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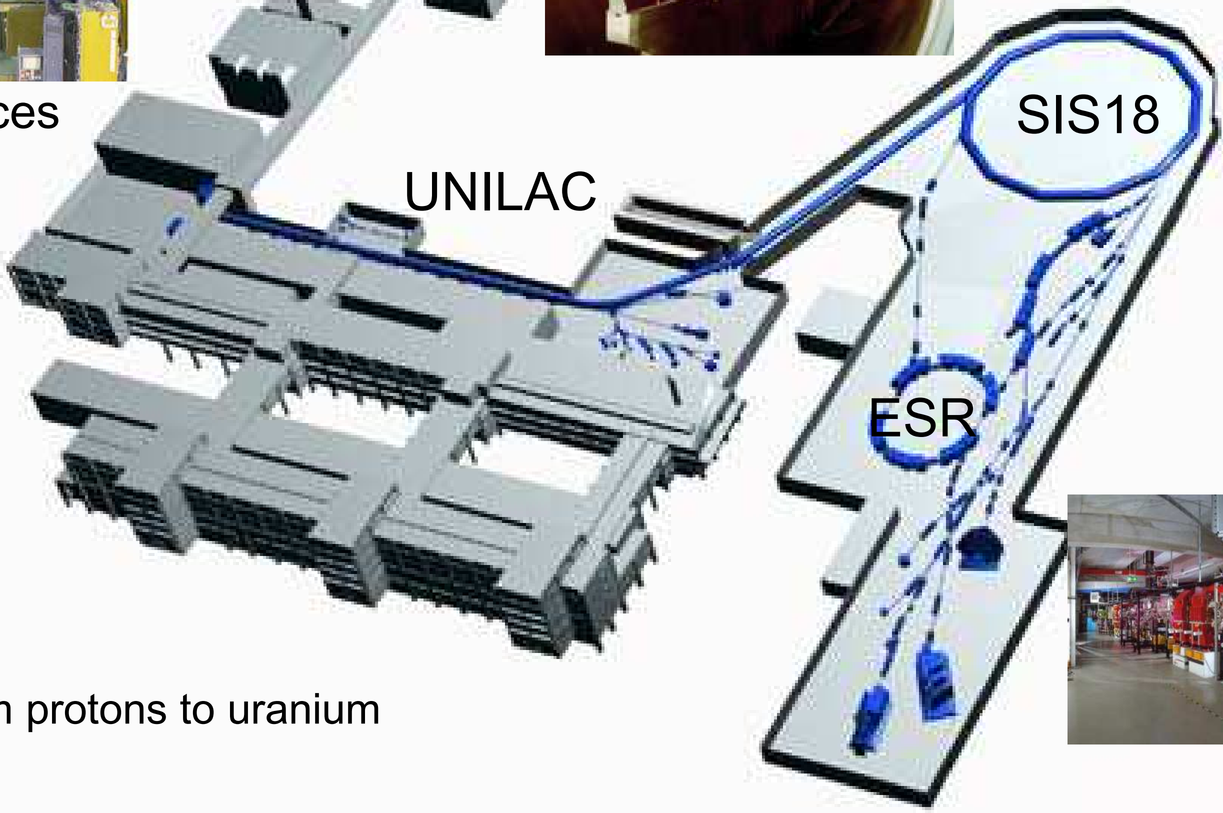
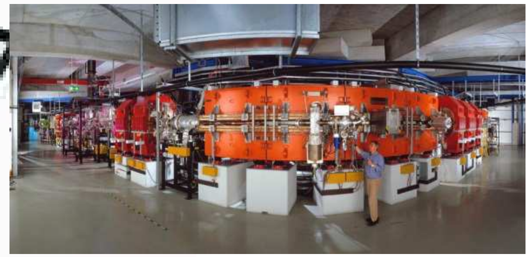
- Provision of charged particles (ions)
- Accelerating cavities
- High voltage sparking
- Radio-frequency power sources
- Focusing and stability of beams
  - longitudinal
  - transverse
- Radio frequency quadrupole
- Beam self forces

GSI Accelerator Facilities

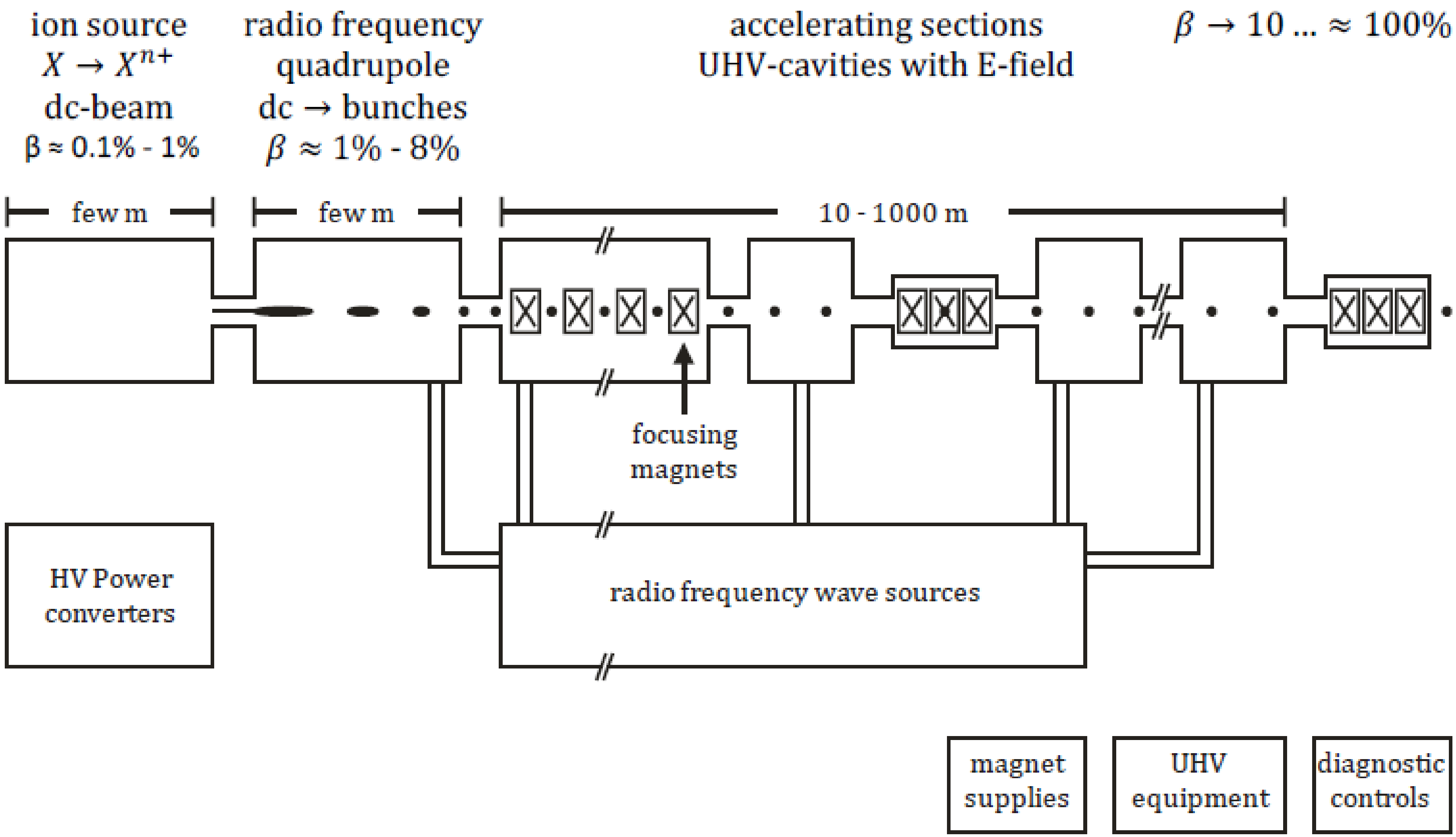
Ion Sources





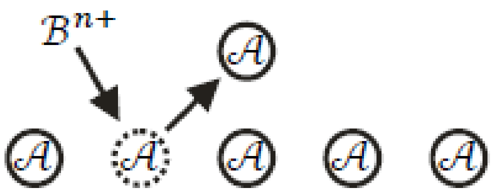
all ions from protons to uranium

Modern Linear Accelerator (schematic)



Ion Creation

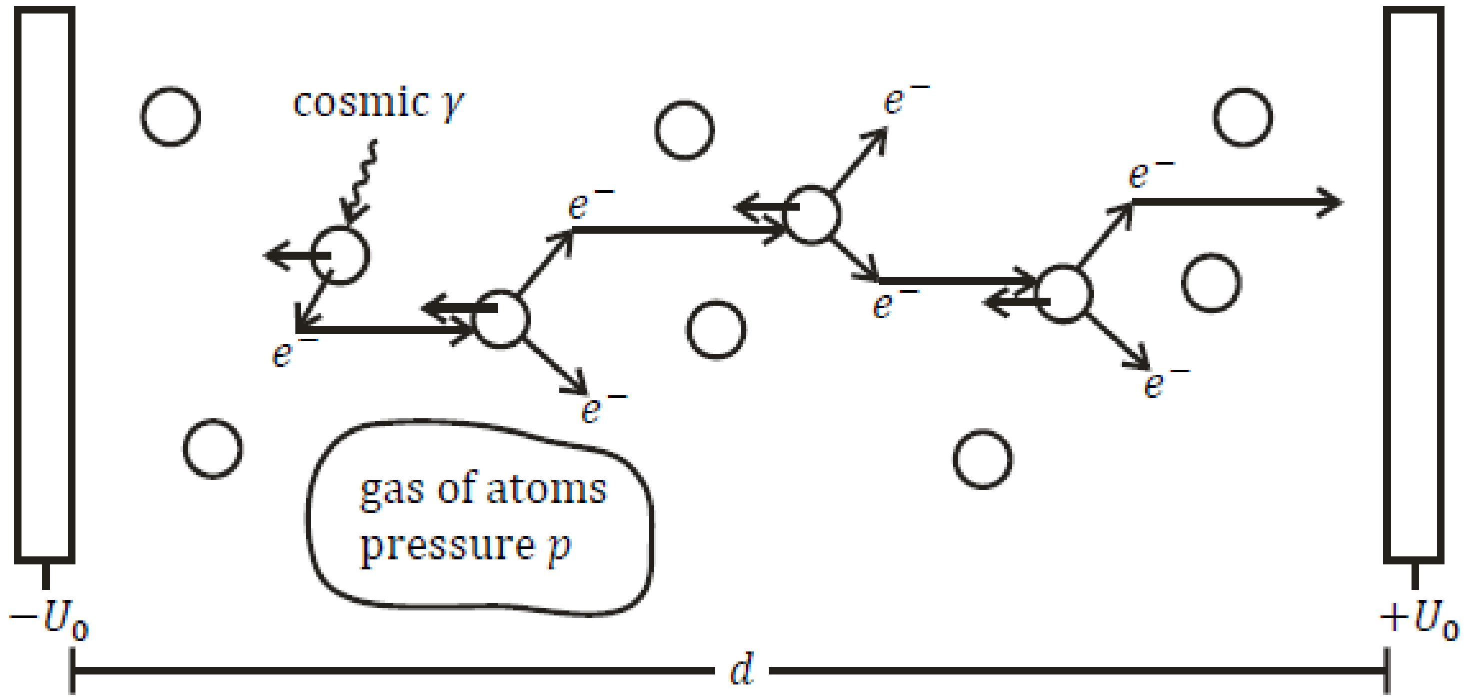
- supply neutral atoms $\mathcal{A}$ by:
  - injection of gas
  - vaporing
  - sputtering, i.e. bombard surface with other ion species $\mathcal{B}^{n+}$



