

# Uncertainty-aware BNNs for topo-cluster calibration

What theorists can teach experimentalists about calorimeter calibration

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Precision calibration of calorimeter signals in the ATLAS experiment  
using an uncertainty-aware neural network, arXiv:2412.04370 [hep-ex]  
ATLAS Collaboration (including T. Heimel, P. Loch, T. Plehn and L. Vogel)



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To be honest, we had a little help...

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What theorists **with very little help from the ATLAS calorimeter expert Peter Loch** can teach experimentalists about calorimeter calibration

# Today's outline

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1. Motivation
2. BNN-calibration performance
3. BNN-learned uncertainties
4. Repulsive ensembles
5. Summary and outlook

## Motivation

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# Topo-cluster calibration

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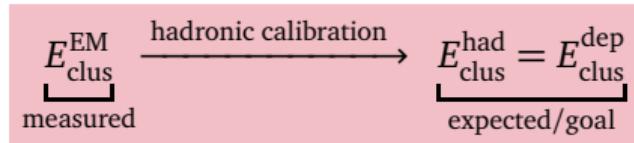
## Why topo-cluster calibration?

[arXiv:1603.02934, ATL-PHYS-PUB-2023-019]

- clusters of topologically connected cell signals principal calorimeter signals
- calibrated to correctly measure energy deposited by EM showers
- do not compensate for invisible energy losses in complex hadronic showers



### multi-dimensional correlated calibration



# Topo-cluster calibration



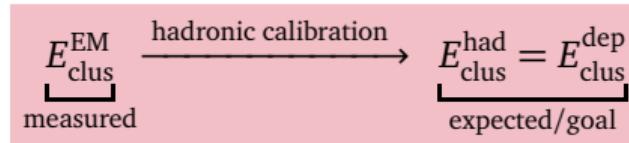
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### multi-dimensional correlated calibration



Standard ATLAS approach: **local cluster weighting (LCW)**

- four-step sequence with multi-dimensional, binned look-up tables
- non-smooth, step-like transitions between scale factors,  
no feature correlations, no pile-up measures



# Topo-cluster calibration

## Why topo-cluster calibration?

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**regression network:**  
response over phase space

$$\mathcal{R}_{\text{clus}}^{\text{BNN}}(\mathcal{X}_{\text{clus}}) \stackrel{\text{train}}{\approx} \mathcal{R}_{\text{clus}}^{\text{EM}} = \frac{E_{\text{clus}}^{\text{EM}}}{E_{\text{clus}}^{\text{dep}}}$$

15 topo-cluster features → training dataset  $D_{\text{train}}$  given by pairs  $(\mathcal{X}_{\text{clus}}, \mathcal{R}_{\text{clus}}^{\text{EM}})$

$$\mathcal{X}_{\text{clus}} = \left\{ \underbrace{E_{\text{clus}}^{\text{EM}}, y_{\text{clus}}^{\text{EM}}, \zeta_{\text{clus}}^{\text{EM}}}_{\text{kinematics}}, \underbrace{\text{Var}_{\text{clus}}(t_{\text{cell}}), \lambda_{\text{clus}}, |\vec{c}_{\text{clus}}|, \langle \rho_{\text{cell}} \rangle, \langle m_{\text{long}}^2 \rangle, \langle m_{\text{lat}}^2 \rangle, p_{\text{T}} D, f_{\text{emc}}, f_{\text{iso}}, t_{\text{clus}}, N_{\text{PV}}, \mu}_{\text{shower nature (position, compactness, signal density, internal time structure)}}, \underbrace{f_{\text{topo}}}_{\text{topology}} \right\}$$



# Topo-cluster calibration

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Modern **(B)NNs** for  
local topo-cluster calibration  
correcting for this non-compensation

- single-step training
- exploiting correlations
- smooth and multi-dimensional
- **control and uncertainties key**  
(access to bottom-up systematics)

[Ph.D. Thesis of Y. Gal, arXiv:2211.01421]



# Topo-cluster calibration

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## Why topo-cluster calibration?

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Modern **(B)NNs** for  
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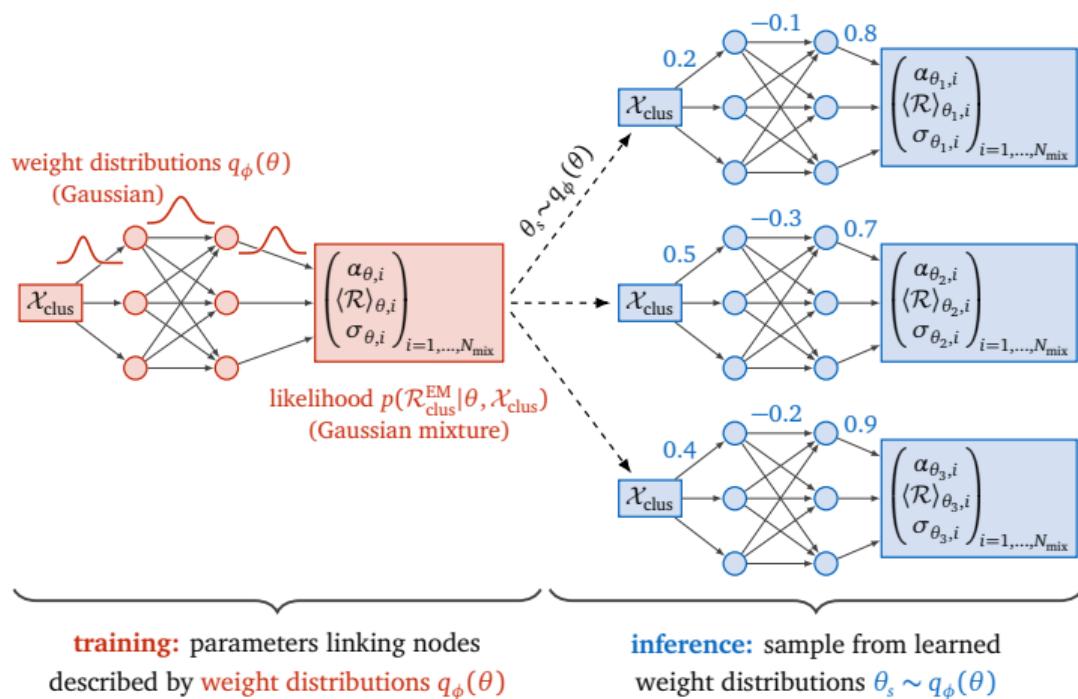
- single-step training
- exploiting correlations
- smooth and multi-dimensional
- **control and uncertainties key**  
(access to bottom-up systematics)

[Ph.D. Thesis of Y. Gal, arXiv:2211.01421]

**learn uncertainty associated with calibration function:**  
better understanding of detector-signal features, signal-quality issues  
in data, and limitations associated with network training



# BNN — Bayesian neural network



BNNs **learn distributions of network parameters**, defining output distribution

[arXiv:2003.11099, arXiv:2206.14831, arXiv:2211.01421]

- weights are not trained as fixed values
- training:  
parameters  $\theta$  described by weight distributions  $q_\phi(\theta) \approx p(\theta | D_{\text{train}})$
- inference:  
sample from weight distributions to get **ensemble of networks**

# BNN — learning weight distributions

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BNNs learn distributions of network parameters, defining output distribution

$\mathcal{R}(x)$  given by probability  $p(\mathcal{R})$  encoded in weight configurations:

$$p(\mathcal{R}) = \int d\theta p(\mathcal{R}|\theta) p(\theta|D_{\text{train}})$$

# BNN — variational inference

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BNNs learn distributions of network parameters, defining output distribution

$\mathcal{R}(x)$  given by probability  $p(\mathcal{R})$  encoded in weight configurations:

$$p(\mathcal{R}) = \int d\theta p(\mathcal{R}|\theta) p(\theta|D_{\text{train}}) \approx \int d\theta p(\mathcal{R}|\theta) q_\phi(\theta)$$

training by **variational approximation** of  
 $p(\theta|D_{\text{train}})$  with simplified and tractable  $q_\phi(\theta)$

# BNN — loss function

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BNNs learn distributions of network parameters, defining output distribution

