda} \left(\partial_t \chi - \mathcal{D}^2 \chi\right)$$



$$= \delta_{ij} \frac{-i\not{p} + m}{p^2 + m^2} e^{-(t+s)p^2}$$

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$$\int_0^\infty \mathrm{d}t \, \bar{\lambda} \left(\partial_t \chi - \mathcal{D}^2 \chi\right)$$

$$\int_0^\infty \mathrm{d}t \, L_\mu \left(\partial_t B_\mu - \mathcal{D}_\nu G_{\nu\mu}\right)$$

$$s, j \xrightarrow{\; p \;} t, i \; = \delta_{ij} \frac{-i\not{p} + m}{p^2 + m^2} e^{-(t+s)p^2}$$

$$s, \nu, b \overset{p}{\sim\!\sim\!\sim} t, \mu, a = \frac{\delta^{ab}}{p^2} \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) e^{-(t+s)p^2}$$

+ vertices & mixed propagators

Application

Energy-momentum tensor

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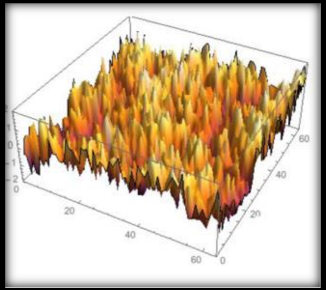
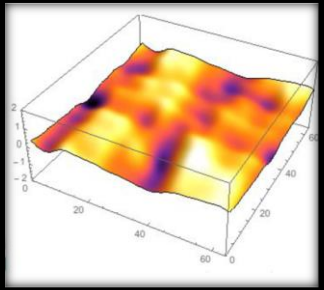


$$G_{\mu\nu}(x) \xrightarrow{A(x) \to B(t,x)} \mathcal{G}_{\mu\nu}(t,x)$$



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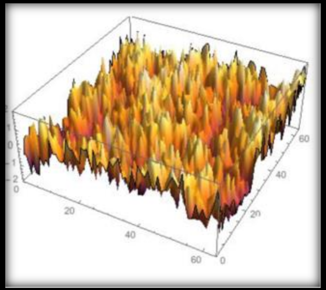

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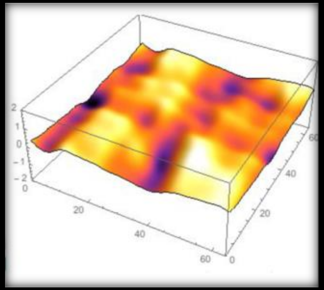

Short flow-time expansion (SFTX):
[Lüscher, Weisz 2011]

$$\tilde{\mathcal{O}}_i(t,x) = \zeta_{ij}(t)\mathcal{O}_j(x) + \dots$$

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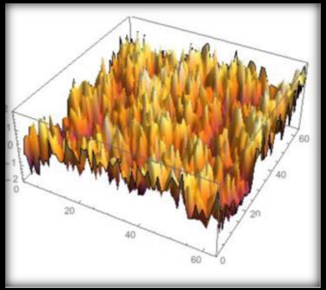

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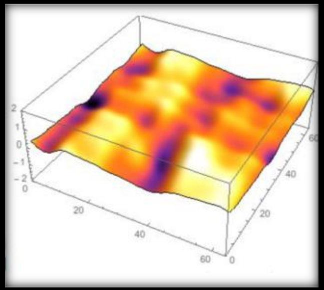

Short flow-time expansion (SFTX):
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$$\tilde{\mathcal{O}}_i(t,x) = \zeta_{ij}(t)\mathcal{O}_j(x) + \dots$$

$$T^{\mathrm{YM}}_{\mu\nu}(x) = c_1(t)\left[\mathcal{G}_{\mu\rho}\mathcal{G}_{\nu\rho}(t,x)\right] + c_2(t)\left[\delta_{\mu\nu}\mathcal{G}_{\rho\sigma}\mathcal{G}_{\rho\sigma}(t,x)\right]$$