- collide atoms with photons or electrons e^-

$$\begin{matrix} e^- & & & & \mathcal{A}^+ + e^- + e^- \text{ , most efficient} \\ \text{or} & \longrightarrow & \mathcal{A} & \longrightarrow & \\ \gamma & & & & \mathcal{A}^+ + e^- \text{ , creates free } e^- \end{matrix}$$

Electron & Ion Production by controlled Discharge

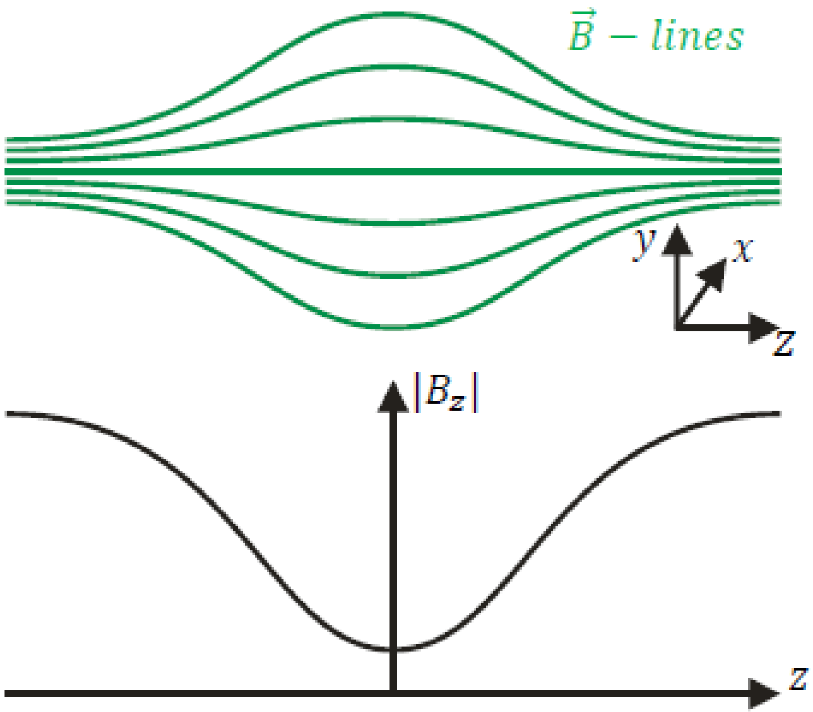


if p & d are properly chosen

→ contineous e^- & ion production

Confinement of Charged Particles

- without confinement ions & e^- move towards electrodes and get lost
- higher ion intensities require to confine ions inside "production" volume
- confinement by magnetic bottle:



$$\frac{{v_x}^2 + {v_y}^2}{|B_z|} = \text{const}$$

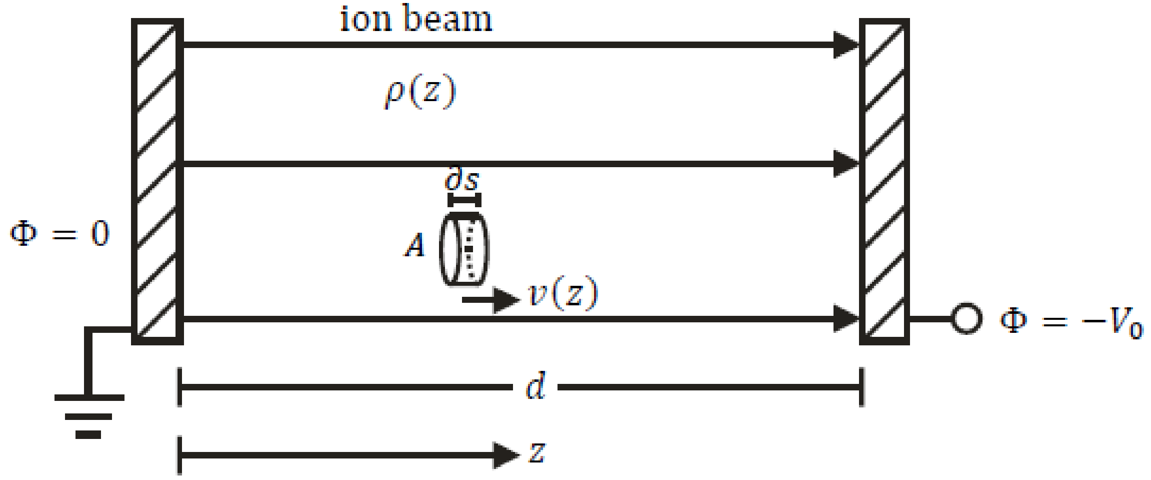
(see Jackson "class el. dyn.")

$${v_{tot}}^2 - {v_z}^2 = |B_z| \cdot \text{const}$$

$${v_z}^2 = \frac{2}{m} E_{kin} - |B_z| \cdot \text{const}$$

$\rightarrow |B_z| > B_c \rightarrow$ ion reflection along z-axis

Extraction of Charged Particles



$I(z) = \frac{\partial q}{\partial t} = \frac{\rho(z)\cdot A\cdot \partial s}{\partial t} = \text{const!}$

$\frac{I}{A} := J = \rho(z)\cdot v(z) = \rho\cdot\dot{z} = \text{const!}\ \ (*)$

$\Longrightarrow \frac{\partial J}{\partial z} = 0$ continuity equation (1)

$\vec{\nabla}\vec{E} = \frac{\rho}{\varepsilon_0}\ ,\ \vec{E} = -\vec{\nabla}\Phi$

$\Longrightarrow \Delta\Phi = \frac{\delta^2\Phi}{\delta z^2} = -\frac{\rho}{\varepsilon_0}$ Poisson equation (2)

$\frac{m}{2}\dot{z}^2 = -q\Phi(z)$ energy preservation (3)

$\Longrightarrow$ $J = \frac{4}{9}\varepsilon_0\sqrt{\frac{2q}{m}}\cdot\frac{V_0^{3/2}}{d^2}$ **Child-Langmuir Law**

- used simplification of planar plates (electrodes)
- real electrodes:
  - are curved
  - have beam holes (extraction electrode @ -V_0)

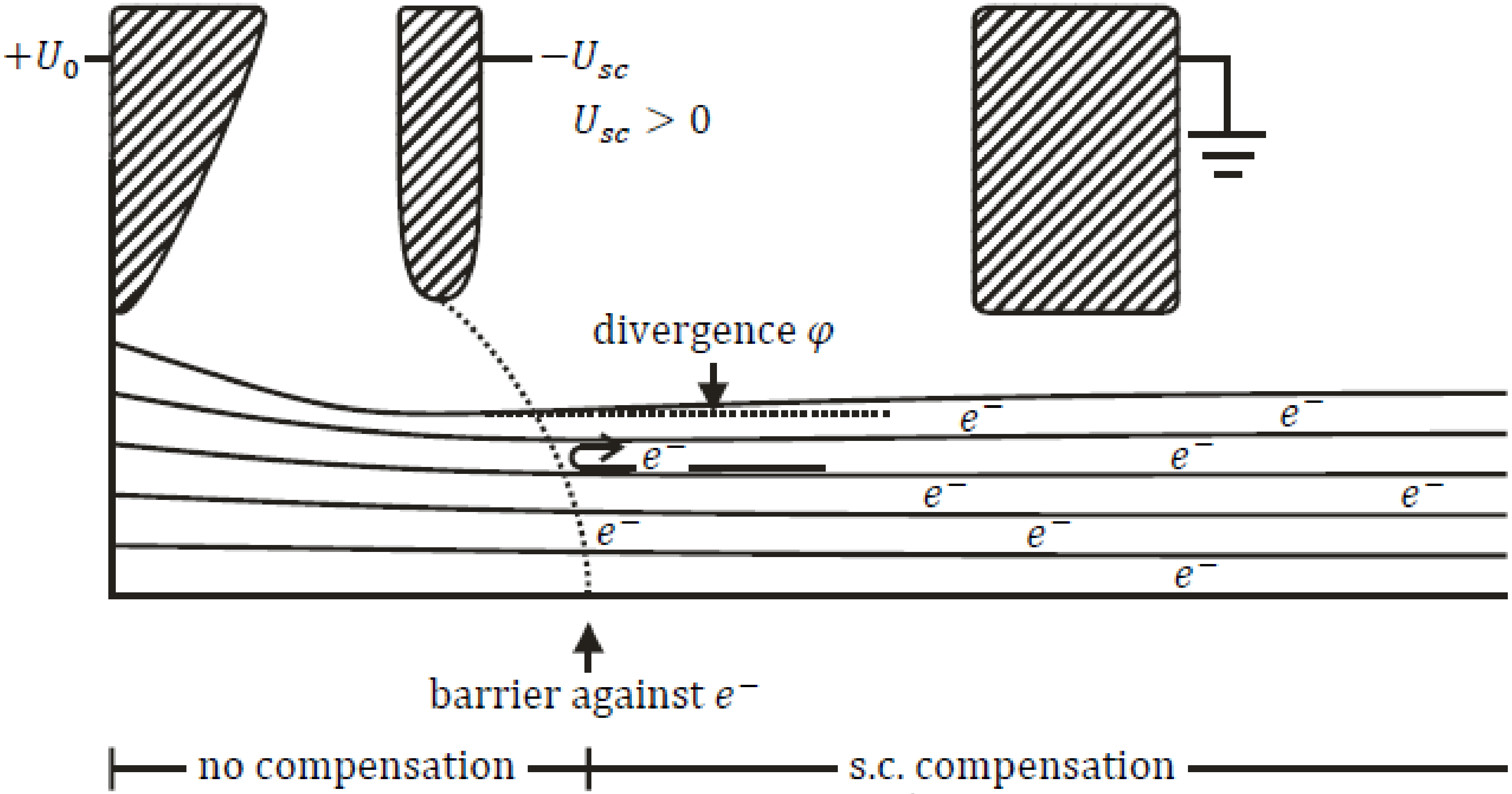
→ real currents may be lower (ions)

but scaling law $J \sim V_0^{3/2}\ d^{-2}$ holds

$I = A\cdot J := P\cdot\frac{A_{\text{cathode}}}{d^2}\cdot V^{3/2}$, P is "Perveance"

