Three-loop QCD corrections to heavy-to-light form factors and applications to inclusive B decays

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Collaborative Research Center TRR 257

P H

Particle Physics Phenomenology after the Higgs Discovery

in collaboration with Matteo Fael, Tobias Huber, Fabian Lange, Kay Schönwald and Matthias Steinhauser;

based on arXiv: 2406.08182, published in Phys. Rev. D. 110, 056011 and work in progress.

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Motivation and applications

- Heavy-to-light form factors are important ingredients for many particle physics processes. We focus on B physics.

- Factorisation theorem for the photon energy spectrum in $B \to X_s\gamma$: $\mathrm{N}^3\mathrm{LL}'$ analysis.

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\gamma} \propto \boxed{H} \int J \times S + \mathcal{O}\left(\frac{\Lambda_{QCD}}{m_b}\right)$$

- Jet and soft function J and S known at $\mathrm{N}^3\mathrm{LO}$.
- Hard function H up to now at NNLO.

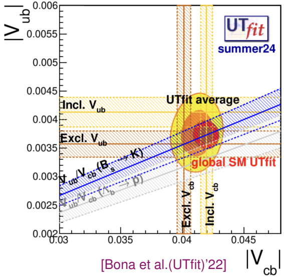
[Becher,Neubert'05'06] [Ali,Greub,Pecjak'07] [Bell,Beneke,Huber,Li'10] [Ligeti,Stewart,Tackmann'08] [Brüser,Liu,Stahlhofen'18'19] [SIMBA'20] [Dehnadi,Novikov,Tackmann'22]

- Determination of $|V_{ub}|$ from inclusive semi-leptonic $B \to X_u \ell \nu$ decays.
  - Puzzle in the extraction from inclusive and exclusive decays:

$$|V_{ub}^{\text{excl.}}|/|V_{ub}^{\text{incl.}}| = 0.84 \pm 0.04$$

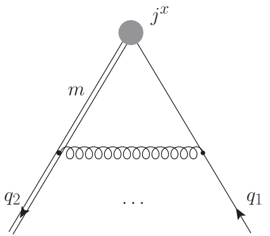
[HFLAV'22]

  - However, most recent extraction by Belle compatible with unity. [Belle'23]



[Bona et al.(UTfit)'22]

Setup of the calculation



- Kinematics: $q_1^2 = 0,\ q_2^2 = m^2$

  $s \equiv q^2 = (q_1 - q_2)^2$

- External currents:

$$j^x = \bar{\psi}_Q\{1, \mathrm{i}\gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, \mathrm{i}\sigma^{\mu\nu}\}\psi_q$$

- Depending on the phenomenological application: momentum transfer $s = 0$ or $s \neq 0$.

- General structure of the amplitude:

$$\int \frac{\mathrm{d}^4 y}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}q\cdot y}\langle\psi_Q^{\text{out}}(q_2, s_2)|j^x(y)|\psi_q^{\text{in}}(q_1, s_1)\rangle = \bar{u}(q_2, s_2)\Gamma(q_1, q_2)u(q_1, s_1)\delta^{(4)}(q - q_1 - q_2)$$

- Example: Vertex function for the tensor current:

$$\Gamma^t_{\mu\nu}(q_1, q_2) = \mathrm{i}F_1^t(q^2)\sigma_{\mu\nu} + \frac{F_2^t(q^2)}{m}(q_{1,\mu}\gamma_\nu - q_{1,\nu}\gamma_\mu) + \frac{F_3^t(q^2)}{m}(q_{2,\mu}\gamma_\nu - q_{2,\nu}\gamma_\mu) + \frac{F_4^t(q^2)}{m^2}(q_{1,\mu}q_{2,\nu} - q_{1,\nu}q_{2,\mu})$$

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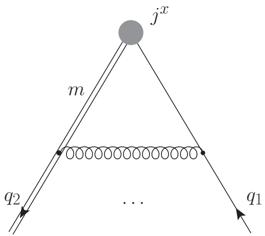

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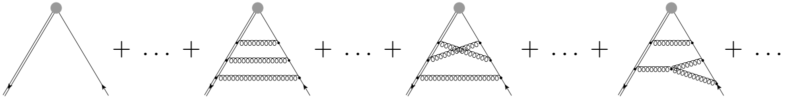
- Depending on the phenomenological application: momentum transfer $s = 0$ or $s \neq 0$.