$\mathcal{R}(x)$  given by probability  $p(\mathcal{R})$  encoded in weight configurations:

$$p(\mathcal{R}) = \int d\theta p(\mathcal{R}|\theta) p(\theta|D_{\text{train}}) \approx \int d\theta p(\mathcal{R}|\theta) q_\phi(\theta)$$

Similarity by minimizing Kullback-Leibler (KL) divergence:

$$\min_{\phi} D_{\text{KL}}[q_\phi(\theta), p(\theta|D_{\text{train}})] \xrightarrow{\text{Bayes}} \mathcal{L}_{\text{BNN}} = \underbrace{D_{\text{KL}}[q_\phi(\theta), p_{\text{prior}}(\theta)]}_{\text{regularization}} - \underbrace{\langle \log p(D_{\text{train}}|\theta) \rangle_{\theta \sim q_\phi(\theta)}}_{\text{negative log-likelihood}}$$

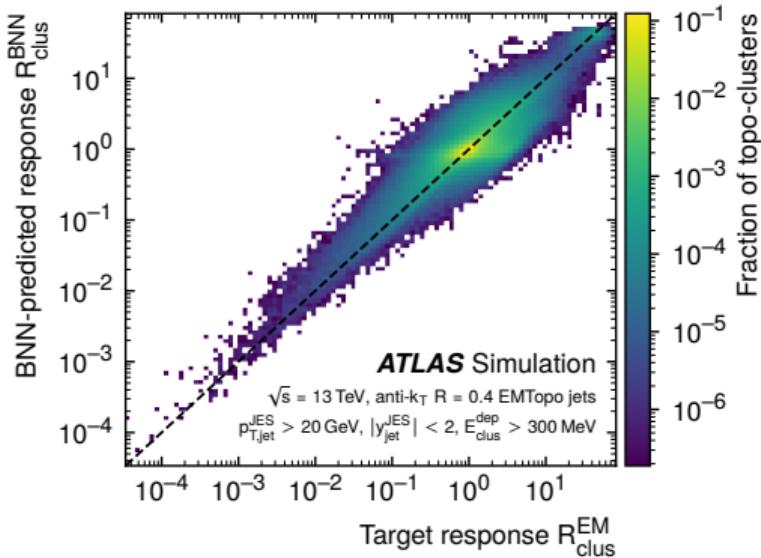
## BNN-calibration performance

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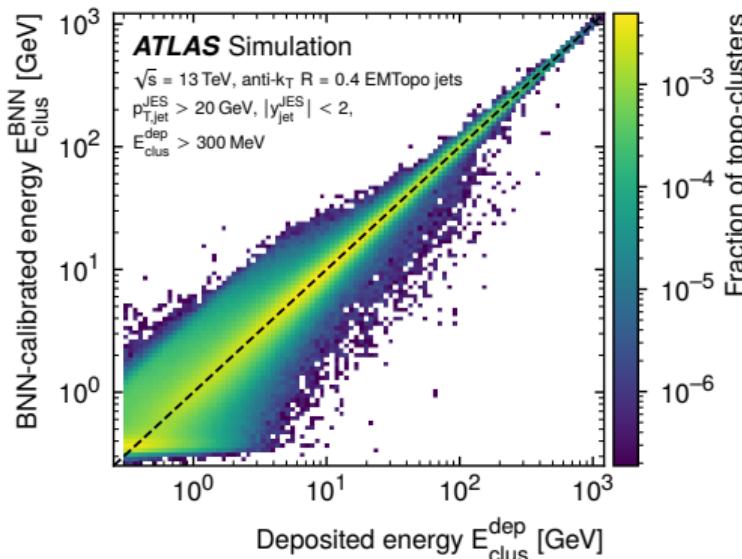
# BNN — response prediction and energy calibration



$$\mathcal{R}_{\text{clus}}^{\text{BNN}} \xrightarrow{\text{train}} \mathcal{R}_{\text{clus}}^{\text{EM}} = \frac{E_{\text{clus}}^{\text{EM}}}{E_{\text{clus}}^{\text{dep}}}$$



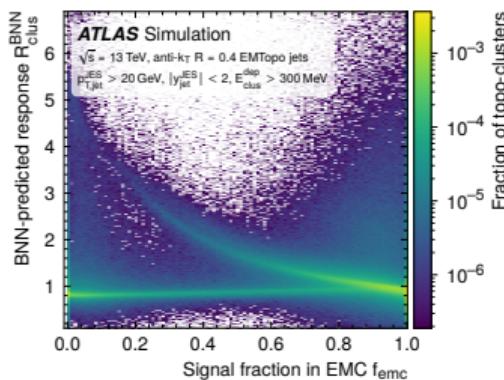
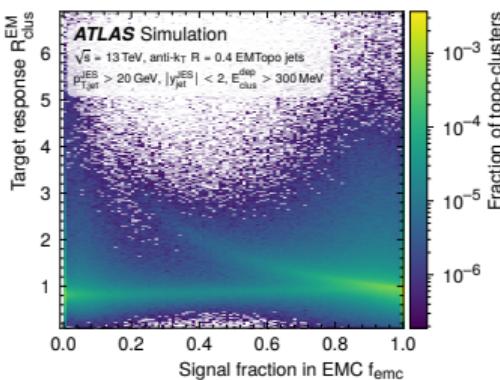
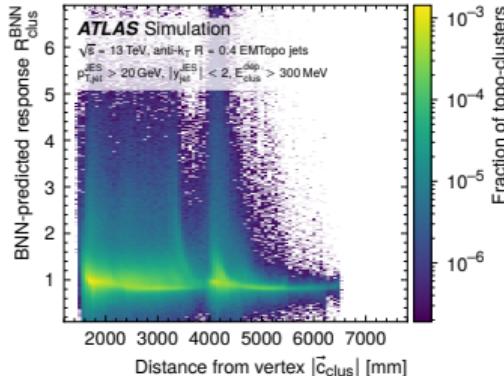
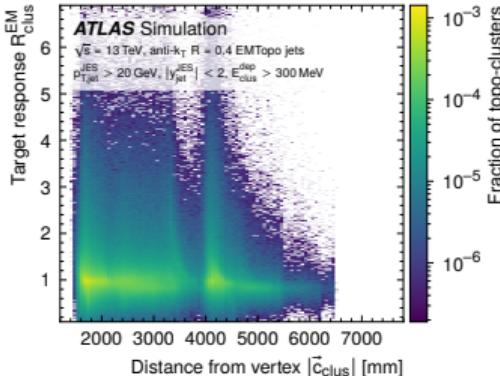
$$E_{\text{clus}}^{\text{BNN}} = \frac{E_{\text{clus}}^{\text{EM}}}{\mathcal{R}_{\text{clus}}^{\text{BNN}}} \longrightarrow E_{\text{clus}}^{\text{dep}}$$



agreement of BNN prediction and regression target:  
correlation curves for predicted response and calibrated energy look **promising**



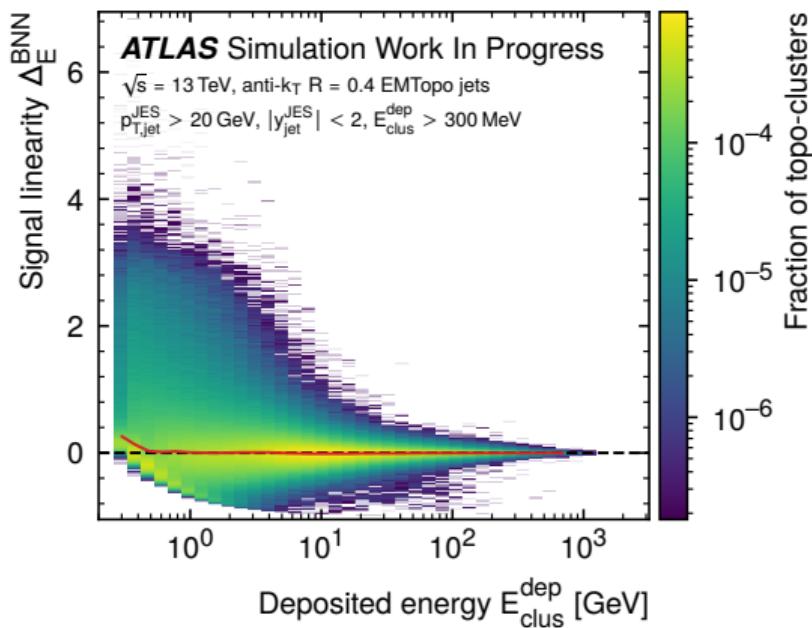
# BNN — response vs features



very complex target distributions (i.e. due to calorimeter geometry) are reproduced well by the BNN

example: central-barrel to endcap-calorimeter transition in  $|\vec{c}_{\text{clus}}|$