Triode Extraction:

insert a screening (sc) electrode that keeps e^- off from emitter



space charge

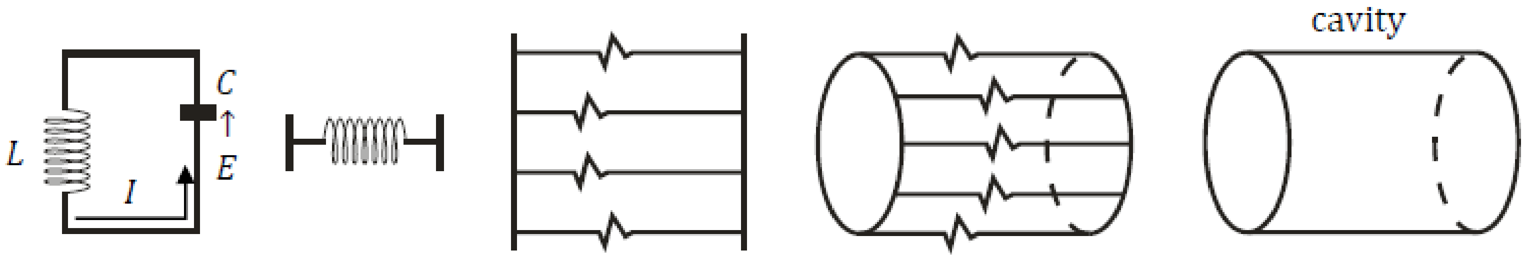
Accelerating Cavities

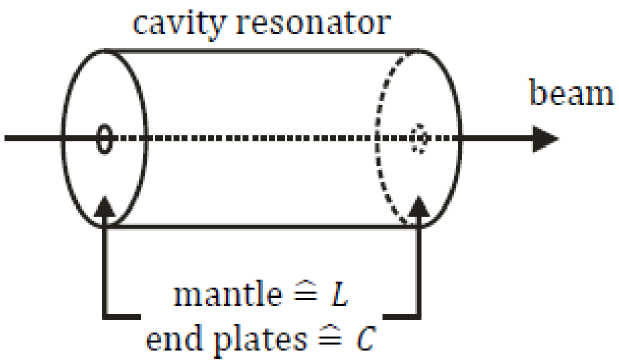
dc E-field strength $\leq 2\frac{kV}{m}$, stronger → arc discharge

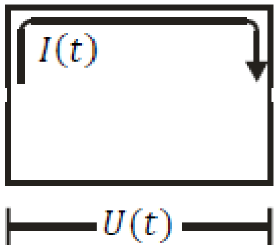
max. accelerating E-field strength $\approx \omega^n$

n: $n(\omega$, material, geometry, rf pulse shape)

n is under research!







cavity houses resonating em fields $E_0 \gg 2\frac{kV}{m}$ and can be evacuated !!!

field description → Maxwell's equations + boundaries

Maxwell's equations in vacuum:

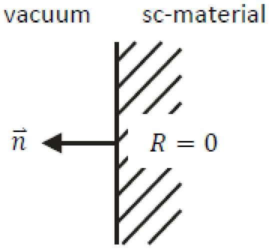
$$\vec{\nabla}\vec{E} = \frac{\rho}{\varepsilon_0}; \qquad \vec{\nabla}\times\vec{E} = -\dot{\vec{B}}$$

$$\vec{\nabla}\vec{B} = 0; \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dot{\vec{E}}$$

$$c^2 = \frac{1}{\mu_0\varepsilon_0}; \; \rho = 0; \; \vec{J} = 0$$

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\ddot{\vec{E}} = 0$$

boundaries:



$\vec{E}\times\vec{n} = 0, \quad E_{\parallel} = 0$

suppose $\vec{E}$

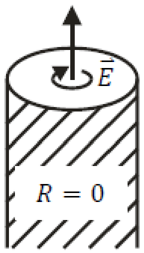
$\vec{E} \rightarrow \vec{J}$

$\rightarrow$ redistribution of charges

$\rightarrow$ $\vec{E}$ cancelled

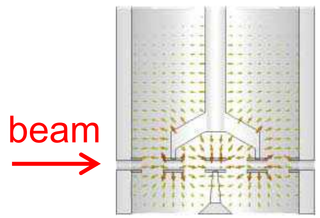
$\vec{B}\cdot\vec{n} = 0, \; B_{\perp} = 0$:

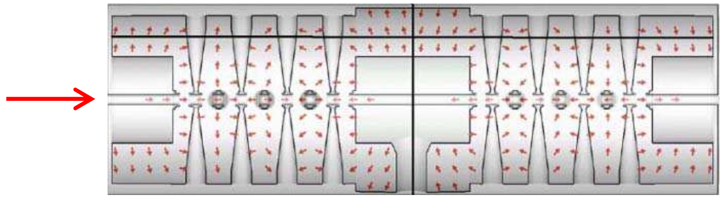
suppose $B_{\perp} \neq 0 \rightarrow$ at some $t < t_0$: $\qquad \dot{B}_{\perp} \neq 0, \; \dot{B}_{\perp} = -\vec{\nabla}\times\vec{E}$



$\vec{E}$ will cause $\vec{J}\sim\vec{E}$

$\vec{J}$ will cause $\vec{B}_{\perp}$ opposite to $\vec{B}_{\perp} \rightarrow \vec{B}_{\perp}$ cancelled

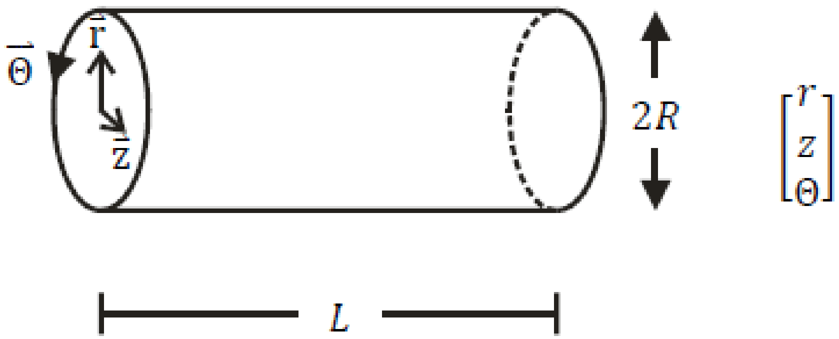




e.m. field configuration in cavity ruled by:

- wave equation from Maxwell
- boundaries imposed by cavity inner surface shape

Simplest Cavity (Pillbox)



<u>boundaries at material surface impose:</u>

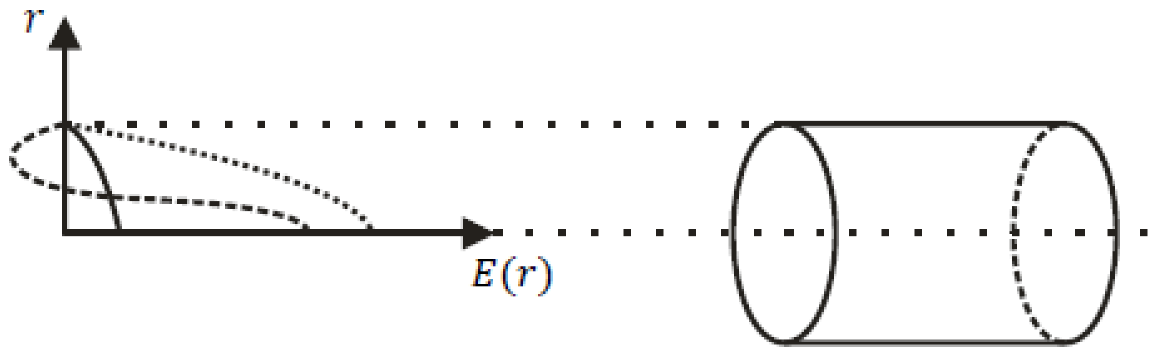
$B_r(r = R) = 0 \qquad B_z(z = 0, z = L) = 0$

$E_z(r = R) = E_\Theta(r = R) = 0$

$E_r(z = 0, z = L) = 0$

simplest Ansatz:

$$\vec{E} = \vec{e}_z \cdot E(r) \cdot e^{i\omega t}$$



Simplest Cavity (Pillbox)

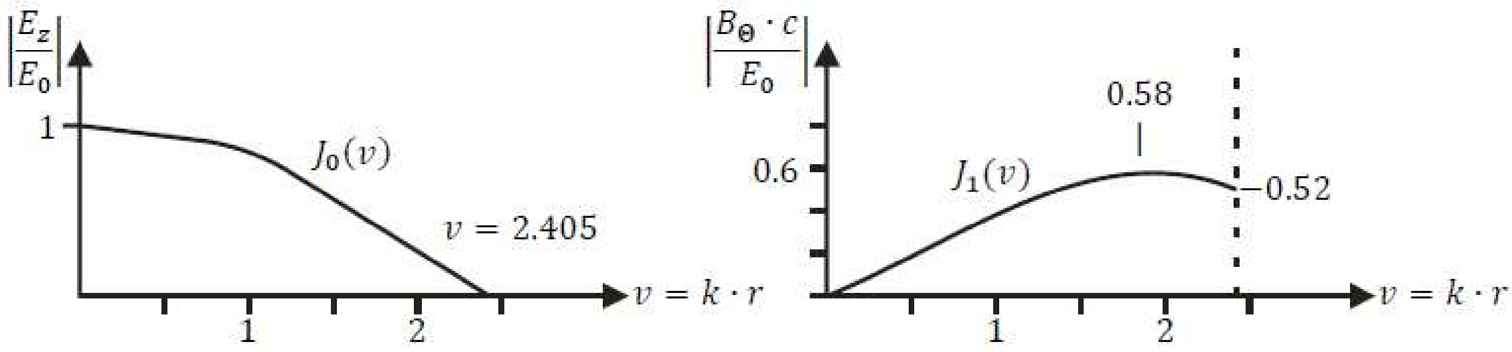
Wave equation + pillbox-geometry boundaries → Bessel diff equ. in r

$$\Longrightarrow \quad E(v) = E_0 \cdot J_0(v) \cdot e^{i\omega t} \quad = E_z = \vec{E}(t,r)$$