Perturbative

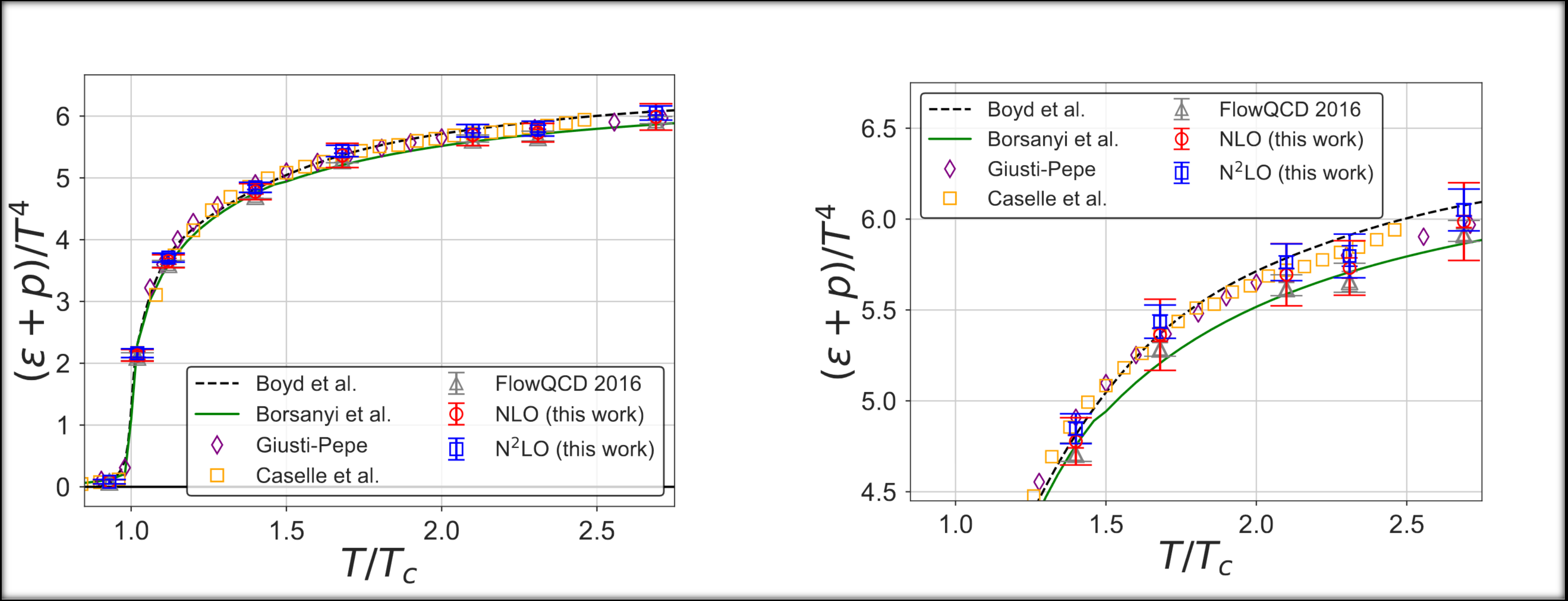
Lattice: *No renormalization required!*

[Suzuki 2013]

[Harlander, Kluth, Lange 2018]

Application

2-loop PT + lattice



Entropy density: $\varepsilon + p = -\frac{4}{3}\left\langle T_{00}(x) - \frac{1}{4}T_{\mu\mu}(x)\right\rangle$ [Iritani, Kitazawa, Suzuki, Takaura 2019]

Other Applications?

Other Applications!

- Strong coupling
- Renormalization of EFTs
- Neutron electric dipole moment
- Hadronic vacuum polarization
- Meson mixing and lifetimes
- QCD static force

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Applications

Strong coupling

[Lüscher 2010; Lüscher 2014]
[Harlander, Neumann 2016]

$$\langle G^a_{\mu\nu} G^a_{\mu\nu} \rangle = 0 \quad \longrightarrow \quad E(t) = \frac{1}{4} \mathcal{G}^a_{\mu\nu}(t) \mathcal{G}^a_{\mu\nu}(t)$$

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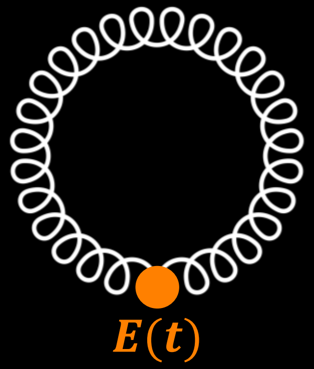
$$\langle t^2 E(t) \rangle = \frac{3\alpha_s}{4\pi} \frac{N_A}{8} + \mathcal{O}(\alpha_s^2)$$



$$\sim \int \frac{d^D p}{(4\pi)^2} e^{-tp^2} \neq 0$$

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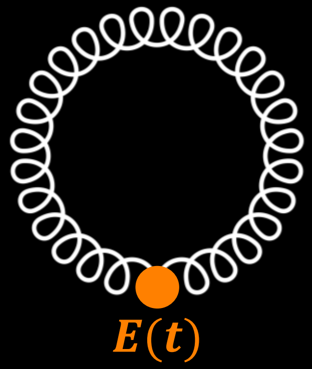
$$\alpha_s = \frac{32\pi}{3N_\mathrm{A}} \langle t^2 E(t) \rangle + \ldots$$