# BNN — signal linearity

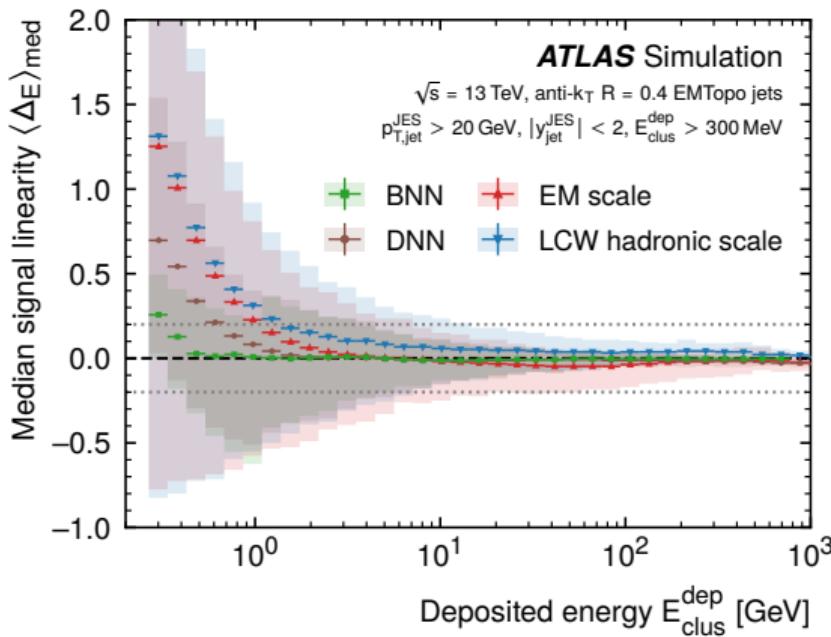


**Signal linearity:** ratio of calibrated over deposited energy

$$\Delta_E^\kappa = \frac{E_{\text{clus}}^\kappa}{E_{\text{clus}}^{\text{dep}}} - 1 \quad \text{with} \quad E_{\text{clus}}^\kappa = \frac{E_{\text{clus}}^{\text{EM}}}{\mathcal{R}_{\text{clus}}^\kappa}$$

- scales  $\kappa \in \{\text{EM}, \text{LCW}, \text{DNN}, \text{BNN}\}$
- should peak at zero after successful calibration
- evaluated as function of features and deposited energy
- **BNN better over whole energy range, most significant at low energies**

# BNN — median signal linearity

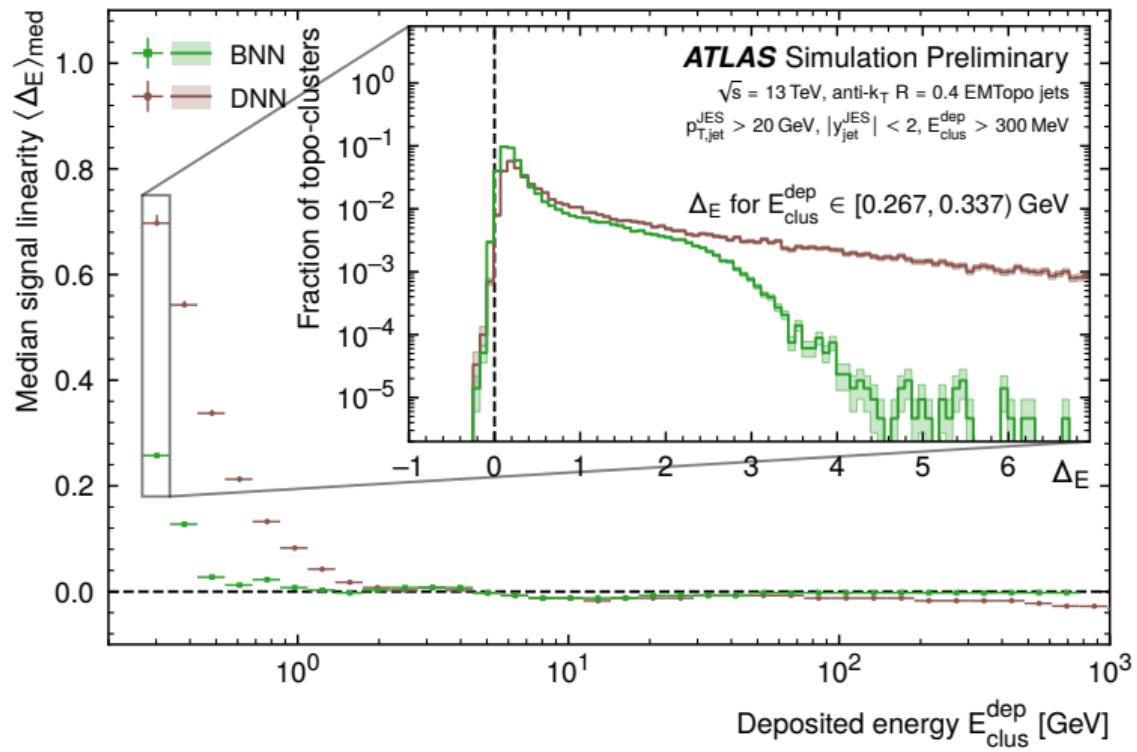


**Signal linearity:** ratio of calibrated over deposited energy

$$\Delta_E^\kappa = \frac{E_{\text{clus}}^\kappa}{E_{\text{clus}}^{\text{dep}}} - 1 \quad \text{with} \quad E_{\text{clus}}^\kappa = \frac{E_{\text{clus}}^{\text{EM}}}{\mathcal{R}_{\text{clus}}^\kappa}$$

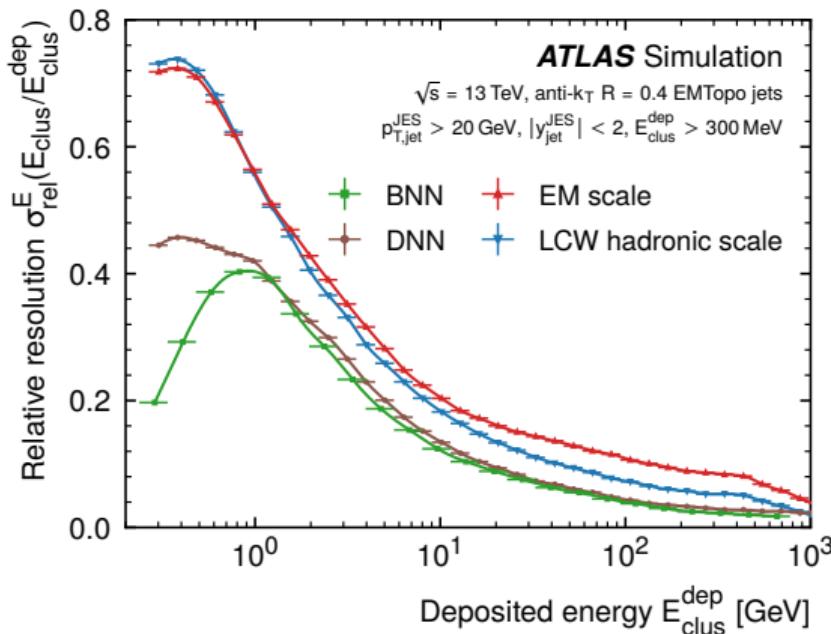
- scales  $\kappa \in \{\text{EM}, \text{LCW}, \text{DNN}, \text{BNN}\}$
- should peak at zero after successful calibration
- evaluated as function of features and deposited energy
- **BNN better over whole energy range, most significant at low energies**

# BNN — bin-wise signal linearity



BNN-derived  
calibration shows  
significantly suppressed  
tails compared to DNN