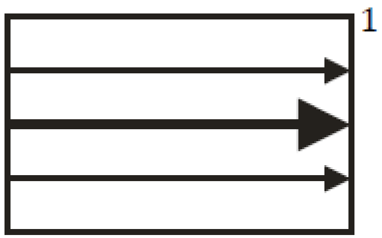
$$\vec{B}(v) = -\frac{E_0}{c} i \cdot J_0(v) \cdot e^{i\omega t} \quad = B_\Theta = \vec{B}(t,r), \qquad \text{phase shifted } i \mathrel{\widehat{=}} 90°$$

$$k^2 := \frac{\omega^2}{c^2}, \quad r = \frac{v}{k}$$

simplest case:



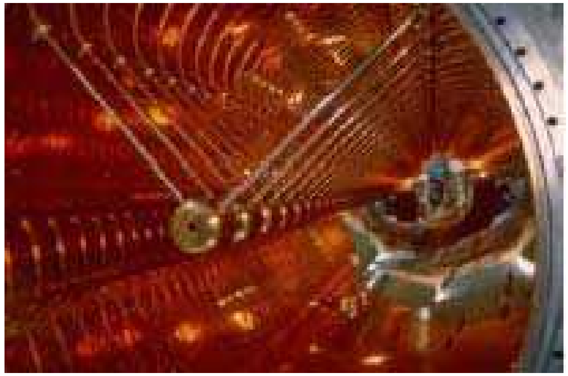
$\vec{E}$: $\vec{E} = E_z$, i.e.longitudinal:



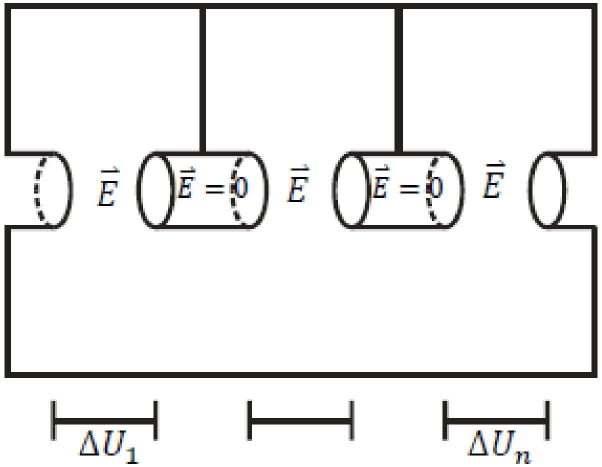
$$f(R) = 114.75\ MHz \cdot \frac{1}{R[m]}, \quad \frac{\partial f}{\partial L} = 0 \ !$$

- sign of $\vec{E}$ oscillates → how to accelerate?
- avoid "bad polarity"

long pill box not suitable for acceleration



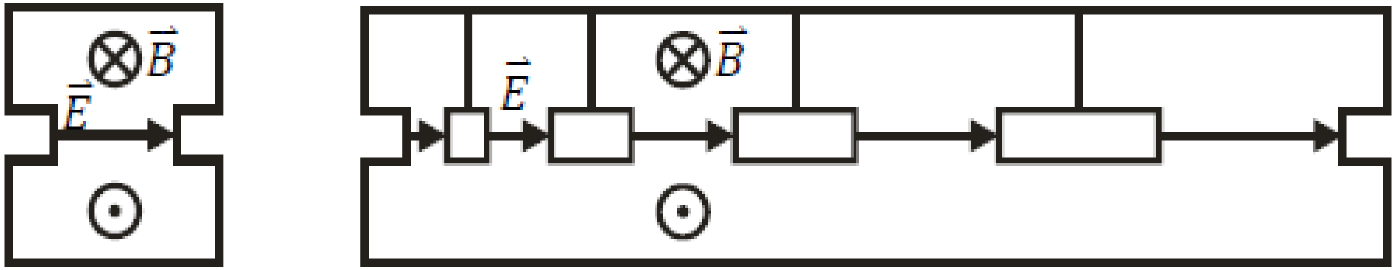
drift tubes:



$U_{tot} = \sum U_i$

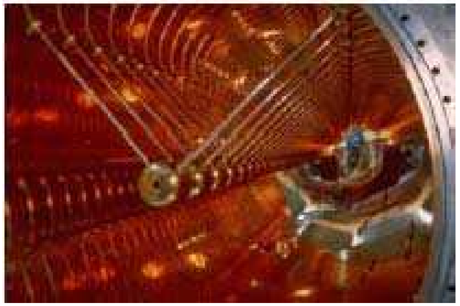
- no $\vec{E}$ inside tubes
- $\vec{E}$ between tubes
- tubes are voltage-dividers
- $\vec{E}$ → : part between tubes
- $\vec{E}$ ← : part inside tubes
- DTL: Drift Tube Linac

Alvarez-Cavity Drift Tube Linac



Pillbox   Alvarez

lengths of drift tubes increases since : ω_{rf} = const, but part. velocity increases

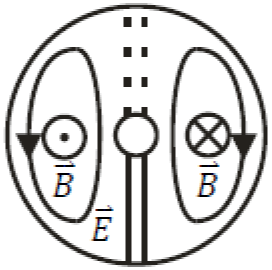


Other Cavity Types (examples)

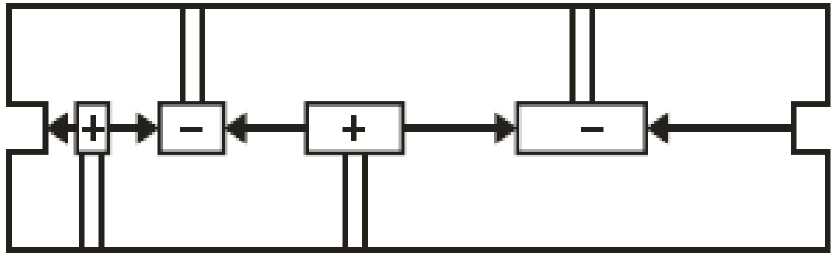
standing e.m. wave cavities non-relativistic part. velocities

H-Field Mode Cavity

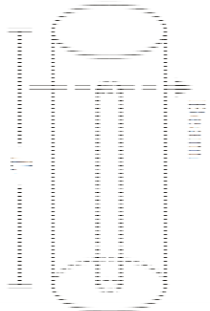
from entrance:



from side:



Quarter – Wave Resonator (QWR):



boundaries:

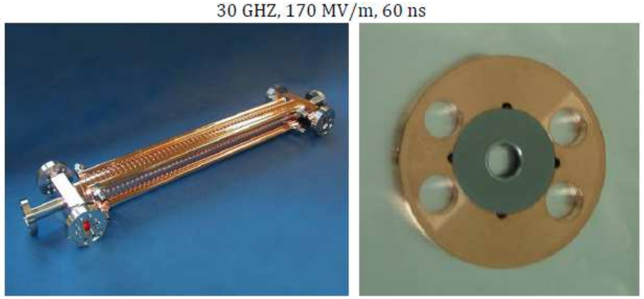
$$E_r(z = L) = 0 \rightarrow L = (2n-1) \cdot \frac{\lambda}{4}$$

$$\downarrow$$

$$f = 75MHz \cdot \frac{2n-1}{L[m]}$$

travelling wave, relativistic

30 GHZ, 170 MV/m, 60 ns

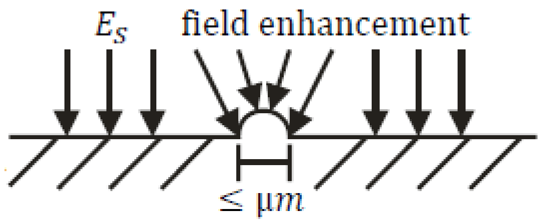


2.9978 GHz Trav.Wave Cavity, SLAC, California, U.S.A.

NLC/GLC, SLAC/KEK 11 GHz, 65 MV/m, 400 ns

High Voltage Sparking

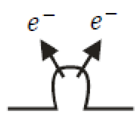
surface field E_S is limited



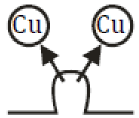
nc-cavities:
- dirt
- impurities
- scratches
- ⋮

break down process:

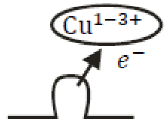
- local field → $E_{S,L} = \beta_L \cdot E_S$
- local current, e^- pulled up, heating, P_{loss}



- emission of neutrals

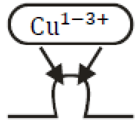


- ionization of neutrals

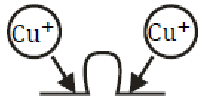


- ion layer increases $E_{S,L}$

  $E_{S,L} > 10$ GeV/m



- exp. process
- plasma (conducting)
  - el. short cut, which absorbs cavity energy
  - ions impact on surface
  - → damage



$t = 0$

≈ 0.1 ns

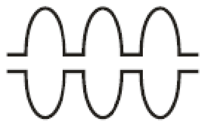
5 ns ...