Lattice



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Applications

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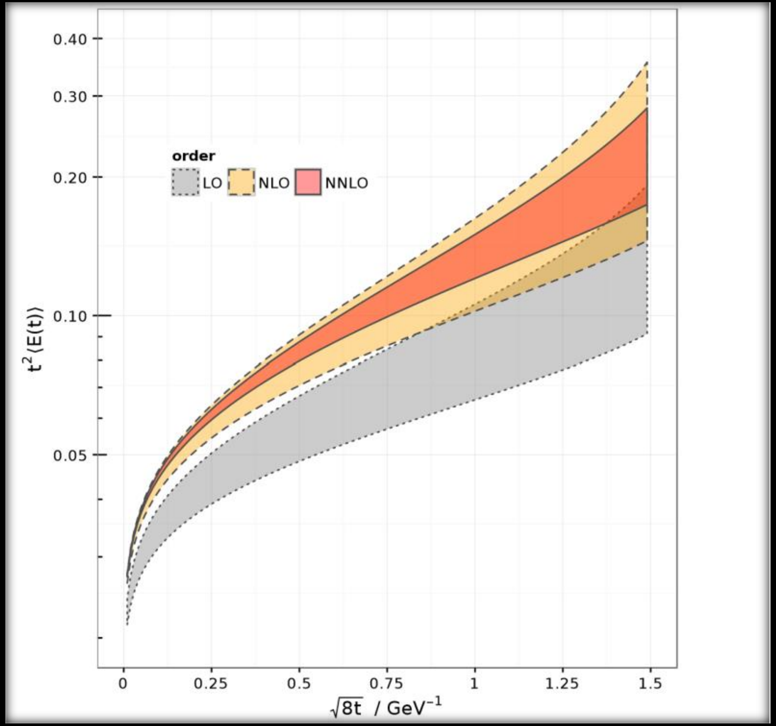
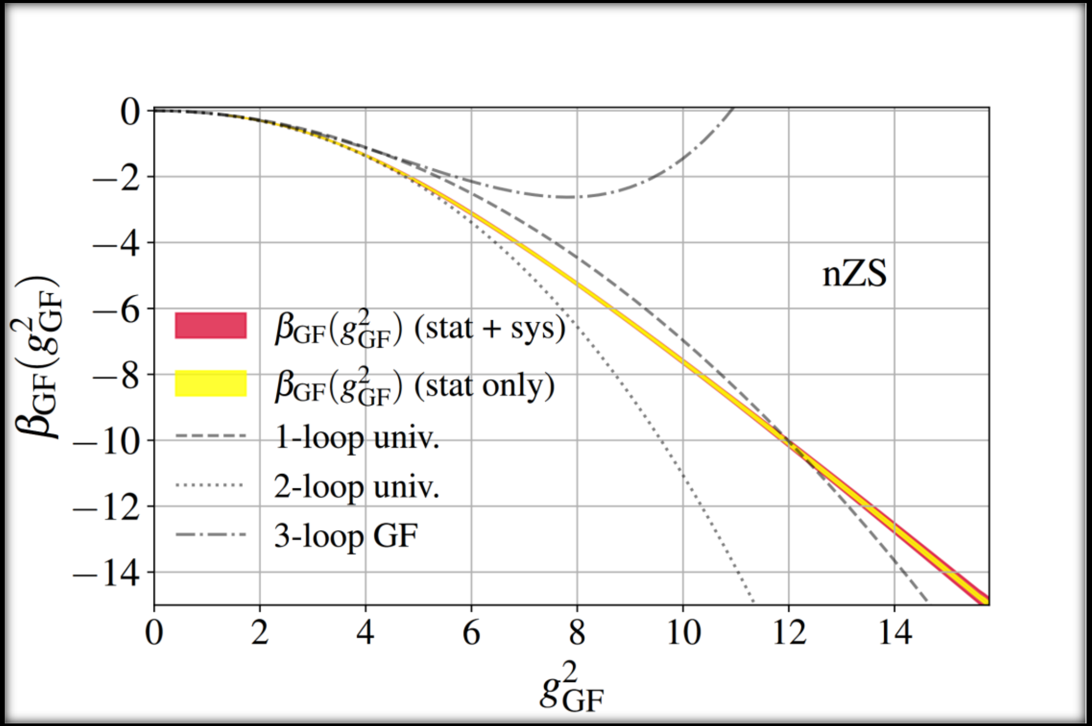


$N_f = 3, \qquad 3/\sqrt{8t} < \mu < 1.15/\sqrt{8t}$

[Harlander, Neumann 2016]