# BNN — relative local energy resolution

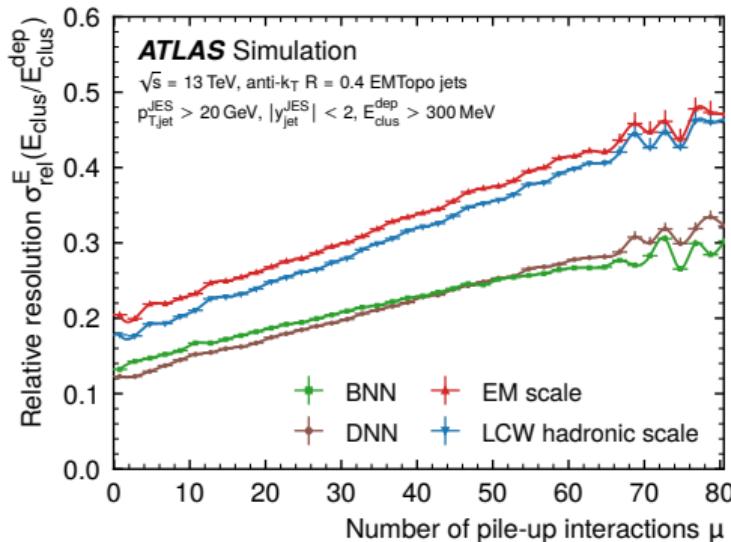
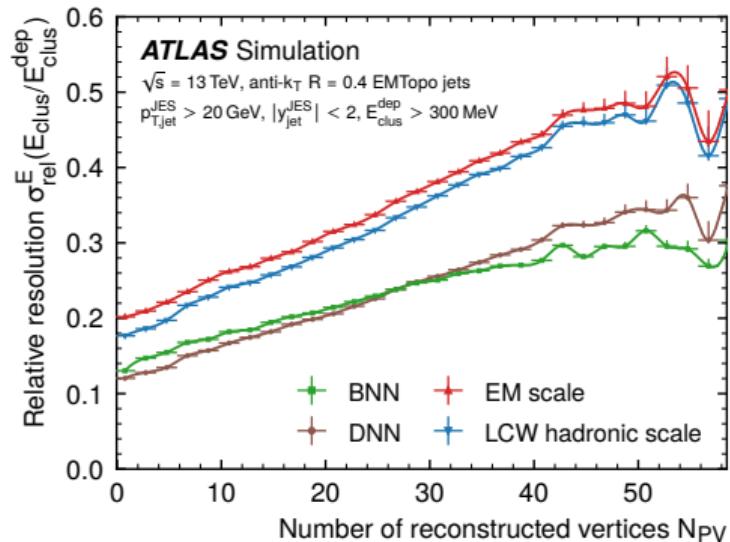


Relative local energy resolution:

$$\sigma_{\text{rel}}^E = \frac{Q_{f=68\%}^w}{2\langle E_{\text{clus}}^\kappa / E_{\text{clus}}^{\text{dep}} \rangle_{\text{med}}}$$

- $Q_{f=68\%}^w \equiv 68\%$  inter-quantile range
- BNN better over whole energy range, **spectacular at low energies**
- BNN learns signal-source transition from inelastic hadronic interactions to ionisation-dominated signals

# BNN — relative local energy resolution

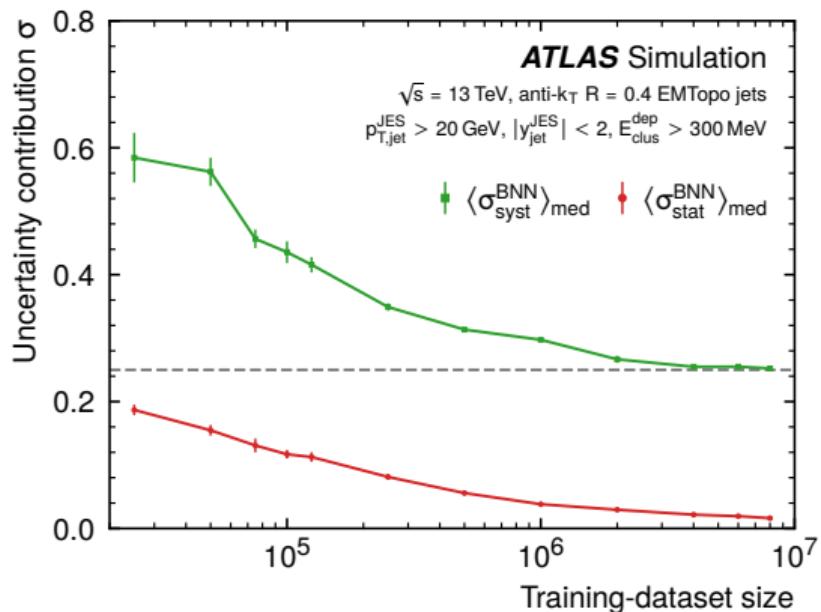


relative local energy resolution vs in-time and out-of-time pile-up activity  
→ BNN shows cluster-by-cluster pile-up mitigation

## BNN-learned uncertainties

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# BNN — systematic and statistical uncertainties



Learned  $\sigma_{\text{tot}}$  with two origins:

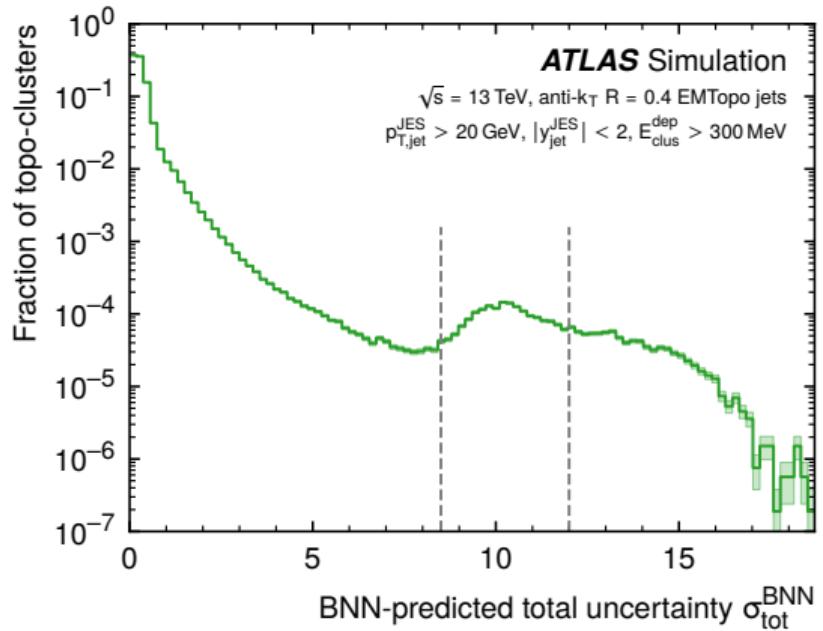
[arXiv:1904.10004, arXiv:2003.11099, arXiv:2206.14831]

- **statistics  $\sigma_{\text{stat}}$**   $\rightarrow p(\theta | D_{\text{train}})$   
limited training statistics,  
vanishing for good training statistics
- **systematics  $\sigma_{\text{syst}}$**   $\rightarrow p(\mathcal{R}_{\text{clus}}^{\text{EM}} | \theta, \mathcal{X}_{\text{clus}})$   
stochastic training data (pile-up),  
limited network expressivity,  
bad hyper-parameters,  
plateau for good training statistics

For well-trained LHC models:

$$\sigma_{\text{tot}} \equiv \sqrt{\sigma_{\text{syst}}^2 + \sigma_{\text{stat}}^2} \approx \sigma_{\text{syst}} \gg \sigma_{\text{stat}}$$