- Two-loop corrections to heavy-to-light form factors are known. [Asatrian et al.'06] [Ali,Greub,Pecjak'07] [Asatrian,Greub,Pecjak'08] [Ligeti,Stewart,Tackmann'08] [Bell'08] [Bonicani,Ferroglia'08] [Beneke,Huber,Li'08] [Huber'09] [Bell,Beneke,Huber,Li'10]
- Analytical three-loop $\propto N_c^3$ corrections to heavy-to-light form factors appeared in 2023. [Chen,Wang'18] [Datta,Rana,Ravindran,Sarkar'23]
- Subsequent analytical calculation of all fermionic pieces (except linear in n_h) in [Datta,Rana'24] confirms our results.

Calculation steps

1. Generate all possible Feynman diagrams.



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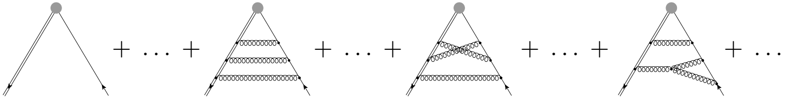

- We have to calculate 259 Feynman diagrams (times the number of currents).

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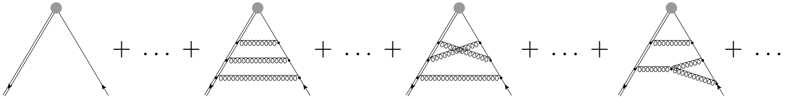

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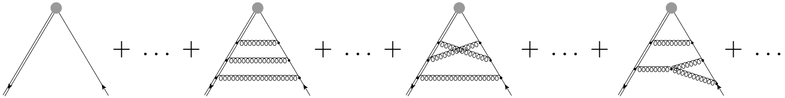

2. Apply Feynman rules, simplify the colour, tensor and Dirac structure and obtain scalar integral topologies.
   - Main calculation ($s \neq 0$) using projectors: (Toolchain: `qgraf`, `tapir`, `exp`, `calc`).
     [Nogueira'93] [Gerlach,Herren,Lang'22][Harlander,Seidensticker,Steinhauser'97] [Seidensticker'99]
   - Cross-check: Tensor current at $s = 0$ in Feynman gauge: (Toolchain: `qgraf`, `FeynHelpers` (`Fermat`), `FEYNSON`).
     [Shtabovenko'16][Lewis'86][Magerya'22]
   - We find several hundred thousand scalar integrals in 47 families.

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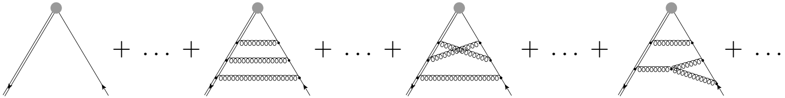

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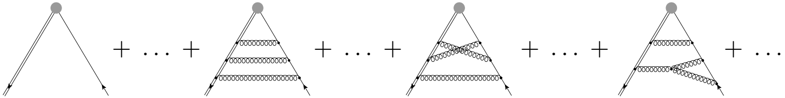

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    - IBP reduction to MIs: Automated implementation in `Kira`.

      [Maierhöfer,Usovitsch,Uwer'17] [Klappert,Lange,Maierhöfer,Usovitsch'20]
    - We obtain 429 MIs for all form factors ($s \neq 0$) and 246 MIs (tensor, $s = 0$) at three-loop level. (Full two-loop: 18 MIs).

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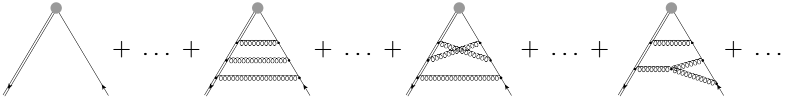

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4. Calculate the Master Integrals up to the desired order in the dimensional regulator ϵ.

5. Perform UV renormalization and IR subtraction (matching onto SCET).

6. Use results for a phenomenological analysis.

Master Integrals

- Three-loop MIs for $s \neq 0$:
  - Limit $s \to 0$ of the full amplitude possible.
  - Differential equations with respect to $x = s/m^2$: LiteRed and subsequent reduction with Kira. [Lee'23]

$$\frac{\mathrm{d}}{\mathrm{d}x} M_n = A_{nm}(\epsilon, x) M_m$$

[Kotikov'91]

- Boundary conditions:
  - Direct integration (at $x = 0$).
  - Mellin-Barnes techniques (at $x = 0$).
  - PSLQ on numerical results obtained from AMFlow (at $x = 0$). [Bailey,Ferguson'18] [Liu,Ma'22]
  - Regularity conditions (in $x = 0$ and $x = 1$).