$N_f = 0$

[Hasenfratz, Peterson, van Sickle, Witzel 2023]

Applications

Renormalizing EFTs

Let's recall our approach to the EMT …

Applications

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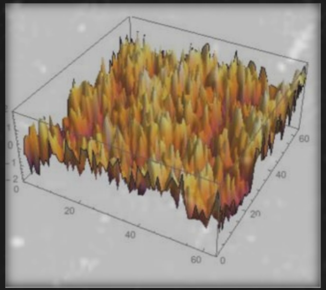

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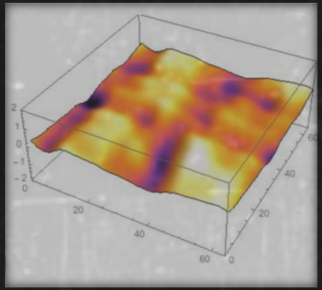

Short flow-time expansion (SFTX):
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$$\tilde{\mathcal{O}}_i(t,x) = \zeta_{ij}(t)\mathcal{O}_j(x) + \ldots$$

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$$\tilde{\mathcal{O}}(t) = \zeta^{B}(t,\epsilon)\mathcal{O}^{B}(\epsilon) = \underbrace{\zeta^{B}(t,\epsilon)Z^{-1}(\epsilon)}_{\zeta^{R}(t)\overset{!}{=}\text{fin.}}\mathcal{O}^{R} \quad \Rightarrow \quad \boxed{Z(\epsilon)}$$

Applications

Renormalizing EFTs

Let's have a closer look: $\tilde{\mathcal{O}}(t) = \boxed{\zeta^B(t,\epsilon)}\mathcal{O}^B(\epsilon)$

?

Applications

Renormalizing EFTs

Let's have a closer look:

$$\tilde{O}(t) = \zeta^B(t,\epsilon)\,\mathcal{O}^B(\epsilon)$$

Method of Projectors

[Gorishny, Larin, Tkachov 1983; Gorishny, Larin 1987]

Only tree-level contributions at t = 0

$$P[X] \sim \Pi\left(\frac{\partial}{\partial p_i^\nu}, \frac{\partial}{\partial m}\right)\langle A_\mu^a(p_1)\psi(p_2)\ldots|X|0\rangle\Big|_{p_i=m=0}$$

Applications

Renormalizing EFTs

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$$P_i[\mathcal{O}_j] = \delta_{ij} \quad \Rightarrow \quad \zeta_{ij}^B(t,\epsilon) = P_j[\tilde{\mathcal{O}}_i(t)]$$

Higher orders for t > 0

Applications

Renormalizing EFTs

$$\mathcal{O} \;\rightarrow\; \tilde{\mathcal{O}}(t) \;\rightarrow\; \zeta^B(t,\epsilon) \;\rightarrow\; Z(\epsilon)$$

Applications

Renormalizing EFTs

$$\mathcal{O} \rightarrow \tilde{O}(t) \rightarrow \zeta^B(t,\epsilon) \rightarrow Z(\epsilon)$$

No need to separate UV and IR divergences!

$$0 = \frac{1}{\epsilon_{UV}} + \frac{1}{\epsilon_{IR}} \sim \int \mathrm{d}^D k \, \frac{1}{(k^2)^n} \xrightarrow{\text{flow}} \int \mathrm{d}^D k \, \frac{e^{-tk^2}}{(k^2)^n} \sim \frac{1}{\epsilon_{IR}} = -\frac{1}{\epsilon_{UV}}$$

$Z(\epsilon)$ cancels IR divergences

Applications

Renormalizing EFTs

Flowed standard model?

- Known: SU(N) + fermions
- SU(3) ⊗ SU(2)
- U(1) ⟶ linear flow equation
- Higgs
- All the flowed renormalization constants

$$B_\mu^B = B_\mu \,, \qquad \chi^B = Z_\chi^{1/2}\chi$$

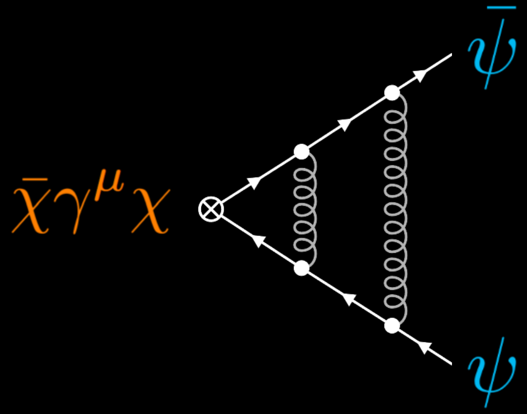
[Lüscher 2011; Lüscher 2013]

Applications

Renormalizing EFTs

How to find Z_χ ?