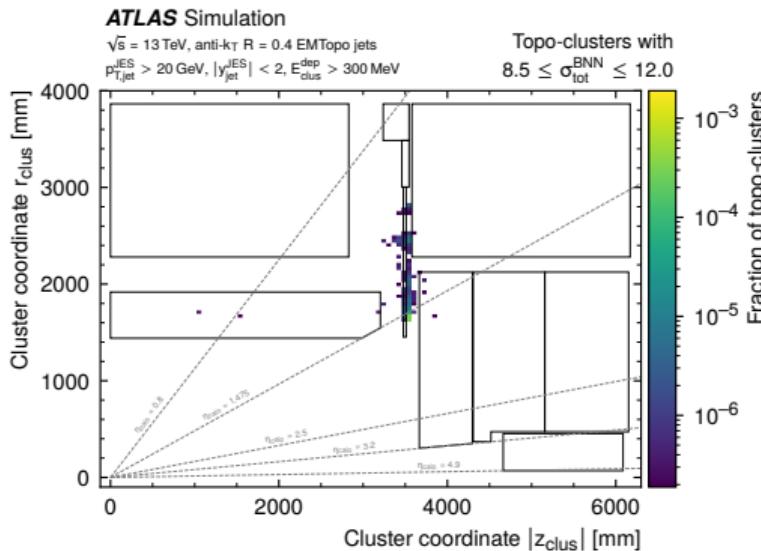
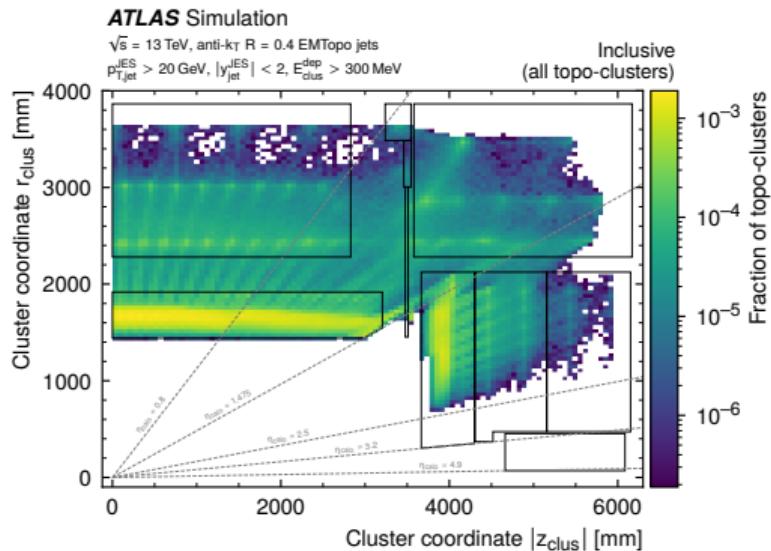
# BNN — uncertainties as explainable AI



Use BNN uncertainty to **understand** data

- uncertainty spectrum shows distinctive secondary maximum
- what feature leads the BNN to flag these topo-clusters with **large learned uncertainties?**
- analyze anomalous clusters in terms of **detector geometry**

# BNN — uncertainties as explainable AI

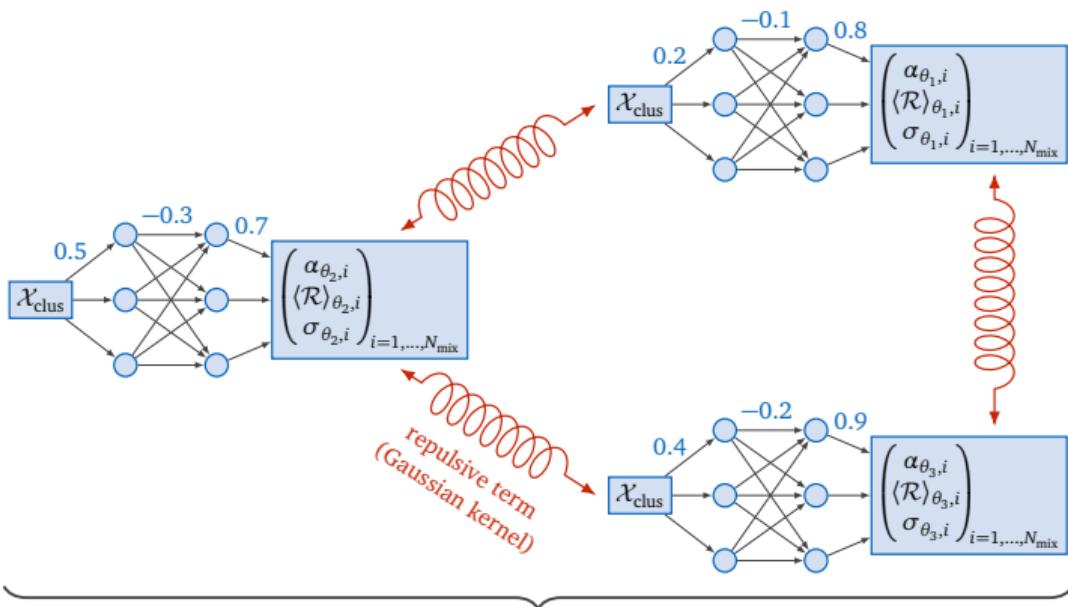


large uncertainties from tile-gap scintillator region:  
not a regular calorimeter → feature quality in this region is insufficient

## Repulsive ensembles

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# RE — repulsive ensemble



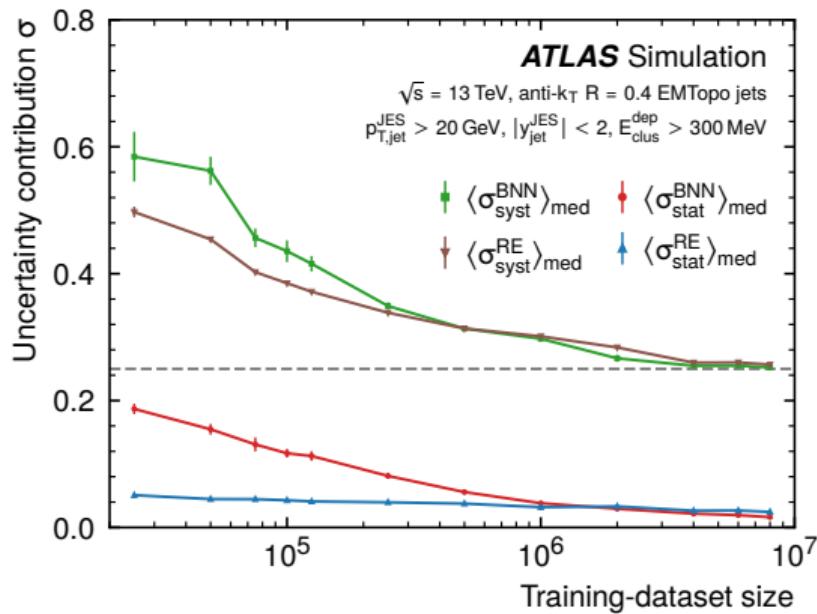
**training:** repulsive term connecting function space of all simultaneously trained networks forces ensemble to spread out and cover loss around actual minimum

Alternative way  
for uncertainty estimation

[arXiv:2106.11642, arXiv:2211.01421, arXiv:240313899]

- regular ensembles do not sample from weight posterior
- introduce repulsive force between ensemble members during optimization such that  $\theta \sim p(\theta | D_{\text{train}})$
- **repulsive term** ensures that uncertainty covers probability distribution over space of network functions

# RE — systematic and statistical uncertainties

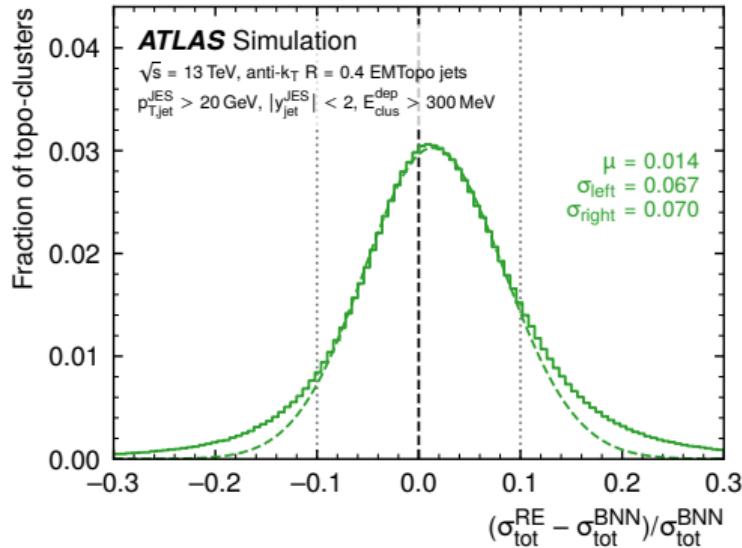
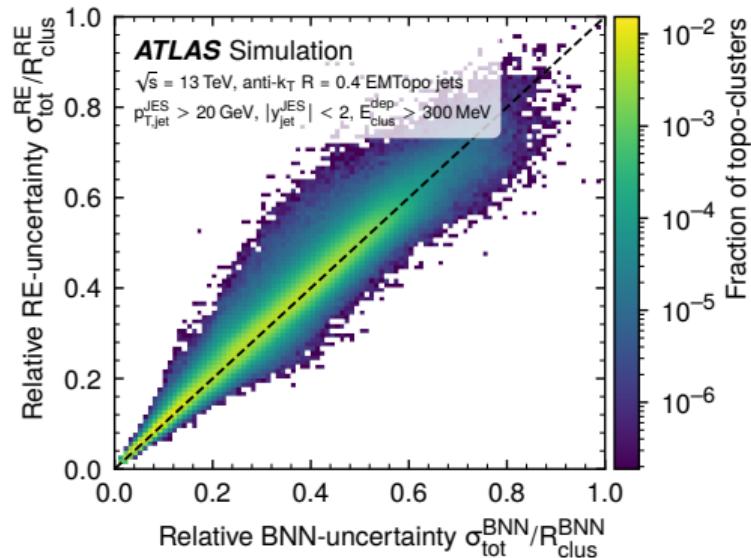