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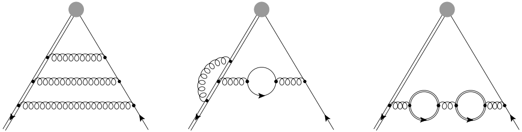


Topology	Results
all	Semi-analytically
N_C^3	Analytically
$C_F T_F^2 n_l^2$	Analytically
$C_F T_F^2 n_h^2$	Analytically
$C_F T_F^2 n_l n_h$	Analytically
$C_F^2 T_F n_l$	Analytically
$C_F C_A T_F n_l$	Analytically

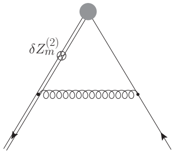
UV renormalization and IR subtraction

- Form factors exhibit poles in ϵ up to $1/\epsilon^6$.
- Standard UV renormalisation procedure.
- Form factors F are still IR divergent!
- Universal renormalization constant Z stemming from the SCET approach for any of the UV renormalized form factors F:

$$\boxed{C = Z^{-1} F}$$

- Matching coefficients C are finite!
- Cancellation of poles in $1/\epsilon$:
  In the physical region cancellation of at least 16 digits for each colour of each form factor and each $1/\epsilon$ pole.

- Example: Diagram for mass renormalization:



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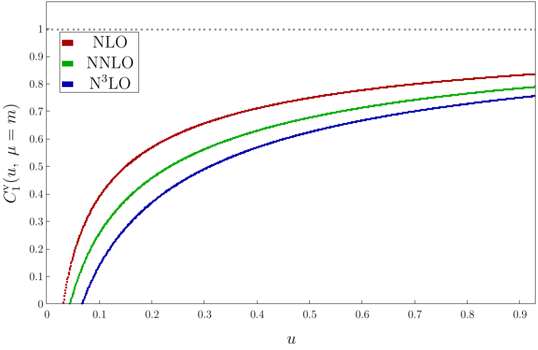
- Example: Matching coefficient $C_1^{\mathrm{v}}(u, \mu = m)$ with $u = 1 - x$.



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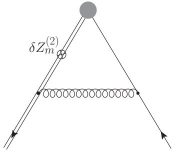

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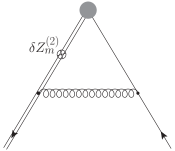

- The two-fold structure of the RGE

$$\frac{d}{d\ln(\mu)} C(s,\mu) = \left[\gamma^{\text{cusp}}(\alpha_s^{(n_l)}) \ln\left(\frac{(1-x)m}{\mu}\right) + \gamma^H(\alpha_s^{(n_l)}) + \gamma^{\text{QCD}}(\alpha_s^{(n_f)})\right] C(s,\mu)$$

can be used to distinguish two scales μ (SCET) and ν (QCD).

- The dependence of the matching coefficients C on $L_\mu = \ln(\mu^2/m^2)$ and $L_\nu = \ln(\nu^2/m^2)$ is then predicted from lower loops.
- Cross-check of the genuine three-loop calculation.

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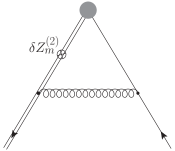

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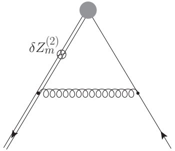

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Hard function in $B \to X_s\gamma$ to three-loops

- SCET-based approach for the photon energy spectrum of $B \to X_s\gamma$:
  $\mathrm{N^3LL'}$ analysis requires the hard function H to three-loops.

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}E_\gamma} \propto \boxed{H} \int J \times S + \mathcal{O}(\frac{\Lambda_{QCD}}{m_b})$$

- We have to consider the electromagnetic dipole operator Q_7:

$$Q_7 = -\frac{e\,\overline{m}_b(\mu)}{4\pi^2}\,(\bar{s}_L \sigma_{\mu\nu} F^{\mu\nu} b_R) \xrightarrow[\mathrm{SCET}]{\mathrm{matching}} J^A = (\bar{\xi} W_{hc}) \not{\epsilon}_\perp (1-\gamma_5) h_v$$

- On-shell matching yields for momentum transfer $s = 0$ (after IR-subtraction):

$$\langle s\gamma | Q_7 | b\rangle = -\frac{e\,\overline{m}_b\, 2E_\gamma}{4\pi^2} \underbrace{\left(C_1^t - \frac{1}{2}\, C_2^t\right)\Big|_{s=0}}_{\equiv C_\gamma} \times J^A$$