$\beta \;\hat{=}$ enhancement factor

$\neq$ particle velocity

usual cavities: $E_S \leq 6E_{acc}$

E_S minimized: elliptical shape



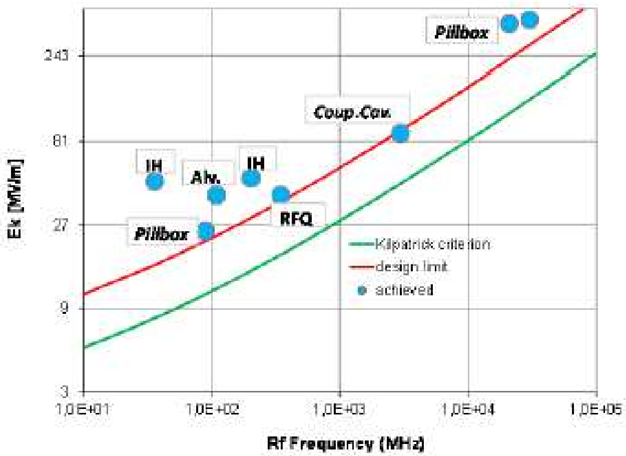
$\frac{E_S}{E_{acc}} \approx 2$

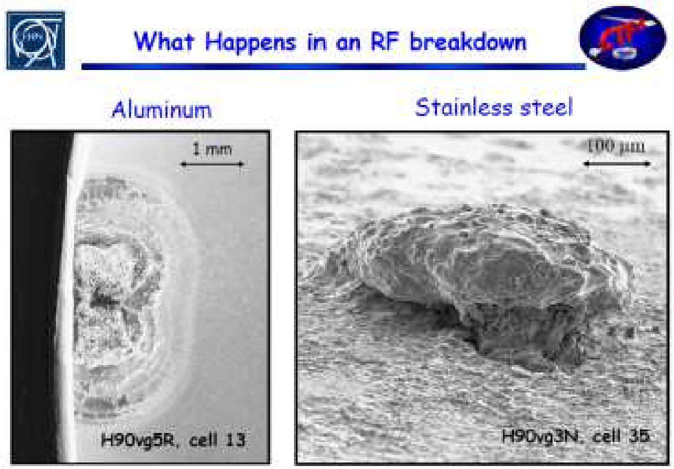
High Voltage Sparking

$E_{S,max}$ [material, ω, geometry, rf-pulse shape, surface treatment, "history"] under investigation!

Empirical (from late 40ies):

Kilpatrick-Criterion: $f\,[\mathrm{MHz}] = 1.64 \cdot {E_k}^2 \cdot e^{-\frac{8.5}{{Ez}^2}}$



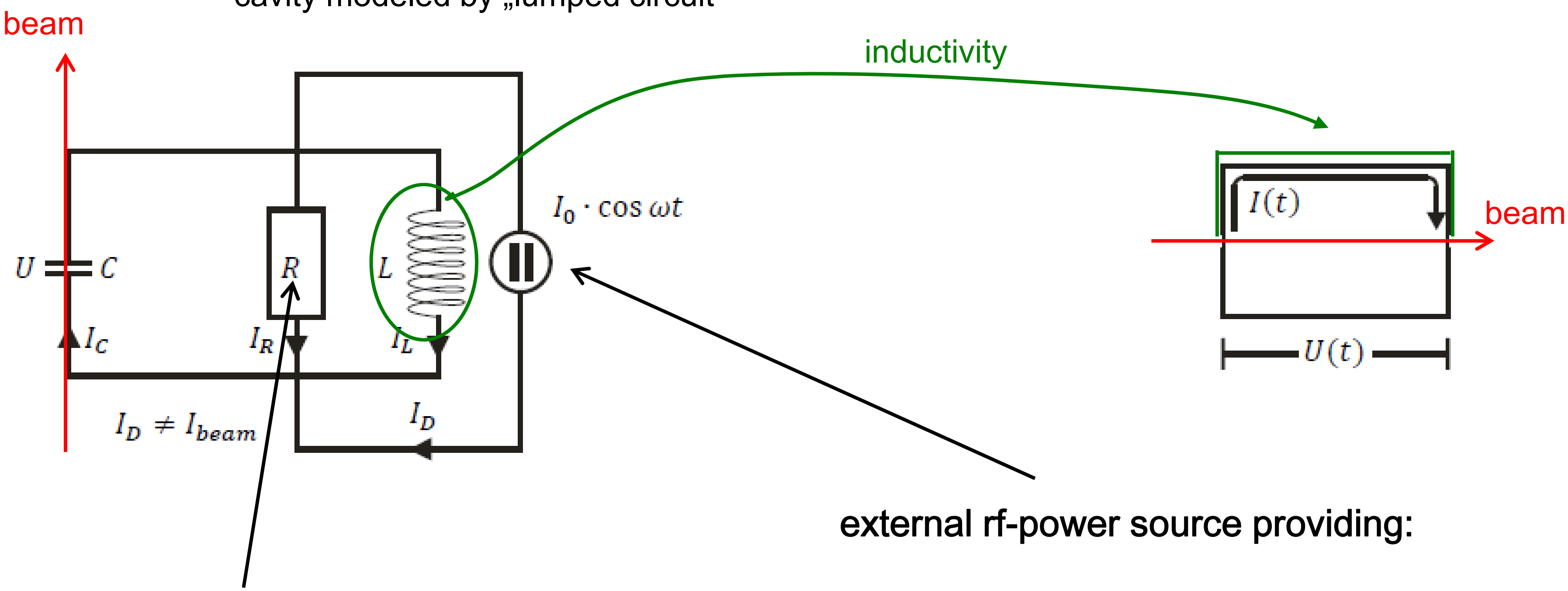


Conditioning:

1. increase E_S until break down
2. stop rf after break down

→ burning away imperfections ≈ several days!

cavity modeled by „lumped circuit“



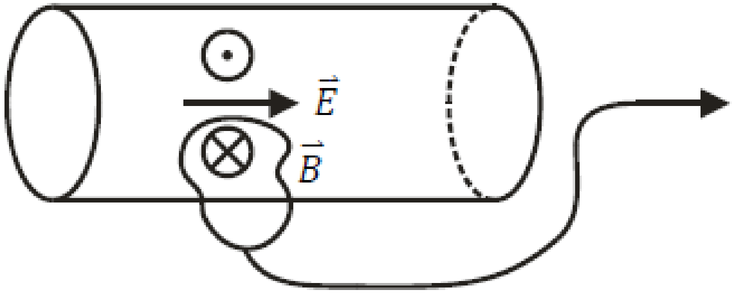
external rf-power source providing:

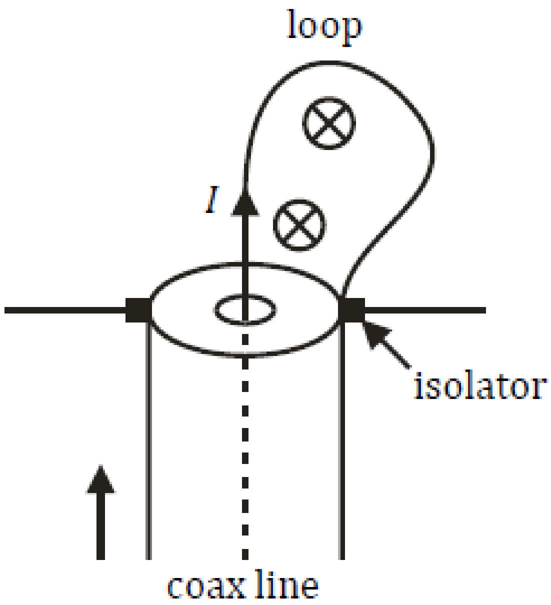
- ohmic losses (surface currents)
- R: the higher, the better !!

- compensation of ohmic losses (kW – MW)
- energy gain of particle beam (kW – MW)

Rf-Coupling:

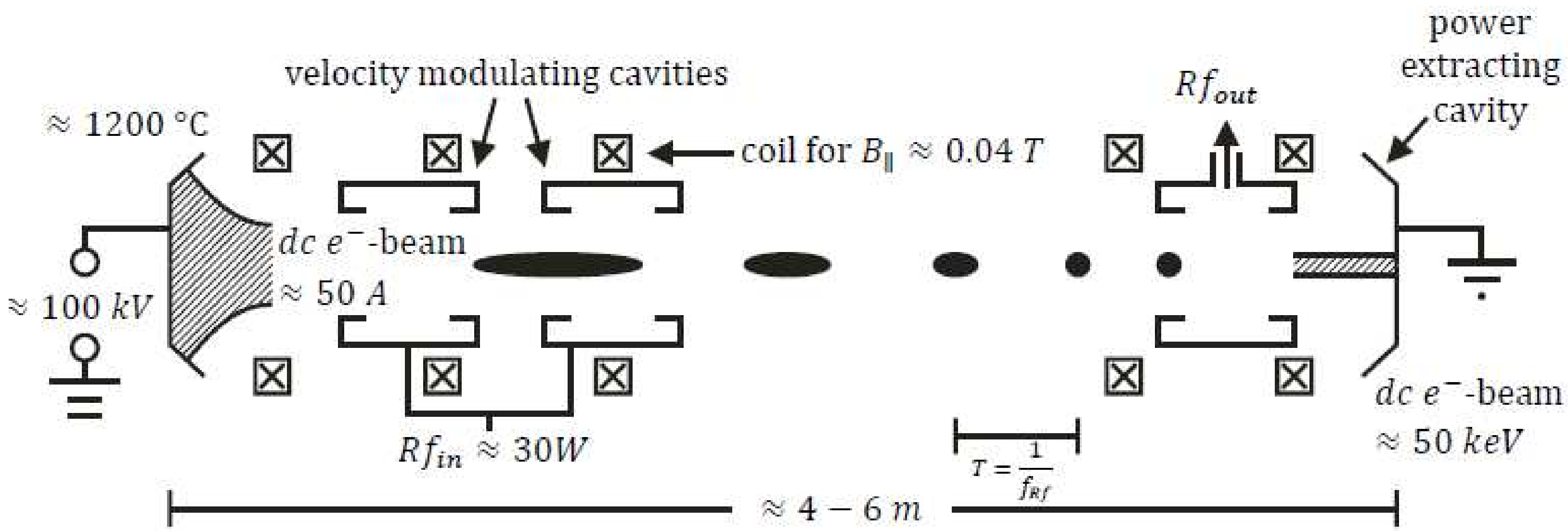
standing wave cavity:





inductive coupling: Alvarez, IH, CH, Pillbox

Klystron (example for rf-power source)

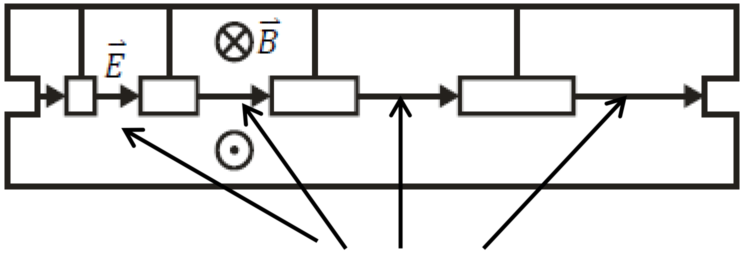


Gain $G \approx \frac{\text{Power Rf, out}}{\text{Power Rf, in}}$ $\approx 50\,dB$

measured in dB $1\,dB \mathrel{\widehat{=}}$ factor 10 $n\,dB \mathrel{\widehat{=}}$ factor 10^n

