Extension of the Nested Soft-Collinear Subtraction Scheme to multi-partonic final states

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Introduction

Consider the cross section for $pp \to X + Nj$

$$2s\,\mathrm{d}\sigma = \underbrace{f_a \otimes f_b}_{\text{NP}} \otimes \boxed{\mathrm{d}\hat{\sigma}_{ab}} \longrightarrow \text{UV and IR divergent}$$

The partonic cross section is expanded in perturbative QCD

$$\mathrm{d}\hat{\sigma}_{ab} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{LO}} + \underbrace{\boxed{\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}}}}_{\text{Solved problem}} + \underbrace{\boxed{\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NNLO}}}}_{\text{Open problem}} + \dots$$

At NLO the cross section reads

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}} \xrightarrow{\text{KLN}} \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO,fin}}$$

GOAL: A general process-independent formula for the finite NNLO contribution.
The state of the art is $pp \to 3j$ at NNLO [Czakon '22].

Implicit and explicit poles

Poles can be implicit and explicit

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \overbrace{\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}}}^{\text{Implicit poles in } 1/\epsilon} + \underbrace{\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}}_{\text{Explicit poles in } 1/\epsilon}$$

From collinear renormalization of the PDFs one gets

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}} = \frac{\alpha_s}{2\pi}\frac{1}{\epsilon}\sum_{x}\left[\hat{P}_{xa}^{(0)} \otimes \mathrm{d}\hat{\sigma}_{xb}^{\mathrm{LO}} + \mathrm{d}\hat{\sigma}_{ax}^{\mathrm{LO}} \otimes \hat{P}_{xb}^{(0)}\right]$$

The virtual contribution is

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} = \frac{\alpha_s}{2\pi}\frac{1}{2}\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\sum_{\substack{i,j\\ i\neq j}}\left[\frac{1}{\epsilon^2} + \frac{\gamma_i}{\epsilon \boldsymbol{T}_i^2}\right]\boldsymbol{T}_i \cdot \boldsymbol{T}_j \left(\frac{\mu^2}{2p_i \cdot p_j}\right)^{\epsilon} e^{i\pi\lambda_{ij}\epsilon}\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{LO}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V,fin}}$$

Emergence of infrared poles

Infrared singularities appear because massless propagators diverge when particles become soft ($E \to 0$) and/or collinear ($\theta_{ij} \to 0$)

$$\left|\mathcal{M}_{(0)}(\{p\},k)\right|^2 \xrightarrow{k\to 0} \sum_{i,j} \underbrace{\frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)}}_{\text{Singular when } E_k \to 0 \text{ and } \theta_{ik},\theta_{jk} \to 0} \times \boldsymbol{T}_i \cdot \boldsymbol{T}_j \overbrace{\left|\mathcal{M}_{(0)}(\{p\})\right|^2}^{\text{Independent of singularities}}$$

The infrared behaviour of QCD amplitudes is universal and factorizes [Catani '98; Catani,Grazzini '99, '00].

- Divergences from phase-space integration

$$\int_0^{E_{\max}} \mathrm{d}E_k \frac{E_k^{\mathrm{d}-3}}{E_k^2} \sim \frac{1}{\mathrm{d}-4} \sim \frac{1}{\epsilon}$$

- $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$ is process dependent

QUESTION: How can this knowledge be used to extract and cancel the infrared poles?

Subtraction schemes are an answer

The poles of the real contribution need to become explicit to observe local cancellation of the IR poles

$$\int \mathrm{d}\Phi_{N+1}\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{ab} = \int \mathrm{d}\Phi_{N+1}\mathrm{d}\hat{\sigma}_{ab\to X+(N+1)j} \longrightarrow \int \mathrm{d}\Phi_{N}\left[\frac{c_{-2}}{\epsilon^2}+\frac{c_{-1}}{\epsilon^1}\right]\mathrm{d}\hat{\sigma}^{\mathrm{LO}}_{ab} + \mathcal{O}(\epsilon^0)$$