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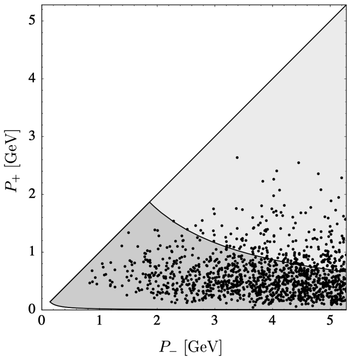
- The hard function is given via $H(\mu) = \left| C_\gamma \Big|_{L_\nu = 0} \right|^2$:

$$\boxed{H(m_b) = 1 - 4.5483 \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right) - 19.2861 \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^2 - 181.1617 \left(\frac{\alpha_s^{(4)}(m_b)}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4)}$$

Semileptonic $B \to X_u l \nu_l$ decays

[Fael,Huber,Lange,Müller,Schönwald,Steinhauser,w.i.p.]

- Kinematic variables $P_\pm = E_X \mp |\vec{P_X}|$.
- Experiment requires cut on P_+, E_l or $M_X = P_+ P_-$ to suppress the background.
- Experimental cuts constrain kinematics to the shape-function region of small P_+ and large P_-.



- In this region the double differential decay width factorises

[Bosch,Lange,Neubert,Paz'04'05]

$$\frac{\mathrm{d}^2\Gamma_u(B \to X_u l \nu_l)}{\mathrm{d}P_+ \mathrm{d}P_-} \propto |V_{ub}|^2 \, H \int J \times S + \mathcal{O}(\frac{\Lambda_{QCD}}{m_b})$$

[Korchemsky,Sterman'94] [Bauer,Fleming,Pirjol,Stewart'00]

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- Factorisation theorem for Γ_u at leading power in the $1/m_b$-expansion in the relevant phase space region (for the incl. determination of $|V_{ub}|$):

$$\frac{\mathrm{d}\Gamma_u^{(0)}(B \to X_u l \nu_l)}{\mathrm{d}P_+ \mathrm{d}P_-} \propto |V_{ub}|^2 \exp\left[2S(\mu_h,\mu_i) - 2S(\mu_i,\mu_0) + 2a_{\gamma^J}(\mu_i,\mu_0) - 2a_{\gamma'}(\mu_h,\mu_0) - 2a_\Gamma(\mu_h,\mu_i)\ln\frac{m_b}{\mu_h}\right]$$
$$\times H_{ui}(y,m_b,\mu_h)\, y^{-2a_\Gamma(\mu_h,\mu_i)}$$
$$\times \widetilde{j}\left(\ln\frac{m_b y}{\mu_i} + \partial_\eta, \mu_i\right)\frac{e^{-\gamma_E \eta}}{\Gamma(\eta)}\int_0^{P_+} d\hat{\omega}\left[\frac{1}{P_+ - \hat{\omega}}\left(\frac{P_+ - \hat{\omega}}{\mu_i}\right)^\eta\right]_* \hat{S}(\hat{\omega},\mu_0)$$

[Bosch,Lange,Neubert,Paz'04'05][Greub,Neubert,Pecjak'09]

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- Hard functions H_{ui} are given via the vector matching coefficients C_j^{v} to three-loops.
- Jet function $\widetilde{j}$ is calculated to three-loops in [Brüser,Liu,Stahlhofen'18] .

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$$\times \widetilde{j}\left(\ln\frac{m_b y}{\mu_i} + \partial_\eta, \mu_i\right) \frac{e^{-\gamma_E \eta}}{\Gamma(\eta)} \int_0^{P_+} d\hat{\omega} \left[\frac{1}{P_+ - \hat{\omega}}\left(\frac{P_+ - \hat{\omega}}{\mu_i}\right)^\eta\right]_* \hat{S}(\hat{\omega},\mu_0)$$

[Bosch,Lange,Neubert,Paz'04'05][Greub,Neubert,Pecjak'09]

- Hard functions H_{ui} are given via the vector matching coefficients C_j^{v} to three-loops.
- Jet function $\widetilde{j}$ is calculated to three-loops in [Brüser,Liu,Stahlhofen'18] .
- Shape function $\hat{S}$: Exponential model as in [Greub,Neubert,Pecjak'09]: $\hat{S} \propto \hat{\omega}^{b-1}\exp\left(-\frac{b}{\Lambda}\hat{\omega}\right)$

Semileptonic $B \to X_u l \nu_l$ decays

[Fael,Huber,Lange,Müller,Schönwald,Steinhauser,w.i.p.]