Repulsive force ensures that uncertainty covers probability distribution over space of network functions

[arXiv:2106.11642, arXiv:2211.01421, arXiv:2403.13899]

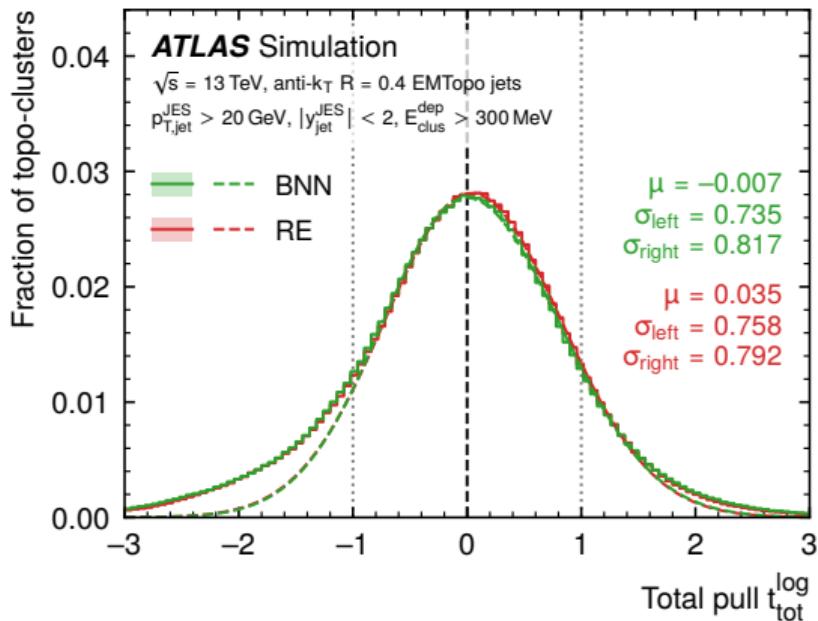
- gives two uncertainties
- systematics  $\sigma_{\text{syst}}$   
plateau for good training statistics,  
part of likelihood (same as for BNN)
- statistics  $\sigma_{\text{stat}}$   
vanishing for good training statistics  
(but with flatter slope)

# BNN and RE — consistency check



10% agreement between uncertainty estimates  
and both uncertainty predictions track each other well

# BNN and RE — uncertainties vs data spread



**Pull:** central prediction and uncertainty  
Does uncertainty cover data spread?

$$t_{\text{tot}}^{\kappa}(x) = \frac{\mathcal{R}_{\text{clus}}^{\kappa}(x) - \mathcal{R}_{\text{clus}}^{\text{EM}}(x)}{\sigma_{\text{tot}}^{\kappa}(x)}$$

- evaluated in  $\log_{10} \mathcal{R}_{\text{clus}}$  space
- stochastic data defining shape, Gaussian with order-one width
- BNN and RE errors consistent
- per-cluster error **conservative**  
(reliable upper limit for deviation)

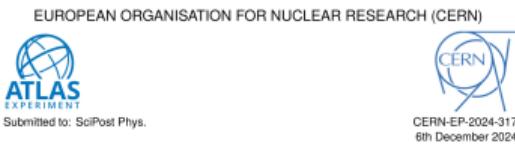
## **Summary and outlook**

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# Summary and outlook

arXiv:2412.04370v1 [hep-ex] 5 Dec 2024



## Precision calibration of calorimeter signals in the ATLAS experiment using an uncertainty-aware neural network

The ATLAS Collaboration<sup>1</sup>

The ATLAS experiment at the Large Hadron Collider explores the use of modern neural networks for a multi-dimensional calibration of its calorimeter signal defined by clusters of topologically connected cells (topo-clusters). The Bayesian neural network (BNN) approach not only yields a continuous and smooth calibration function that improves performance relative to the standard calibration but also provides uncertainties on the calibrated energies for each topo-cluster. The results obtained by using a trained BNN are compared to the standard local hadronic calibration and to a calibration provided by training a deep neural network. The uncertainties predicted by the BNN are interpreted in the context of a fractional contribution to the systematic uncertainties of the trained calibration. They are also compared to uncertainty predictions obtained from an alternative estimator employing repulsive ensembles.

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<sup>1</sup> The full author list can be found at:

<https://atlas.web.cern.ch/Atlas/PUBNOTES/ATL-PHYS-PUB-2024-XXX/authorlist.pdf>

## Modern uncertainty-aware BNNs for multi-dimensional calorimeter-signal calibration

- continuous and smooth calibration of topo-clusters
- improved performance relative to LCW and DNN
- meaningful per-cluster systematics
- BNNs and REs: learn reliable uncertainties

## Next steps:

- try deterministic NN with heteroscedastic loss instead of BNN (to speed up inference) [arXiv:2412.12069]
- further tune (B)NN performance
- full performance study within ATLAS (apply trained calibration to data)

Thanks for your attention — merry Christmas and happy holidays!



*Download full paper:*  
arXiv:2412.04370



INSPIRE HEP

# References and further reading

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## ML-based topo-cluster calibration and ML with uncertainties

- ❑ ATLAS Collaboration  
*The application of neural networks for the calibration of topological cell clusters in the ATLAS calorimeters*  
ATLAS PUB Note (2023)
- ❑ ATLAS Collaboration (including T. Heimel, P. Loch, T. Plehn and L. Vogel)  
*Precision calibration of calorimeter signals in the ATLAS experiment using an uncertainty-aware neural network*  
arXiv:2412.04370 [hep-ex]
- ❑ Y. Gal  
*Uncertainty in Deep Learning*  
Ph.D. Thesis, University of Cambridge (2016)
- ❑ T. Plehn, A. Butter, B. Dillon, T. Heimel, C. Krause and R. Winterhalder  
*Modern Machine Learning for LHC Physicists*  
arXiv:2211.01421 [hep-ph] (continuously updated on website)



# References and further reading

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## Bayesian neural networks (BNNs) and repulsive ensembles (REs)

- 📄 G. Kasieczka, M. Luchmann, F. Otterpohl and T. Plehn  
*Per-object systematics using deep-learned calibration*  
SciPost Phys. 9, 089 (2020), arXiv:2003.11099 [hep-ph]
- 📄 H. Bahl, N. Elmer, L. Favaro, M. Haußmann, T. Plehn and R. Winterhalder  
*Accurate Surrogate Amplitudes with Calibrated Uncertainties*  
arXiv:2412.12069 [hep-ph]
- 📄 F. D'Angelo and V. Fortuin  
*Repulsive Deep Ensembles are Bayesian*  
arXiv:2106.11642 [cs.LG]
- 📄 L. Röver, B. M. Schäfer and T. Plehn  
*PINNferring the Hubble Function with Uncertainties*  
arXiv:2403.13899 [astro-ph.CO]