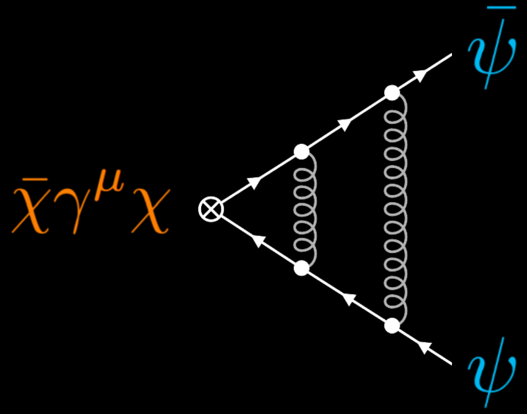
- Vector currents $\bar{\psi}\gamma^\mu\psi$ finite
  see also: [JB, Harlander, Kohnen, Lange 2023]
- No operator renormalization



Applications

Renormalizing EFTs

How to find Z_χ *?*

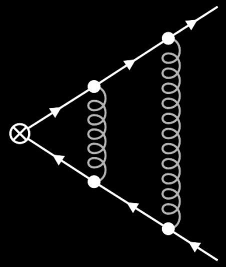


- Vector currents $\bar{\psi}\gamma^\mu\psi$ finite
  see also: [JB, Harlander, Kohnen, Lange 2023]
- No operator renormalization
- Get Z_χ from SFTX

$$Z_\chi P_{\bar{\psi}\gamma^\mu\psi}[\bar{\chi}\gamma^\mu\chi] \overset{!}{=} \text{fin.}$$

Applications

Renormalizing EFTs



≤ 4125 diagrams

208 flowed integrals

qgraf → tapir → exp → form

qgraf: Generate Diagrams

tapir: Insert abstract Feynman rules

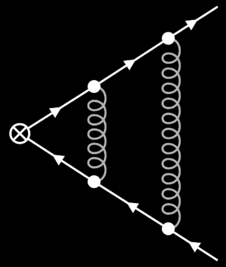
exp: Implement topology

form: Feynman Rules Integral Identification

Applications

Renormalizing EFTs

$$\int_{p,k}\int_{[0,1]^3} \mathrm{d}^3u\, u^c\, \frac{e^{-\left[P_1(u)p^2+P_2(u)k^2+P_3(u)(p+k)^2\right]t}}{(p^2)^{a_1}(k^2)^{a_2}((p+k)^2)^{a_3}}$$



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qgraf → **tapir** → **exp** → **form**

- **qgraf**: Generate Diagrams
- **tapir**: Insert abstract Feynman rules
- **exp**: Implement topology
- **form**: Feynman Rules, Integral Identification

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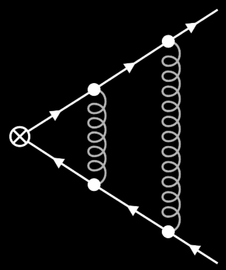

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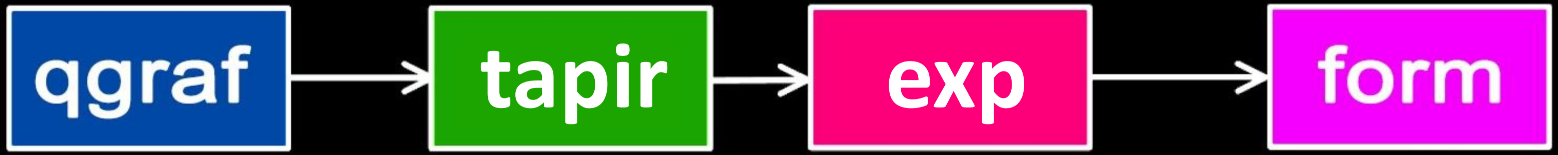

Generate IBP relations [Tkachov 1981; Chetyrkin, Tkachov 1981; Artz, Harlander, Lange, Neumann, Prausa 2019]

Reduction with **kira+FireFly** [Maierhöfer, Usovitsch, Uwer 2018; Klappert *et al.* 2021; Klappert, Lange 2020; Klappert, Klein, Lange 2021]

→ **6 master integrals**

Applications

Renormalizing EFTs

Results *(unbroken phase, vanishing Yukawa couplings)*

- ❖ Z_χ through NNLO for all fermions & Higgs
- ✓ Flowed currents independent of R_ξ gauge
- ✓ Agreement with SM & flowed QCD results
- ✓ Necessary ingredients for SMEFT renormalization

More Applications!