- Factorisation theorem for Γ_u at leading power in the $1/m_b$-expansion in the relevant phase space region (for the incl. determination of $|V_{ub}|$):

$$\frac{d\Gamma_u^{(0)}(B \to X_u l \nu_l)}{dP_+ dP_-} \propto |V_{ub}|^2 \exp\left[2S(\mu_h,\mu_i) - 2S(\mu_i,\mu_0) + 2a_{\gamma^J}(\mu_i,\mu_0) - 2a_{\gamma'}(\mu_h,\mu_0) - 2a_\Gamma(\mu_h,\mu_i)\ln\frac{m_b}{\mu_h}\right]$$
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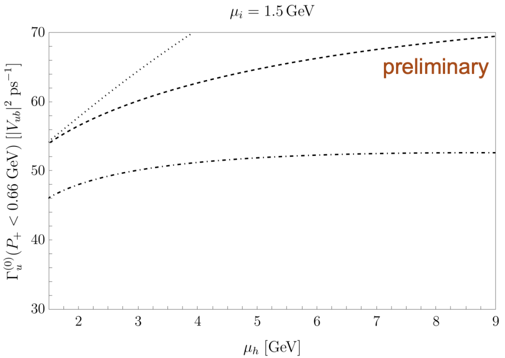
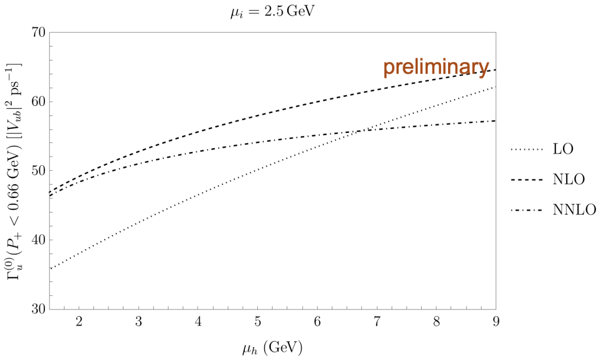
- But: Some of the RGE exponents a_{Γ_j} and S depend on currently unknown higher-order anomalous dimensions. $\Rightarrow$ Padé approximation

Semileptonic $B \to X_u l \nu_l$ decays: Cut on P_+

[Fael,Huber,Lange,Müller,Schönwald,Steinhauser,w.i.p.]

- Dependence of the partial decay rate on the scale μ_h with a cut on $P_+ < 0.66\ \mathrm{GeV}$ in units of $|V_{ub}|^2 \mathrm{ps}^{-1}$:





- The NNLO corrections shift the value of $|V_{ub}^{\text{incl.}}|$ upwards.

[Greub,Neubert,Pecjak'09]

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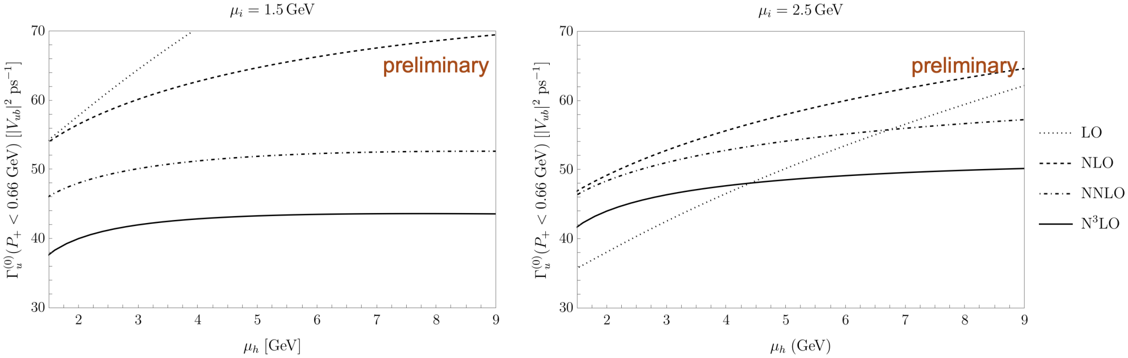
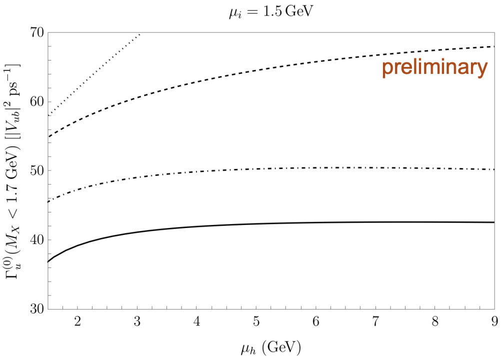