**Backup slides...**

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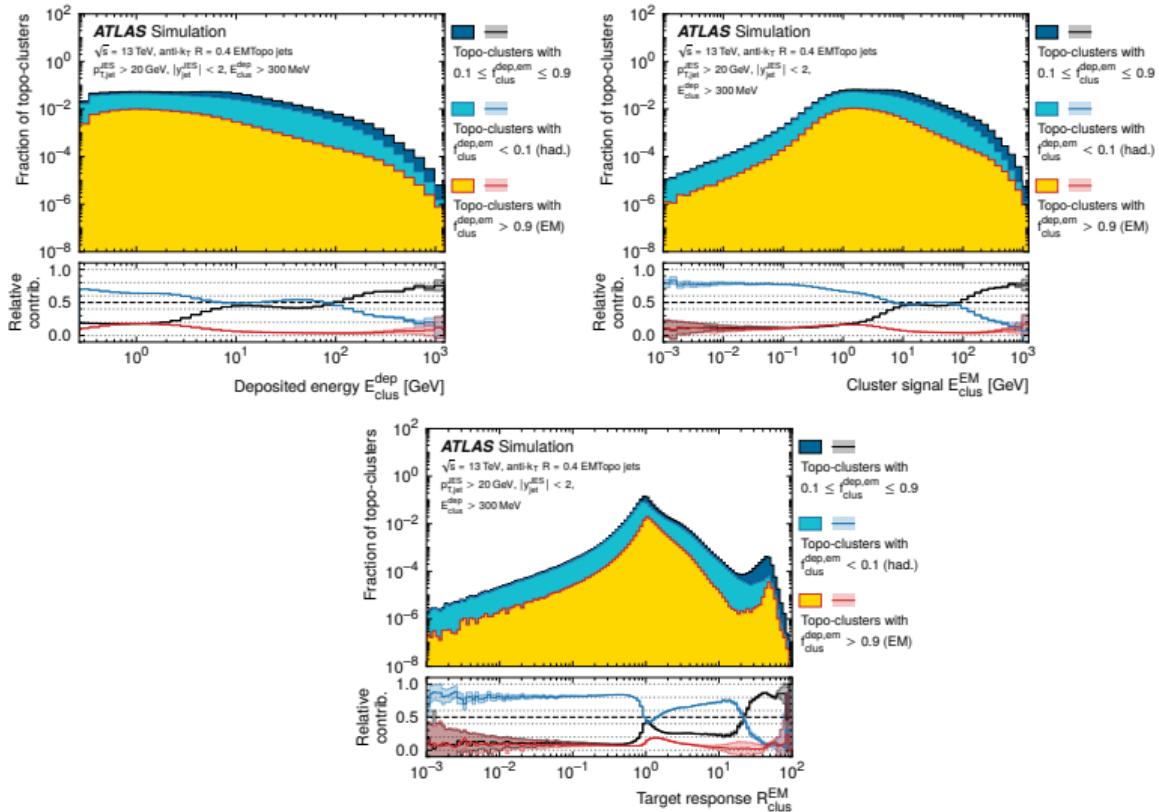
# Dataset — topo-cluster features

**Table 1:** The dataset consists of topo-clusters reconstructed in MC simulations of full proton-proton collision events at  $\sqrt{s} = 13 \text{ TeV}$  (LHC Run 2) with multi-jet final states

| category                     | symbol                                                                                                                                            | description / comment                                                                                                                                      |
|------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------|
| kinematics                   | $E_{\text{clus}}^{\text{EM}}, y_{\text{clus}}^{\text{EM}}$                                                                                        | cluster signal and rapidity at the EM energy scale                                                                                                         |
| signal strength              | $\zeta_{\text{clus}}^{\text{EM}}$                                                                                                                 | signal significance                                                                                                                                        |
| timing<br>time structure     | $t_{\text{clus}}$<br>$\text{Var}_{\text{clus}}(t_{\text{cell}})$                                                                                  | signal timing<br>variance of the cell-time distribution in the cluster                                                                                     |
| shower depth                 | $\lambda_{\text{clus}}$<br>$ \vec{c}_{\text{clus}} $                                                                                              | distance of the CoG from the calorimeter front face<br>distance of the CoG from the nominal vertex                                                         |
| shower shape,<br>compactness | $f_{\text{emc}}$<br>$\langle \rho_{\text{cell}} \rangle, p_{\text{T}} D$<br>$\langle m_{\text{long}}^2 \rangle, \langle m_{\text{lat}}^2 \rangle$ | energy fraction in the EM calorimeter (EMC)<br>cluster signal density and signal compactness<br>energy dispersion along/perpendicular to main cluster axis |
| topology                     | $f_{\text{iso}}$                                                                                                                                  | cluster isolation measure                                                                                                                                  |
| pile-up                      | $N_{\text{PV}}$<br>$\mu$                                                                                                                          | number of reconstructed primary vertices<br>number of pile-up interactions per bunch crossing                                                              |



# Dataset — energy and response distributions



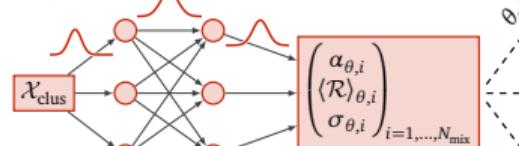
# BNN — network architecture



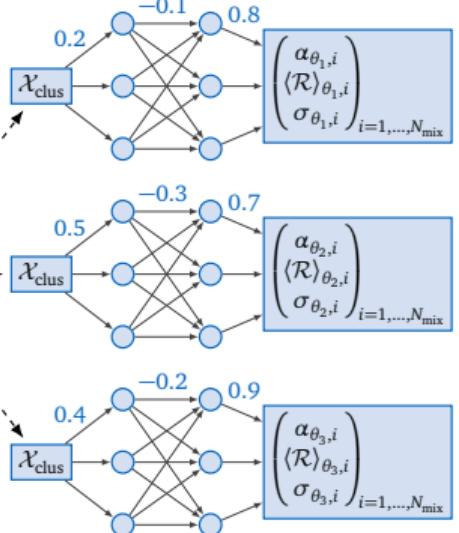
## Bayesian neural network (BNN)

weight distributions  $q(\theta)$

(Gaussian)



$\theta_s \sim q(\theta)$



Central-value prediction (maximum likelihood) and uncertainties for a Gaussian mixture model:

$$\mathcal{R}_{\text{clus}}^{\text{BNN}}(\mathcal{X}_{\text{clus}}) = \arg \max_{\mathcal{R}} \frac{1}{N} \sum_{s=1}^N p(\mathcal{R} | \theta_s, \mathcal{X}_{\text{clus}})$$

$$\sigma_{\text{syst}}^2(\mathcal{X}_{\text{clus}}) = \frac{1}{N} \sum_{s=1}^N \left[ \sum_{i=1}^{N_{\text{mix}}} \alpha_{\theta_s,i} (\sigma_{\theta_s,i}^2 + \langle \mathcal{R} \rangle_{\theta_s,i}^2) - \langle \mathcal{R} \rangle_{\theta_s}^2 \right]$$

$$\sigma_{\text{stat}}^2(\mathcal{X}_{\text{clus}}) = \text{Var}(\langle \mathcal{R} \rangle_{\theta_s}) = \frac{1}{N} \sum_{s=1}^N [\langle \mathcal{R} \rangle - \langle \mathcal{R} \rangle_{\theta_s}]^2$$

with

$$\langle \mathcal{R} \rangle_{\theta_s} = \sum_{i=1}^{N_{\text{mix}}} \alpha_{\theta_s,i} \langle \mathcal{R} \rangle_{\theta_s,i} \quad \text{and} \quad \langle \mathcal{R} \rangle = \frac{1}{N} \sum_{s=1}^N \langle \mathcal{R} \rangle_{\theta_s}$$