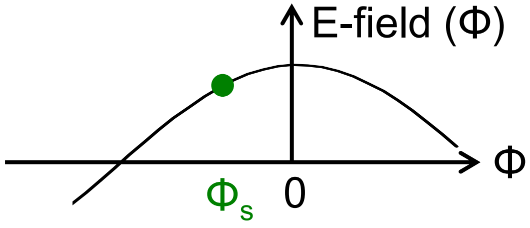
Efficiency $\eta := \frac{\text{Power Rf, out}}{\text{e-Beam Power}}$ $\approx 40\% - 50\%$

power levels: kW– GW (cw – pulsed operation)



design (perfect) particle:

- τ_s from gap to gap
- energy gain ΔE_s per gap ~ $E_0 \cos(\Phi_s)$
- arrival at $\phi_i = \phi_s$ $s \;\hat{=}$ "synchroneous"
- $E = E_{s,n}$ at gap, = kin. + rest energy



all gaps:
- operate at ω_{rf} = const.
- are phase-locked

real particle i:
- arrives at gap n with phase $\Phi_{i,n} \neq \Phi_s$
- arrives at gap n with $E_{i,n} \neq E_{s,n}$
- gains $\Delta E_{i,n} \neq \Delta E_{s,n}$

defining:

- δE := (real energy – design energy) @ gap
- Φ := real phase at gap

one finds the following equations for δE and Φ:

$$\delta E^2 + \frac{\beta^2 \cdot e \cdot q \cdot U \cdot E_s}{\pi \cdot h \cdot \eta} \cdot [\Phi \cdot \cos\Phi_s - \sin\Phi] = C$$

$$\delta E^2 + V(\Phi) = C$$

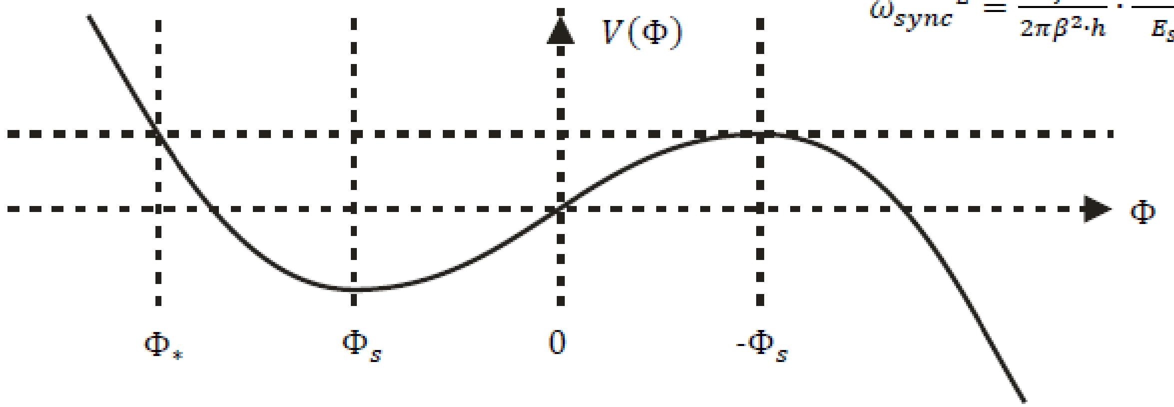
$$\delta\ddot{\Phi} \approx -\frac{2\pi h \cdot \eta}{\tau^2 \cdot \beta^2} \cdot \frac{e \cdot q \cdot U}{E_s} \cdot \sin\Phi_s \cdot \delta\Phi$$

$$\delta\Phi = A \cdot e^{i\omega_{sync} \cdot t}$$

$$\omega_{sync}{}^2 = \frac{\omega_{rf}{}^2 \cdot \eta}{2\pi\beta^2 \cdot h} \cdot \frac{e \cdot q \cdot U}{E_s} \cdot \sin\Phi_s$$

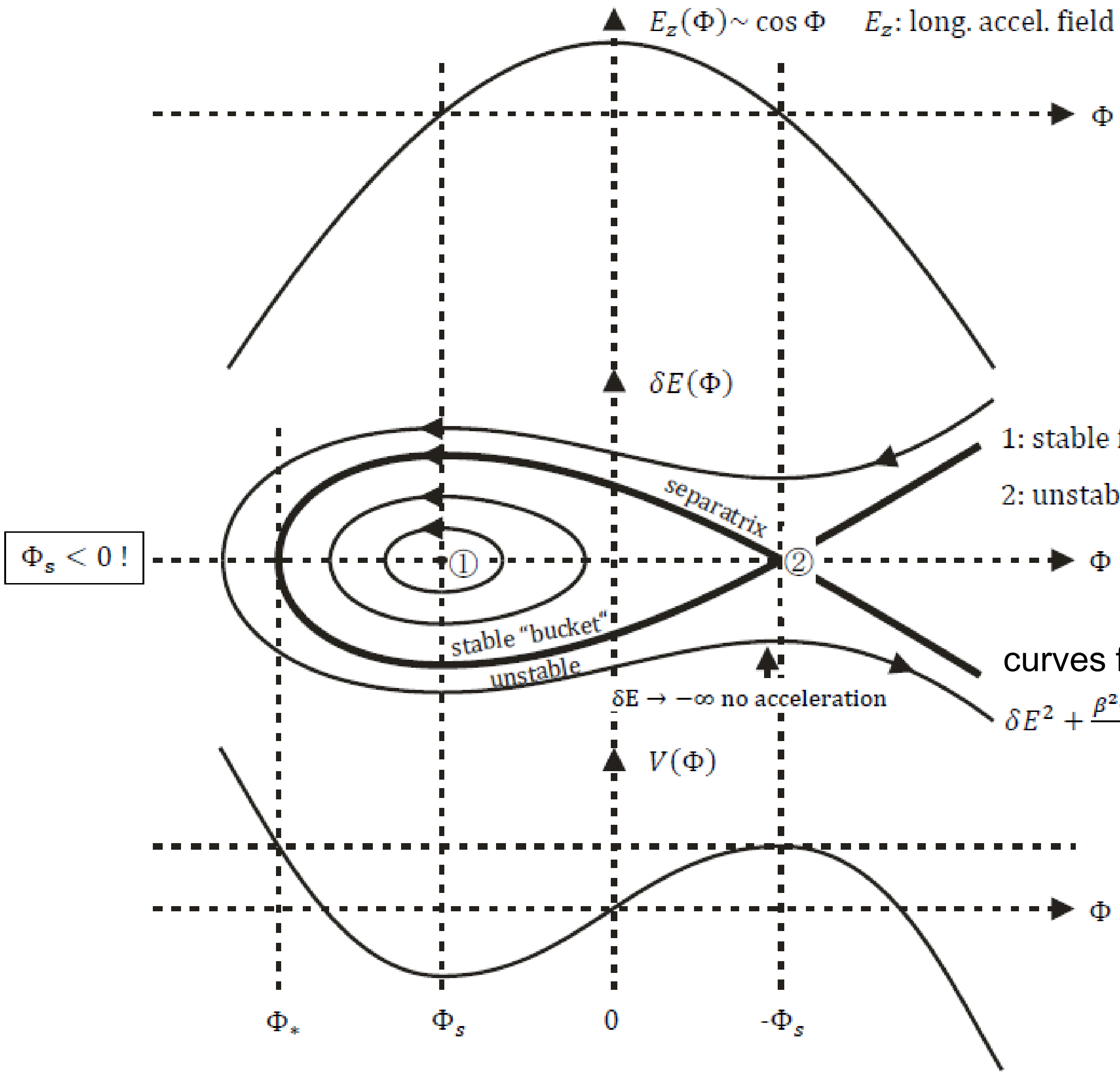
$\eta := -\gamma^{-2}$, h=1

U = gap voltage



- potential $V(\Phi_s)$ allows for stable (quasi-oscillating) motion around Φ_s
- $\Phi_* \approx 2\Phi_s$

Longitudinal Beam Focusing and Stability



curves following:

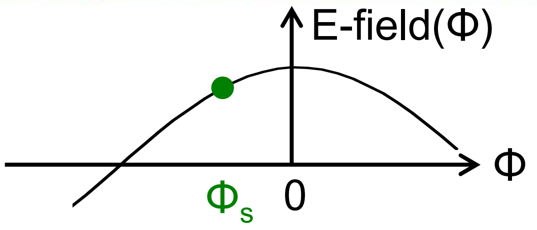
$$\delta E^2 + \frac{\beta^2 \cdot e \cdot q \cdot U \cdot E_s}{\pi \cdot h \cdot \eta} \cdot [\Phi \cdot \cos\Phi_s - \sin\Phi] = C$$

- just particles inside **bucket** are contineously accelerated
- all others will not be accelerated and get lost

acceleration: $\cos\Phi_s > 0 \rightarrow -\frac{\pi}{2} < \Phi_s < \frac{\pi}{2}$

$$\omega_{sync}{}^2 > 0 \rightarrow -\pi < \Phi_s < 0$$

$\rightarrow$ stable acceleration for $0 > \Phi_s > -\frac{\pi}{2}$



Particles sufficiently close to Φ_s are longitudinally focused. But this leads to :

Transverse gap defocusing

inside gap: $\vec{E} = -\vec{\nabla}\Psi;\ \vec{\nabla}\cdot\vec{E} = 0$

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}^2\Psi \Longrightarrow \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} = 0$$

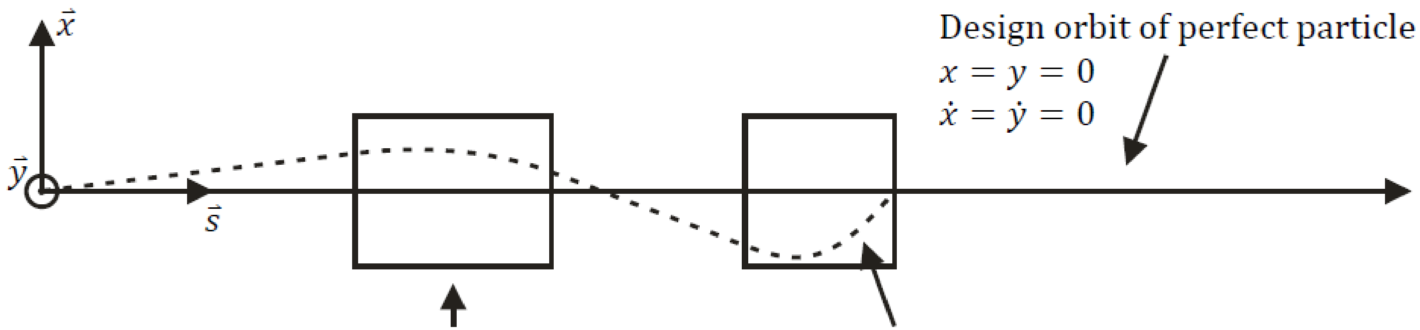
$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} = -\frac{\partial^2\Psi}{\partial z^2}$$

part inside bucket: focusing in $\Phi \mathrel{\hat{=}}$ along $z \rightarrow \frac{\partial^2\Psi}{\partial z^2} > 0$

$$\rightarrow \left[\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2}\right] < 0 \rightarrow \text{transverse defocusing!}$$

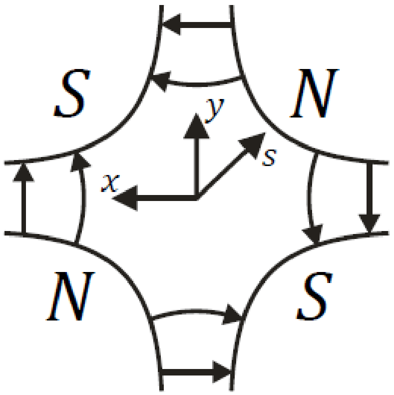
without transverse focusing, stable acceleration will cause transverse particle loss!