The real contribution is regulated by subtracting the divergent parts locally

$$\int \mathrm{d}\Phi_{N+1}\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{ab} = \underbrace{\int \mathrm{d}\Phi_{N+1} S[\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{ab}]}_{\text{Soft singularities } \sim 1/\epsilon^2} + \underbrace{\int \mathrm{d}\Phi_{N+1}(\mathbb{1}-S)\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{ab}}_{\text{Soft regulated}}$$

$$= \int \mathrm{d}\Phi_{N+1} S[\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{ab}] + \underbrace{\int \mathrm{d}\Phi_{N+1}(\mathbb{1}-S)C[\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{ab}]}_{\text{Hard collinear singularities } \sim 1/\epsilon} + \underbrace{\int \mathrm{d}\Phi_{N+1}(\mathbb{1}-C)(\mathbb{1}-S)\mathrm{d}\hat{\sigma}^{\mathrm{R}}_{ab}}_{\text{Integrable}}$$

REMARK: At NLO the problem was solved 30 years ago [Frixione,Kunszt,Signer '95; Catani,Seymour '96]. An answer is still elusive at NNLO.

The Nested Soft-Collinear subtraction scheme at NLO

Let the cross section be

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{LO}} = \left\langle F_{\mathrm{LM}}^{ab} \right\rangle = \mathcal{N} \int \mathrm{d}\Phi \, (2\pi)^{\mathrm{d}} \delta^{(\mathrm{d})}(\{p\}) \, |\mathcal{M}(\{p\})|^2 \, \mathcal{O}(\{p\})$$

and

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} = \left\langle F_{\mathrm{LM,R}}^{ab} \right\rangle$$

Introduce partitions of unity $\Delta^{(\mathfrak{m})}$ to select unresolved partons and angular partitions $\omega^{\mathfrak{m}i}$ to separate collinear singularities. Then, extract soft and collinear singularities sequentially to get

$$\left\langle F_{\mathrm{LM,R}}^{ab} \right\rangle = \sum_{\mathfrak{m}} \left\langle \Delta^{(\mathfrak{m})} F_{\mathrm{LM,R}}^{ab} \right\rangle$$

$$= \sum_{\mathfrak{m}} \left[\left\langle S_{\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{LM,R}}^{ab} \right\rangle + \sum_{i} \left\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \omega^{\mathfrak{m}i} \Delta^{(\mathfrak{m})} F_{\mathrm{LM,R}}^{ab} \right\rangle + \left\langle \mathcal{O}_{\mathsf{NLO}}^{(\mathfrak{m})} \Delta^{(\mathfrak{m})} F_{\mathrm{LM,R}}^{ab} \right\rangle \right]$$

Overview of pole cancellation at NLO (I)

- Soft term

$$\sum_{\mathfrak{m}} \left\langle S_{\mathfrak{m}} \Delta^{(\mathfrak{m})} F_{\mathrm{LM,R}}^{ab} \right\rangle = \left\langle I_{\mathrm{S}}(\epsilon) \cdot F_{\mathrm{LM}}^{ab} \right\rangle \longrightarrow I_{\mathrm{S}}(\epsilon) \sim -\frac{1}{\epsilon^2} \sum_{\substack{i,j \\ i \neq j}} \boldsymbol{T}_i \cdot \boldsymbol{T}_j$$

- Collinear term

$$\sum_{\mathfrak{m}} \sum_{i} \left\langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}} \omega^{\mathfrak{m}i} \Delta^{(\mathfrak{m})} F_{\mathrm{LM,R}}^{ab} \right\rangle = \left\langle I_{\mathrm{C}}(\epsilon) \cdot F_{\mathrm{LM}}^{ab} \right\rangle \longrightarrow I_{\mathrm{C}}(\epsilon) \sim \sum_{i} \frac{1}{\epsilon} \gamma_i$$

REMARK: The collinear anomalous dimensions γ_i do not appear automatically.