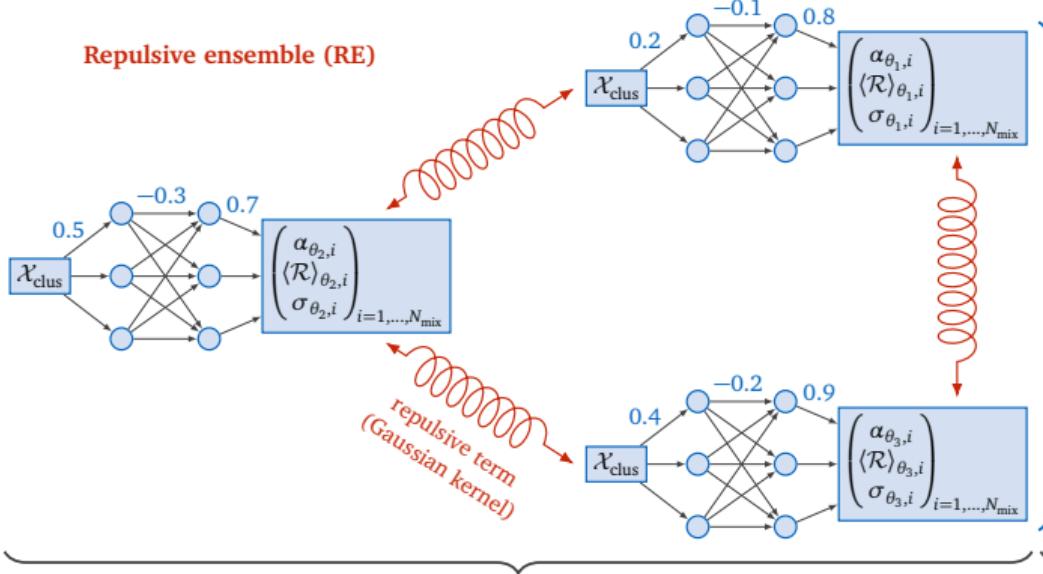
**Training:** Weights linking the nodes of adjacent layers are described by weight distributions  $q(\theta)$

**Inference:** Learned weight distributions  $q(\theta)$  are sampled  $N$  times to generate a set of network parameters  $\theta_s$  and thus an ensemble of networks



# RE — network architecture

Repulsive ensemble (RE)



**Training:** Repulsive term connecting the function space of all  $N$  simultaneously trained networks forces the ensemble to spread out and **cover the loss around the actual minimum**

Central-value prediction (maximum likelihood) and **uncertainties** for a Gaussian mixture model:

$$\mathcal{R}_{\text{clus}}^{\text{RE}}(\mathcal{X}_{\text{clus}}) = \arg \max_{\mathcal{R}} \frac{1}{N} \sum_{s=1}^N p(\mathcal{R}|\theta_s, \mathcal{X}_{\text{clus}})$$

$$\sigma_{\text{syst}}^2(\mathcal{X}_{\text{clus}}) = \frac{1}{N} \sum_{s=1}^N \left[ \sum_{i=1}^{N_{\text{mix}}} \alpha_{\theta_s,i} (\sigma_{\theta_s,i}^2 + \langle \mathcal{R} \rangle_{\theta_s,i}^2) - \langle \mathcal{R} \rangle_{\theta_s}^2 \right]$$

$$\sigma_{\text{stat}}^2(\mathcal{X}_{\text{clus}}) = \text{Var}(\langle \mathcal{R} \rangle_{\theta_s}) = \frac{1}{N} \sum_{s=1}^N [\langle \mathcal{R} \rangle - \langle \mathcal{R} \rangle_{\theta_s}]^2$$

with

$$\langle \mathcal{R} \rangle_{\theta_s} = \sum_{i=1}^{N_{\text{mix}}} \alpha_{\theta_s,i} \langle \mathcal{R} \rangle_{\theta_s,i} \quad \text{and} \quad \langle \mathcal{R} \rangle = \frac{1}{N} \sum_{s=1}^N \langle \mathcal{R} \rangle_{\theta_s}$$

**Inference:** Same formulas as for the BNN, using the  $N$  simultaneously trained ensemble members

# BNN — network setup and hyper-parameters

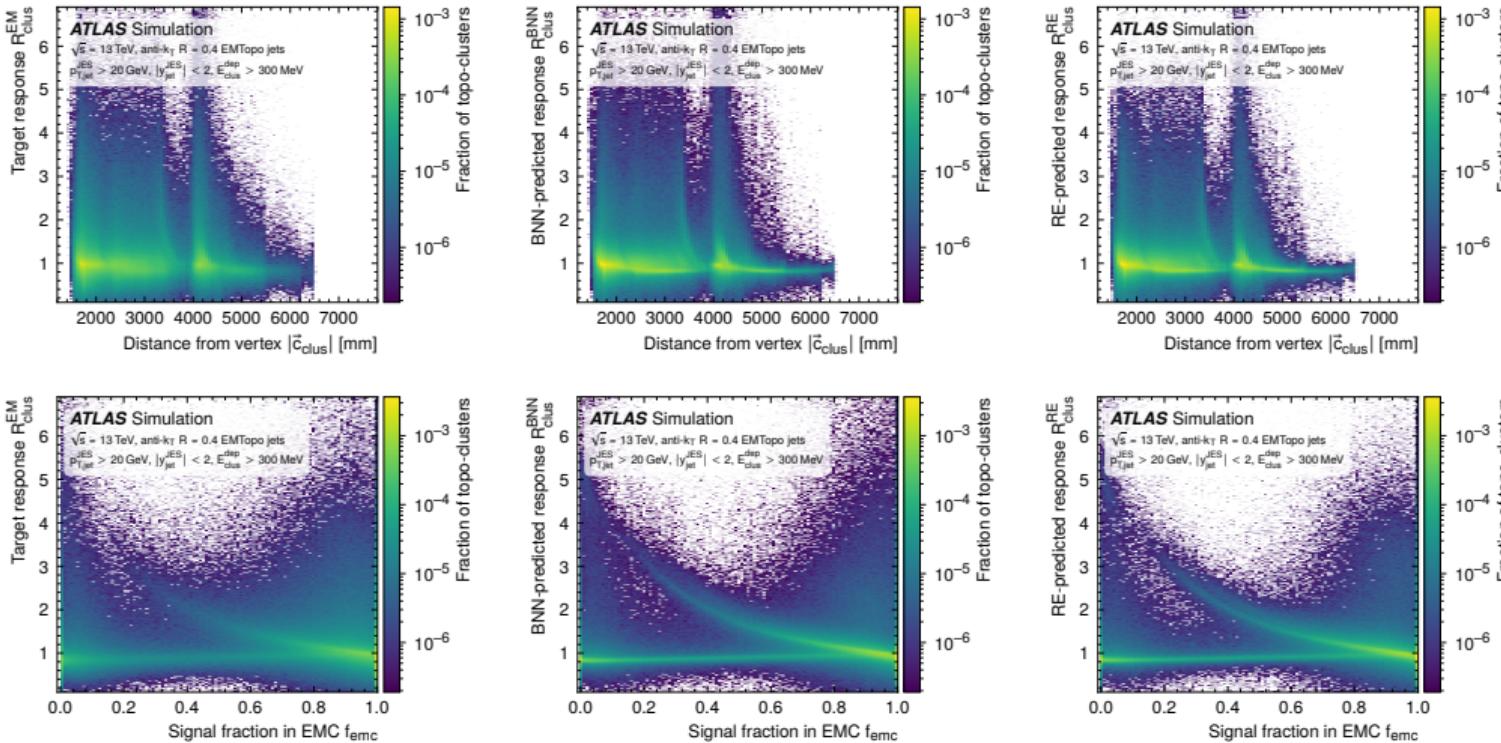
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**Table 2:** BNN and RE setup for the three-mode Gaussian mixture likelihood

| hyper-parameter                                 | network architecture and setup            |
|-------------------------------------------------|-------------------------------------------|
| likelihood model                                | Gaussian mixture model (GMM)              |
| number of mixture modes $N_{\text{mix}}$        | 3 (i.e. 9 output nodes)                   |
| number of hidden layers                         | 4 (with 64 nodes each)                    |
| nodes per layer (input, hidden, and output)     | {15, 64, 64, 64, 9}                       |
| inner activation functions                      | rectified linear unit (ReLU)              |
| central-value prediction                        | maximum of the likelihood (“mode”)        |
| optimizer and learning rate (LR)                | ADAM with $\text{LR} = 10^{-4}$           |
| learning-rate scheduler                         | STEP LR, epochs {25, 100}, $\gamma = 0.1$ |
| number of training epochs                       | 150                                       |
| training (and inference) batch size             | 4096 (512)                                |
| dataset sizes for training, validation, testing | {8.7M, 500k, 5.3M}                        |
| Monte-Carlo samples at inference $N$            | 50                                        |

# EM, BNN and RE — response vs features



# DNN, BNN and RE — signal linearity vs features