Transverse Differential Equations



- correction element for real particles i.e. $x \neq 0, y \neq 0,\ \dot{x} \neq 0,\ \dot{y} \neq 0$
- correction ~ displacement in x & y
- correction by quadrupoles (4-poles)

hyperbolic pole shape



$$\vec{B} = G \cdot \begin{bmatrix} y \\ x \\ 0 \end{bmatrix}$$

$$\vec{\nabla}\vec{B} = 0$$

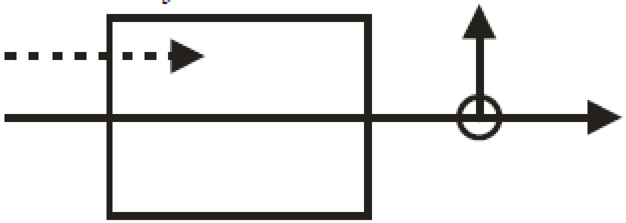
$$\vec{\nabla} \times \vec{B} = 0$$

"Gradient $\left[\frac{\mathrm{T}}{\mathrm{m}}\right]$"

- focuses in x $(G > 0)$
- de-focuses in y $(G > 0)$
- $G < 0 \rightarrow$ vice versa

horizontal:

$$B_y = G \cdot x$$



$$F_x = -q \cdot e \cdot v \cdot B = A \cdot m_0\gamma \cdot \ddot{x}$$

$$-q \cdot e \cdot v \cdot B = A \cdot m_0\gamma \cdot v^2 x'', \quad x'' := \frac{\partial^2 x}{\partial s^2}$$

$$x'' = -\frac{q \cdot e \cdot B}{A \cdot m_0 \gamma \cdot v} = -\frac{q \cdot e \cdot G}{p_0} \cdot x := -kx$$

$$= -kx$$

$$y'' = +ky$$

$$x = x_0 \cdot C + x_0' \cdot S$$

$$\begin{bmatrix} x \\ x' \end{bmatrix} = \begin{bmatrix} C & S \\ C' & S' \end{bmatrix} \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$ Wronsky matrix

$$:= R \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}, \ \det\ R = 1$$

k = const. < 0; $K := |k|$

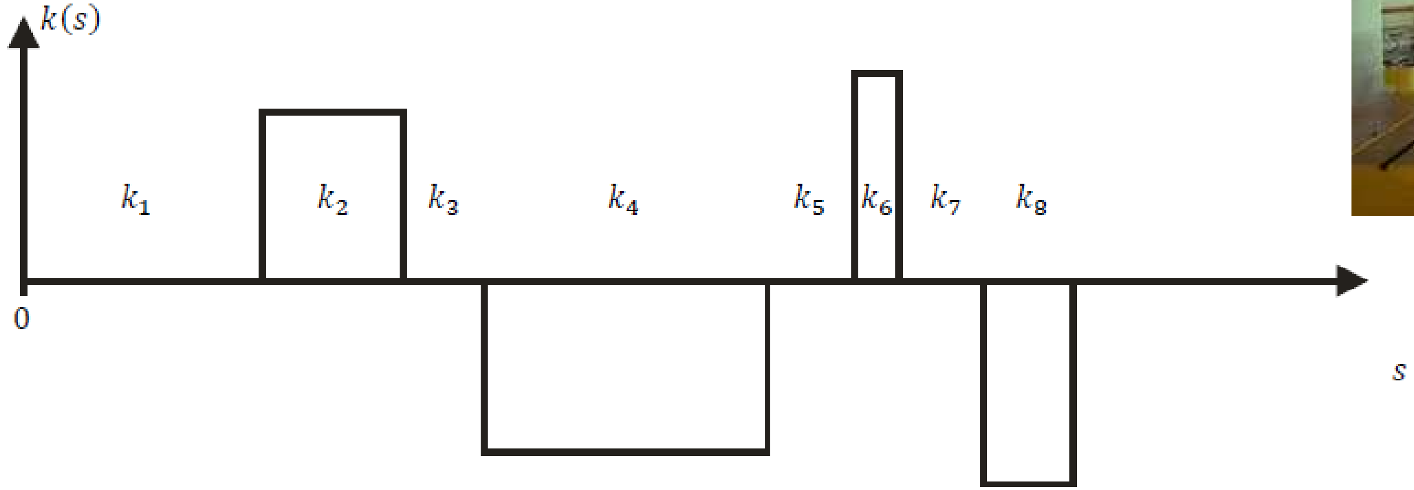
$$C := \cosh[\sqrt{K}s], \ S := \frac{1}{\sqrt{K}} \cdot \sinh[\sqrt{K}s]$$

k = const. > 0

$$C := \cos[\sqrt{k}s], \ S := \frac{1}{\sqrt{K}} \cdot \sin[\sqrt{k}s]$$

Transverse Beam Focusing and Stability

- general solution too hard
- solve piecewise, since k is from finite elements



$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = R[k_n] \cdot R[k_{n-1}] \cdots R[k_2] \cdot R[k_1] \cdot \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}$$

$$= \tilde{R}[k_1 \ldots k_n] \begin{bmatrix} x_0 \\ x_0' \end{bmatrix}, \qquad \det \tilde{R} = \det[k_1] \cdot \det[k_2] \cdots \det[k_{n-1}] \cdot \det[k_n]$$

$$= \quad 1 \quad \cdot \quad 1 \quad \cdots \quad 1 \quad \cdot \quad 1 \quad = \quad 1$$

Transport Matrices

Drift, i.e. $k = 0$ $\quad x'' = 0,\; y'' = 0 \quad C = 1,\; C' = 0,\; S = s,\; S' = 1 \quad R_{Dx} = R_{Dy} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$

$$R_D = \begin{bmatrix} [R_{Dx}] & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & [R_{Dy}] \end{bmatrix} = \begin{bmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

quadrupole:

$$R_q(k>0) = \begin{bmatrix} \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega & 0 & 0 \\ -\sqrt{|k|}\sin\Omega & \cos\Omega & 0 & 0 \\ 0 & 0 & \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega \\ 0 & 0 & \sqrt{|k|}\sinh\Omega & \cosh\Omega \end{bmatrix}$$

(x, x') oscillation

(y, y') exp.growth

$$R_q(k<0) = \begin{bmatrix} \cosh\Omega & \frac{1}{\sqrt{|k|}}\sinh\Omega & 0 & 0 \\ \sqrt{|k|}\sinh\Omega & \cosh\Omega & 0 & 0 \\ 0 & 0 & \cos\Omega & \frac{1}{\sqrt{|k|}}\sin\Omega \\ 0 & 0 & -\sqrt{|k|}\sin\Omega & \cos\Omega \end{bmatrix}$$

(x, x') exp. growth

(y, y') oscillation

rf-gap:

$$k = \frac{E_0 \cdot L_{gap} \cdot T \cdot q \cdot e \cdot 2\pi n}{m_0 c^2 \beta^3 \gamma^3 \lambda} \cdot \sin\Phi_s$$

$$R_{gap} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k}{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{k}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & k & 1 \end{bmatrix}$$

Envelope Equations

second order beam moments:

- $\langle x\ x \rangle \quad := \frac{x_j \cdot x_j}{N}$; square of rms-beam size
- $\langle x\ x' \rangle \quad := \frac{x_j \cdot x'_j}{N}$; rms-beam correlation
- $\langle x'\ x' \rangle \quad := \frac{x'_j \cdot x'_j}{N}$; square of rms-beam divergence

beam moments matrix M := $\begin{bmatrix} <xx> & <xx'> & <xy> & <xy'> \\ <x'x> & <x'x'> & <x'y> & <x'y'> \\ <yx> & <yx'> & <yy> & <yy'> \\ <y'x> & <y'x'> & <y'y> & <y'y'> \end{bmatrix}$

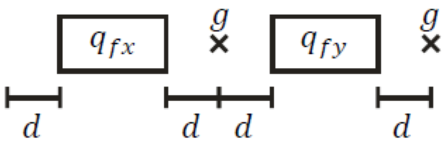
transport of beam moments from initial to final location:

$$M_f := R M_i R^{\mathrm{T}}$$

Periodic Lattice and Stable Envelope Criterion

Lattice: sequence of beam line elements

example:



d: drift

q_{fx}: horizontal focusing quadrupole

g: rf-gap

q_{fy}: vertical focusing quadrupole

Transport matrix: $R = R(g) \cdot R(d) \cdot R(q_{fy}) \cdot R(d) \cdot R(g) \cdot R(d) \cdot R(q_{fx}) \cdot R(d)$

Periodic Lattice $R_P = R^N$; $N = \#$ of lattice periods

when R^N is stable?

stable means $M_f = R^N M_i [R^N]^T \rightarrow$ finite for $N \rightarrow \infty$

$\Longrightarrow$ lattice stable: $|Trace\{R\}| < 2$

example drift: $R(d) = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix}$ $Trace\{R(d)\} = 2 \rightarrow$ unstable