This talk:

- ❖ Energy-momentum tensor [Suzuki 2013; Harlander, Kluth, Lange 2018; ...]
- ❖ Strong coupling [Lüscher 2010, 2014; Harlander, Neumann 2016; ...]
- ❖ Renormalization of EFTs [*in preparation*]
- ❖ Neutron electric dipole moment [Mereghetti, Monahan, Rizik, Shindler, Stoffer 2022; ...] [JB, Harlander, Rizik, Shindler 2022]
- ❖ Hadronic vacuum polarization [Harlander, Lange, Neumann 2022]
- ❖ Meson mixing and lifetimes [Black *et al.* 2024]
- ❖ QCD static force [Brambilla *et al.* 2022]

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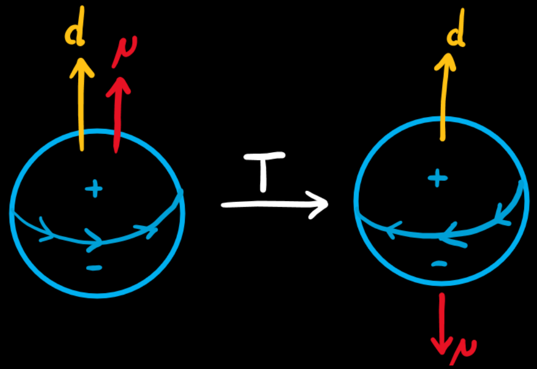
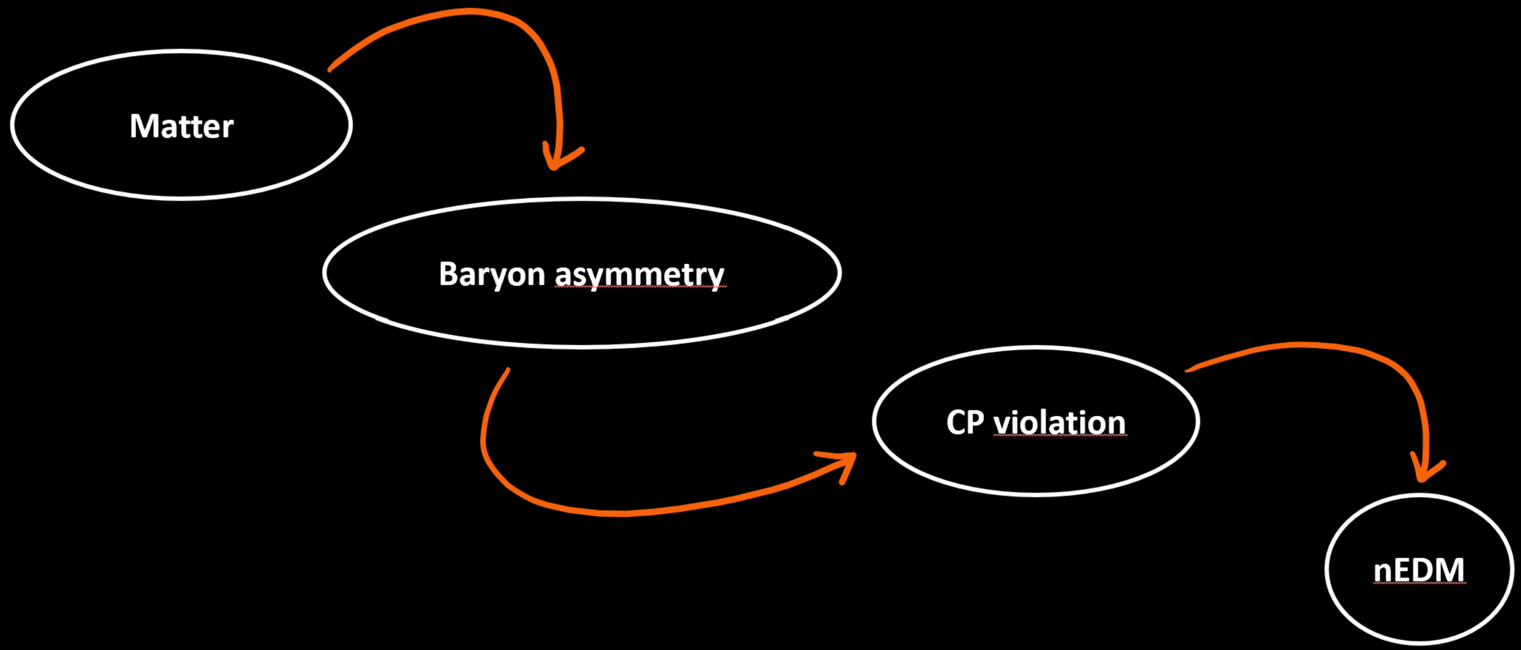
Appendix

nEDM





Experimental upper bound: $d_\mathrm{n} = (0.0 \pm 1.1_{\mathrm{stat}} \pm 0.2_{\mathrm{sys}}) \times 10^{-26} e\ \mathrm{cm}$ [Abel et al. 2020]