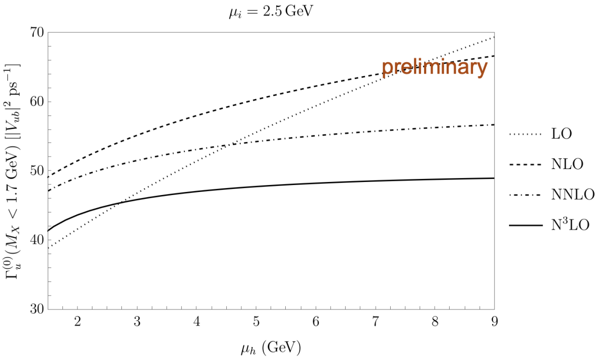
- Preliminary leading power analysis:
The $\mathrm{N^3LO}$ corrections tend to shift the value of $|V_{ub}^{\text{incl.}}|$ further upwards.

Semileptonic $B \to X_u l \nu_l$ decays: Cut on M_X

[Fael,Huber,Lange,Müller,Schönwald,Steinhauser,w.i.p.]

- Dependence of the partial decay rate on the scale μ_h with a cut on $M_X < 1.7\ \mathrm{GeV}$ in units of $|V_{ub}|^2 \mathrm{ps}^{-1}$:

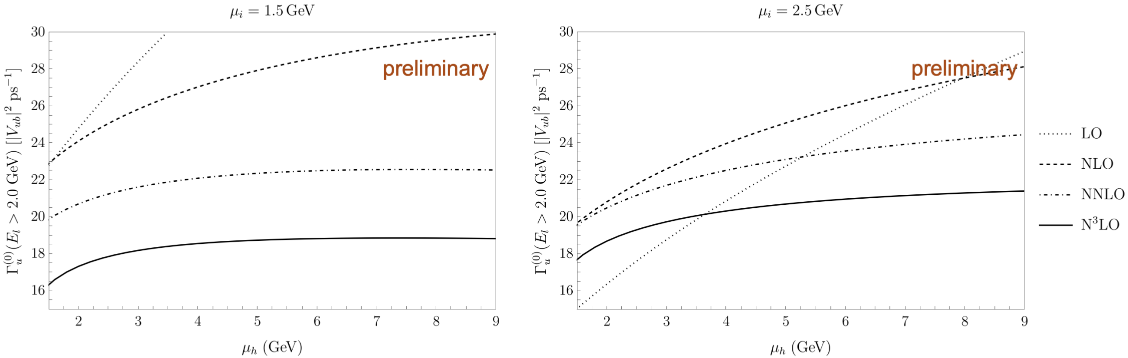




Semileptonic $B \to X_u l \nu_l$ decays: Cut on E_l

[Fael,Huber,Lange,Müller,Schönwald,Steinhauser,w.i.p.]

- Dependence of the partial decay rate on the scale μ_h with a cut on $E_l > 2.0$ GeV in units of $|V_{ub}|^2 \text{ps}^{-1}$:



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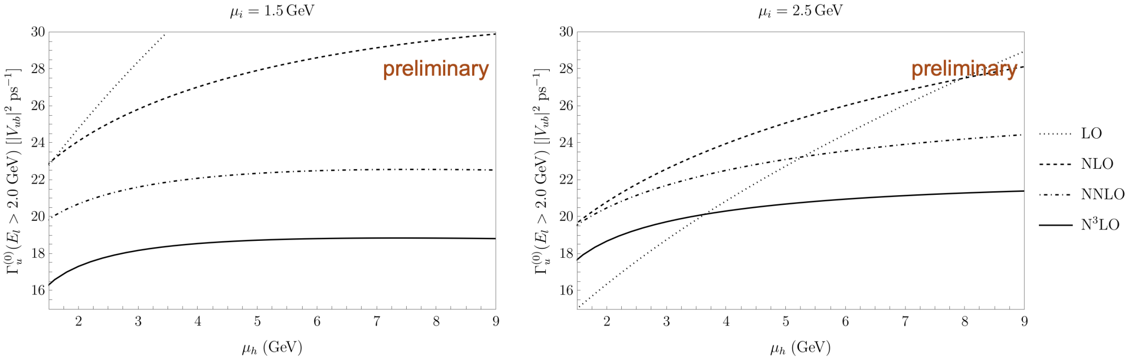

To-do list:

- Shape-function models.
- Renormalization scheme for heavy quark mass.
- Complete estimation of uncertainties.