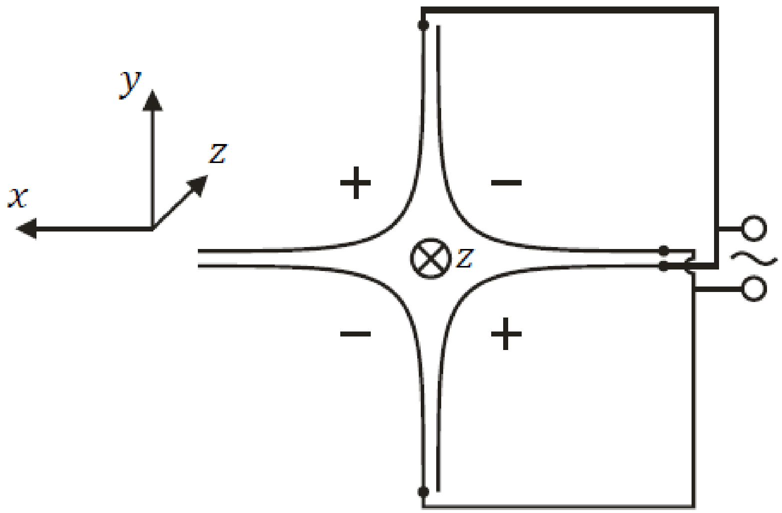
Radio Frequency Quadrupole (RFQ)

- Acceleration with static E-field: maximum beam energy limited by maximum dc-voltage that can be realized within reasonable dimensions. Limit is at some 10 MV.
- Dimensions can be decreased by using fields that oscillate in time $E(t) = E_0 cos(\omega t)$
- Disadvantage of $E(t)$:
  - just particles that arrive within periodically occurring "good time slots" are accelerated
  - particles arriving outside a slot are decelerated, not accelerated, too weakly accelerated, ...
  - duration of "good slot" ΔT depends on frequency ω: $\Delta T \cdot \omega \approx 60°$
  - beam durations from source/LEBT are much longer w.r.t. slot duration
  - particles outside slots will be lost
- Accelerate all particles, i.e. no losses:
  - need to arrange particles inside slots. This process is called "bunching"
  - particles within a "good slot" form the "bunch"
- Beam arranged into slots is "bunched beam"
- Bunched beam will de-bunch, i.e. exceed slot-border, if bunches are not kept together. This is due to tiny differences within individual particle velocities

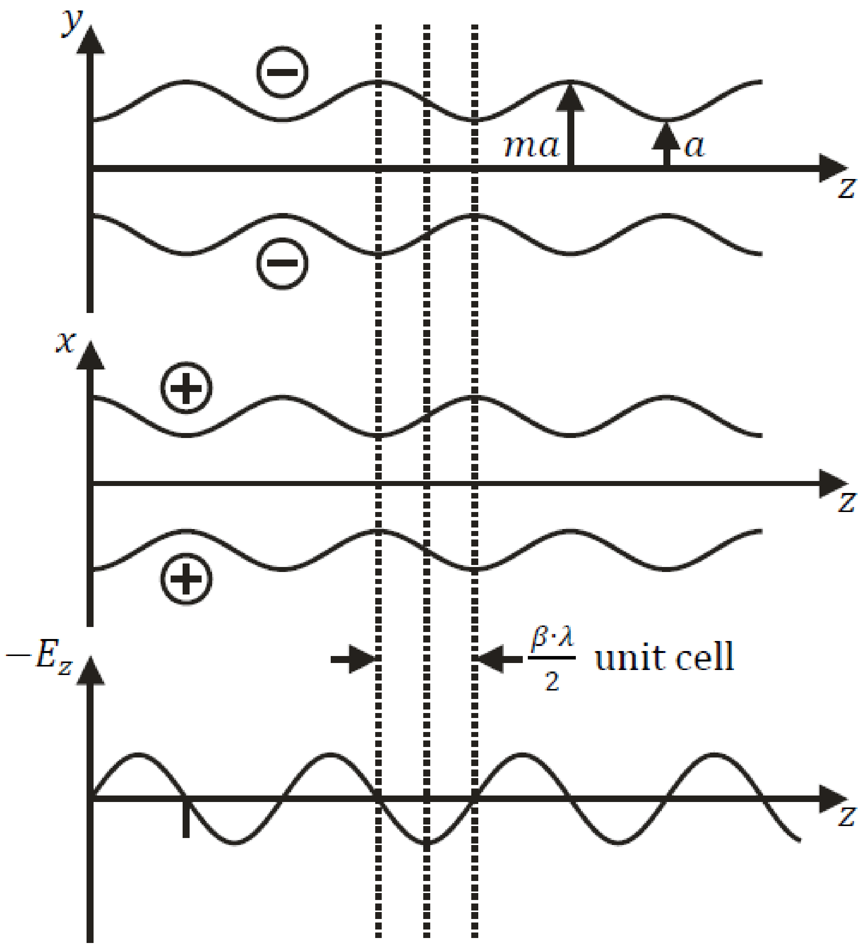
Radio Frequency Quadrupole (RFQ)

Bunching is done most efficiently (hadrons !) with **R**adio **F**requency **Q**uadrupole (RFQ):

bunching + focusing device:



4 Poles, rf-driven → RFQ



- focusing in each plane changes with time
- particles move along z
- focusing varies along z

→ net focussing like F0D0

RFQ Example

HLI 4-rod RFQ, GSI/Germany

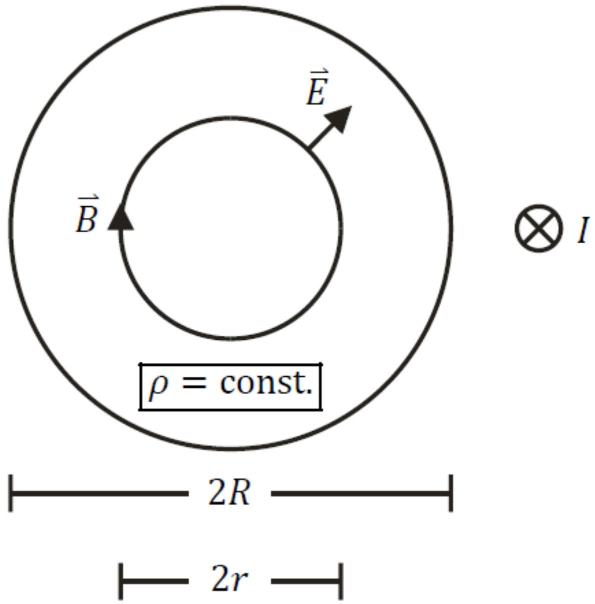
Deuterium – Uranium, length: 2.055 m, $\beta_{in} = 0.0023$, $\beta_{out} = 0.025$, res. frequency: 108.408 MHz

RFQ Example

4-rod RFQ at SARAF/Israel

SARAF-RFQ: 3 MeV D+, res. frequency: 175 MHz, 250 kW CW

Defocusing from Beam Self Forces



$$E(r) = \frac{I \cdot r}{2\pi\varepsilon_0 \cdot R^2 \cdot \beta \cdot c}$$

$$B(r) = \frac{\mu_0 \cdot I \cdot r}{2\pi R^2}$$

$$\ddot{r}_{sc} = \frac{e \cdot q}{A \cdot m_0 \cdot \gamma}\left[\vec{E} - \vec{v} \times \vec{B}\right] = \boxed{\ddot{r}_{sc} = \frac{e \cdot q \cdot I}{A \cdot m_0 \cdot \beta \cdot \gamma^3 \cdot 2\pi\varepsilon_0 R^2 \cdot c} \cdot r}$$

linear in r!

$\vec{E}$: defocusing; $\vec{v} \times \vec{B}$: focusing

- full space charge (sc) is defocusing
- at relativistic energies sc forces vanish
- at low energies sc is very strong !

Reduction of Focusing Strength (Tune Shift)

space charge adds to diff. equation :

$$x'' + \left[k(s) - \frac{P}{R^2(s)}\right] x = 0$$

"Perveance" $P = \frac{e \cdot q \cdot I}{2\pi\varepsilon_0 \cdot A \cdot m_0 \cdot c^3 \beta^3 \gamma^3}$

dimensionless

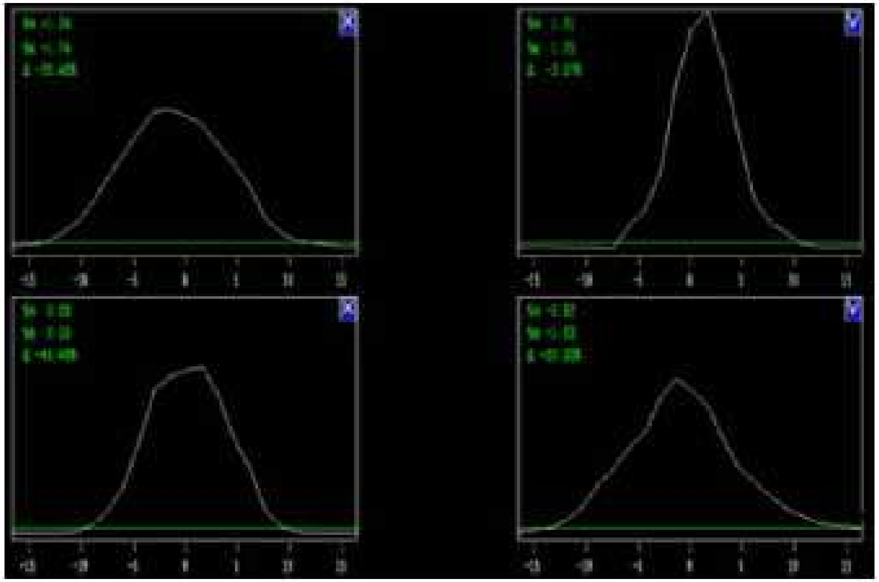
$x(s)$ like harmonic oscillator $\sim e^{i\sqrt{k_{eff}} \cdot s}$

$$\sigma_{eff} := \frac{\text{phase advance}}{\text{length}} = \sqrt{k_{eff}}\,, \qquad I \neq 0$$

$$\sigma_o := \sqrt{k}\,, \qquad I = 0$$

$$\rightarrow \boxed{\sigma_{eff}^2 = \sigma_0^2 - \frac{P}{R^2} := \sigma_0^2 - \text{sc-tune shift}}$$

$\sigma_{eff}^2 > 0$ limits achievable current transverse beam profiles



horizontal vertical

$\rho = \text{const.}$ generally not true, generally ρ decreases with r

$$\rightarrow E(r) = \frac{\int_0^r \rho(r')r'dr'}{\varepsilon_0 \cdot r} \rightarrow x'' + [k(s) + f(x)] \cdot x = 0 \quad \text{non-linear}$$

non-linear forces reduce beam quality