Future experiments estimate: $d_\mathrm{n} \sim 10^{-27} e\ \mathrm{cm}$ [nEDM collaboration 2021, TUCAN collaboration 2022]

Standard model (CKM): $d_\mathrm{n} \sim 10^{-32} e\ \mathrm{cm}$ [Seng 2015, Gavela et al. 1994]

Appendix

nEDM

$$\mathcal{L}_{\mathrm{EFT}} = \mathcal{L}_{\mathrm{SU(N)}} + \sum_{n \geq 5} \frac{C_n}{\Lambda^n} \mathcal{O}^{(n)}$$

$$\mathcal{O}_{\mathrm{CE}} = \bar{\psi} \sigma_{\mu\nu} \gamma_5 G^{\mu\nu} \psi$$

γ_5 ⚡ DimReg

Let's start with

$$\mathcal{O}_{\mathrm{CM}} = \bar{\psi} \sigma_{\mu\nu} G^{\mu\nu} \psi$$

Appendix

nEDM

Flowed operator: $$\tilde{\mathcal{O}}_{\rm CM}(t,x) \equiv \mathcal{O}_{\rm CM}(x)\Big|_{A_\mu\to B_\mu(t)\;\psi\to\chi(t)}$$

Recall SFTX: $$\tilde{\mathcal{O}}_{\rm CM}(t,x) = C_{\rm CM}(t)\mathcal{O}_{\rm CM}(x) + C_{\rm S}(t)\mathcal{O}_{\rm S}(x) + \ldots$$

Results:

$$C_{\rm CM}(t) = 1 + a_s(-4.023 + 0.166 l_{\mu t}) + a_s^2(-11.611 - 10.147 l_{\mu t} + 0.229 l_{\mu t}^2) + \mathcal{O}(a_s^3)$$

$$C_{\rm S}(t)/{\rm i} = -2a_s + a_s^2(6.136 + 3.167) + \mathcal{O}(a_s^3)$$

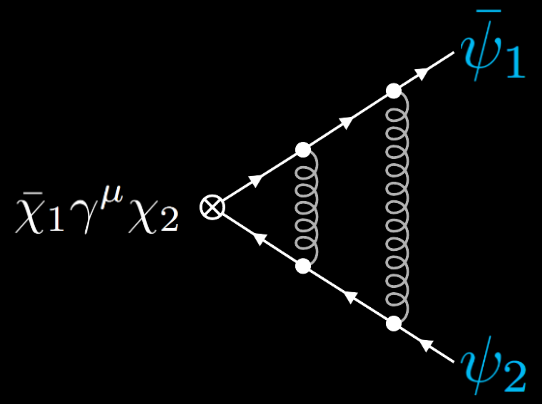
$$a_s = \frac{\alpha_s}{\pi}\,, \qquad l_{\mu t} = \log(8\pi\mu^2 t)$$

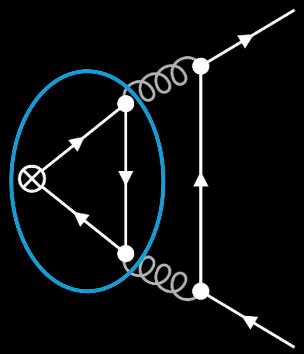
[Mereghetti, Monahan, Rizik, Shindler, Stoffer 2022]
[JB, Harlander, Rizik, Shindler 2022]

Appendix

Flowed Standard Model

$$\tilde{Z}_f P_{\bar{f}_1\gamma^\mu f_2}[\tilde{\bar{\tilde{f}}}_1\gamma^\mu \tilde{f}_2] = \zeta_f \overset{!}{=} \text{fin.}$$





Definition of flowed fields

$$\tilde{f}(t,x)|_{t=0} = f(x)$$
$$\tilde{\phi}(t,x)|_{t=0} = \phi(x)$$
$$\mathcal{G}_\mu^a(t,x)|_{t=0} = G_\mu^a(x)$$
$$\mathcal{W}_\mu^a(t,x)|_{t=0} = W_\mu^a(x)$$
$$\mathcal{B}_\mu(t,x)|_{t=0} = B_\mu(x)$$