Conclusion

- We calculated the three-loop corrections of $\mathcal{O}(\alpha_s^3)$ to heavy-to-light form factors for generic external currents.

- We calculated the hard matching coefficients in SCET for all currents.

- The vector and tensor coefficients are used to extract the hard function in the factorization theorem of $B \to X_u l\nu$ and $B \to X_s\gamma$ to three-loops, respectively.

- Preliminary leading-power results on semileptonic $B \to X_u l\nu_l$ decays:
  - The $\mathrm{N}^3\mathrm{LO}$ corrections to the partial decay rate tend to shift the value of $|V_{ub}^{\text{incl.}}|$ upwards.

Conclusion

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Thank you for your attention!

Backup Slides

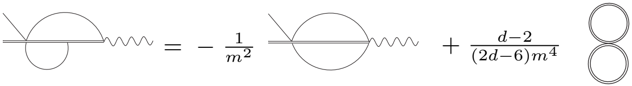
Integration-by-parts reduction

- IBP reduction to MIs: Automated implementation in `Kira`:

  [Maierhöfer,Usovitsch,Uwer'17][Klappert,Lange,Maierhöfer,Usovitsch'20]

  1. IBP reduction (familywise) of sample integrals to find all MIs.
  2. Find "better" MIs using `ImproveMasters.m` to avoid potential "bad" denominators. [Smirnov,Smirnov'20]
     - Denominators should factorize in space-time d and the kinematic variables s and m^2.
  3. IBP reduction (familywise) of the amplitude with the so obtained MIs.
  4. Final IBP reduction of all MIs for all integral families to find further symmetries.

- Two-loop IBP example:



- We obtain 429 MIs for all form factors ($s \neq 0$) and 246 MIs (tensor, $s = 0$) at three-loop level. (Full two-loop: 18 MIs).

Master Integrals

- One- and two-loop MIs:
  - Re-calculated analytically to higher orders in ϵ.
- Three-loop MIs:
  - Different methods depending on the off/on-shellness condition $s \neq 0$ or $s = 0$ and on the topology.
    $\Rightarrow$ Limit $s \to 0$ of the full amplitude possible.
- Three-loop MIs for $s \neq 0$:
  - Differential equations with respect to $x = s/m^2$:
    `LiteRed` and subsequent reduction with `Kira`. [Lee'23]

$$\frac{\partial}{\partial x} M_n = A_{nm}(\epsilon, x) M_m$$

- Method 1: "Expand and Match":

[Fael,Lange,Schönwald,Steinhauser'21,'22,'23]

  - Series expansions about regular and singular points of the DE.
  - Neighboring expansions are then numerically matched at a point where both expansions converge.
  - Here: Expansion points with 50 expansion terms each:

$$x = \{-\infty, -60, -40, -30, -20, -15, -10, -8, -7, -6, -5, -4, -3, -2, -1, -1/2, 0, 1/4, 1/2, 3/4, 7/8, 1\}$$

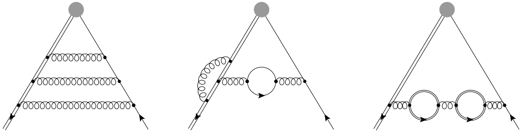
  - Except for $x = 1$ and $x = -\infty$: Taylor expansions, else power-log ansatz.
  - Boundary conditions: `AMFlow` with 100 digits in $x = 0$. [Liu,Ma'23]

Master Integrals

- <u>Method 2:</u> [Ablinger,Blümlein,Marquard,Rana,Schneider'18]
  - Decoupling of blocks of the DE into higher-order ones.
  - Solve these via factorization of the differential operator and variation of constants.
  - No canonical bases.
  - Iterated integrals over the alphabet:

$$\frac{1}{x}, \quad \frac{1}{1 \pm x}, \quad \frac{1}{2-x}$$

  - Boundary conditions:
    - Direct integration (at $x = 0$).
    - Mellin-Barnes techniques (at $x = 0$).
    - `PSLQ` on numerical results obtained from `AMFlow` (at $x = 0$). [Bailey,Ferguson'18]
    - Regularity conditions (in $x = 0$ and $x = 1$).