Flow equations

$$\partial_t \tilde{f} = \Big[\mathcal{D}^2 - \kappa(\partial_\mu \mathcal{D}_\mu - \partial^2)\Big]\tilde{f}\,,$$
$$\partial_t \mathcal{G}_\mu^a = \mathcal{D}_\nu^{ab}\mathcal{G}_{\nu\mu}^b - \kappa \mathcal{D}_\mu^{ab}\partial_\nu \mathcal{G}_\nu^b\,,$$
$$\partial_t \mathcal{W}_\mu^a = \mathcal{D}_\nu^{ab}\mathcal{W}_{\nu\mu}^b - \kappa \mathcal{D}_\mu^{ab}\partial_\nu \mathcal{W}_\nu^b\,,$$
$$\begin{aligned}\partial_t \mathcal{B}_\mu &= \partial_\nu \mathcal{B}^{\nu\mu} + \kappa\partial_\mu\partial_\nu\mathcal{B}_\nu \\ &= \left[\partial^2\delta_{\mu\nu} + (\kappa-1)\partial_\mu\partial_\nu\right]\mathcal{B}_\nu\end{aligned}$$

→ **Linear**

Appendix

QCD flow equations

[Lüscher 2010; Lüscher 2013; Narayanan, Neuberger 2006]
[Lüscher, Weisz 2011]

Flowed gauge-fixing

Flowed QCD: $$\mathcal{L}_\chi = \int_0^\infty \mathrm{d}t\, \bar{\lambda}\left(\partial_t \chi - \mathcal{D}_\mu \mathcal{D}_\mu \chi + \kappa \partial_\mu \mathcal{G}_\mu \chi\right)$$

Lagrange multiplier field
EOM = Flow Equation

Flowed ghost

$$\mathcal{L}_{\mathrm{fQCD}} = \mathcal{L}_{\mathrm{QCD}} + \mathcal{L}_\chi + \mathcal{L}_{\mathrm{B}} + \mathcal{L}_{\mathrm{fl\text{-}gh}}$$

Flowed gauge-fixing

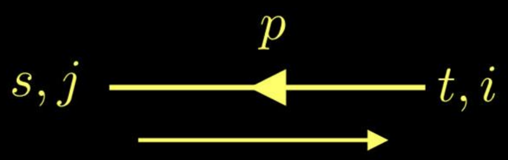
Flowed gluon: $$\int_0^\infty \mathrm{d}t\, L_\mu^a \left(\partial_t \mathcal{G}_\mu^a - \mathcal{D}_\nu^{ab} \mathcal{G}_{\nu\mu}^b + \alpha_0 \mathcal{D}_\mu^{ab} \partial_\nu \mathcal{G}_\nu^b\right), \quad \mathcal{G}_\mu^a(t,x)|_{t=0} = G_\mu^a(x)$$

Appendix

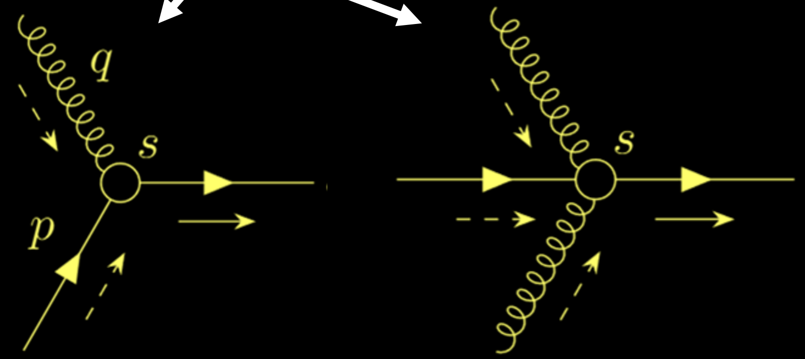
Feynman rules

[Lüscher 2010; Lüscher 2013; Narayanan, Neuberger 2006]
[Lüscher, Weisz 2011]

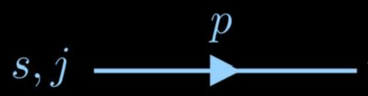
Flowed QCD: $\mathcal{L}_\chi = \int_0^\infty \mathrm{d}t\, \bar{\lambda}\, (\partial_t \chi - \mathcal{D}_\mu \mathcal{D}_\mu \chi + \kappa \partial_\mu \mathcal{G}_\mu \chi)$



$= \delta_{ij}\theta(t-s)e^{-(t-s)p^2}$



$\langle \bar{\chi}(t)\chi(s) \rangle \;=\; e^{-(t+s)p^2} \langle \bar{\psi}\psi \rangle$



$= \delta_{ij} \frac{-i\not{p} + m_0}{p^2 + m_0^2} e^{-(t+s)p^2}$