Topology	Result	Method
all	Semi-analytically	M1
N_C^3	Analytically	M2
$C_F T_F^2 n_l^2$	Analytically	M2
$C_F T_F^2 n_h^2$	Analytically	M2
$C_F T_F^2 n_l n_h$	Analytically	M2
$C_F^2 T_F n_l$	Analytically	M2
$C_F C_A T_F n_l$	Analytically	M2

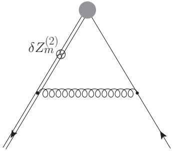
UV renormalization

$$F^x = Z_x \left(Z_{2,Q}^{\rm OS}\right)^{1/2} \left(Z_{2,q}^{\rm OS}\right)^{1/2} F^{x,{\rm bare}}\Big|_{\alpha_s^{\rm bare}=Z_{\alpha_s}\alpha_s^{(n_f)},\, m^{\rm bare}=Z_m^{\rm OS} m^{\rm OS},\, \alpha_s^{(n_f)}=\zeta_{\alpha_s}^{-1}\alpha_s^{(n_l)}}$$

- $\overline{\rm MS}$ scheme for the strong coupling α_s.
- On-shell scheme for the heavy-quark mass m: Explicit mass counterterm insertions in one- and two-loop diagrams.
  (Switch to $\overline{\rm MS}$ scheme possible in the electronic files.)

  [https://www.ttp.kit.edu/preprints/2024/ttp24-017/.]
- Decoupling relation in d dimensions:
  $\alpha_s^{(n_f)}(\mu) \to \alpha_s^{(n_l)}(\mu)$ $(n_f = n_l + n_h)$
- Anomalous dimensions:
  - vector and axialvector current: $Z_v = Z_a = 1$.
  - scalar and pseudoscalar current: related to the mass renormalization: $Z_s = Z_p = Z_m$.
  - tensor current: cannot be related to other renormalization factors.

- Example: Diagram for mass renormalization:



- On-shell wave function renormalization constants:
  - heavy quark: $Z_{2,Q}^{\rm OS}$.
  - light quark: $Z_{2,q}^{\rm OS}$ (starting at two-loops).

IR subtraction

- Form factors F^x are still IR divergent!
- Universal renormalization constant Z stemming from the SCET approach for any of the UV renormalized form factors F^x:

$$\boxed{C = Z^{-1} F}$$

- Z is given by the
  - anomalous dimensions of the light and heavy quark γ^q and γ^Q ($\gamma^H = \gamma^q + \gamma^Q$)
  - light-like cusp anomalous dimension γ^{cusp} and the QCD β function

$$\ln Z = \frac{\alpha_s^{(n_l)}}{4\pi}\left[\frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}\right] + \left(\frac{\alpha_s^{(n_l)}}{4\pi}\right)^2\left[-\frac{3\beta_0\Gamma_0'}{16\epsilon^3} + \frac{\Gamma_1' - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}\right]$$

$$+ \left(\frac{\alpha_s^{(n_l)}}{4\pi}\right)^3\left[\frac{11\beta_0^2\Gamma_0'}{72\epsilon^4} - \frac{5\beta_0\Gamma_1' + 8\beta_1\Gamma_0' - 12\beta_0^2\Gamma_0}{72\epsilon^3} + \frac{\Gamma_2' - 6\beta_0\Gamma_1 - 6\beta_1\Gamma_0}{36\epsilon^2} + \frac{\Gamma_2}{6\epsilon}\right] + \mathcal{O}(\alpha_s^4),$$

$$\Gamma = \gamma^H(\alpha_s^{(n_l)}) - \gamma^{\text{cusp}}(\alpha_s^{(n_l)})\ln\left(\frac{\mu}{m(1-x)}\right), \quad \Gamma' = \frac{\partial}{\partial \ln\mu}\Gamma = -\gamma^{\text{cusp}}(\alpha_s^{(n_l)})$$

- All ingredients for the renormalization procedure are known.

Ward identity, pole cancellations and further checks

- QCD gauge parameter ξ drops out after UV renormalization.
- Equations of motion $\Rightarrow$ Ward identities:

$$-q^\mu \Gamma^v_\mu = m\,\Gamma^s \Rightarrow F^v_1 - \frac{2s}{m^2} F^v_3 = F^s$$

- Cancellation of poles in $1/\epsilon$:
  - In the range $-75 < s < 15/16$: cancellation of at least 16 digits for each colour of each form factor and each $1/\epsilon$ pole
- We find agreement with analytical three-loop $\propto N_c^3$ corrections to heavy-to-light form factors appeared in `2308.12169`. [Chen,Wang'18]

[Datta,Rana,Ravindran,Sarkar'23] [Datta,Rana'24]