- Partitions separate collinear contributions to aid the individual calculations
- Terms have to be recombined to reconstruct γ_i
- Physically meaningful structures are obscured by partitions
- The virtual term is partition-free

Overview of pole cancellation at NLO (II)

The virtual term reads

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{V}} = \left\langle F_{\mathrm{LV}}^{ab} \right\rangle = \left\langle I_{\mathrm{V}}(\epsilon) \cdot F_{\mathrm{LM}}^{ab} \right\rangle + \left\langle F_{\mathrm{LV,fin}}^{ab} \right\rangle$$

where

$$I_{\mathrm{V}}(\epsilon) \sim \frac{1}{\epsilon^2} \sum_{\substack{i,j \\ i \neq j}} \boldsymbol{T}_i \cdot \boldsymbol{T}_j - \sum_i \frac{1}{\epsilon} \gamma_i$$

Combine everything together

$$I_{\mathrm{T}}(\epsilon) = I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon) + I_{\mathrm{V}}(\epsilon) = \mathcal{O}\left(\epsilon^0\right)$$

- Poles $\boldsymbol{T}_i \cdot \boldsymbol{T}_j$ proportional to are cancelled in full generality
- $I_{\mathrm{T}}(\epsilon)$ is process-independent
- PDFs terms cancel too

The Nested Soft-Collinear subtraction scheme at NNLO

The NNLO cross section is

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NNLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{RR}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{RV}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{VV}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{pdf}}$$

- The double-virtual and PDFs terms are well-understood
- The real-virtual term is roughly NLO-like
- The double-real term is challenging

Extract the soft singularities first

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{RR}} = \left\langle F_{\mathrm{LM,RR}}^{ab} \right\rangle$$

$$= \sum_{\mathfrak{m},\mathfrak{n}} \left[\left\langle S_{\mathfrak{mn}} \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM,RR}}^{ab} \right\rangle + \left\langle \overline{S}_{\mathfrak{mn}} S_{\mathfrak{n}} \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM,RR}}^{ab} \right\rangle + \left\langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}} \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM,RR}}^{ab} \right\rangle \right]$$

The double-real term is challenging (I)

Next, extract the collinear singularities

$$\langle F^{ab}_{\mathrm{LM,RR}}\rangle = \sum_{\mathfrak{m},\mathfrak{n}} \Big[\langle S_{\mathfrak{mn}}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{mn}}F^{ab}_{\mathrm{LM,RR}}\rangle + \sum_i \langle \overline{S}_{\mathfrak{m}} C_{i\mathfrak{m}}\omega^{\mathfrak{m}i} S_{\mathfrak{n}}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{mn}}F^{ab}_{\mathrm{LM,RR}}\rangle$$

$$+\langle \mathcal{O}^{(\mathfrak{m})}_{\mathsf{NLO}} S_{\mathfrak{n}}\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{mn}}F^{ab}_{\mathrm{LM,RR}}\rangle + \sum_{i=1}^{4} \langle \overline{S}_{\mathfrak{mn}}\overline{S}_{\mathfrak{n}}\Omega_i\Delta^{(\mathfrak{mn})}\Theta_{\mathfrak{mn}}F^{ab}_{\mathrm{LM,RR}}\rangle \Big]$$

The collinear subtractions Ω_i are complicated. This method can be applied to any process, but poles have to calculated and cancelled one process at a time [Melnikov, et al. '17,'19,'20].

GOAL: A general calculation of the poles of this term that is valid for any process.

The double-real term is challenging (II)

There are many complications associated with two unresolved particles.

- Double-soft and triple-collinear terms are difficult to calculate [Melnikov, et al. '18,'19]
- Double-collinear and triple-collinear singularities overlap → angular partitions $\omega^{\mathfrak{m}i,\mathfrak{n}j}$ do not separate all singularities
- Phase-space decomposed in sectors to disentangle collinear singularities [Czakon '10]
- Quartic and triple color-correlations: $(\boldsymbol{T}_i \cdot \boldsymbol{T}_j)(\boldsymbol{T}_k \cdot \boldsymbol{T}_l)$ and $f_{abc}\boldsymbol{T}_i^a\boldsymbol{T}_j^b\boldsymbol{T}_k^c$

REMARK 1: The reorganization of the NNLO infrared structure is much more complicated. Issues addressed in two recent papers [Melnikov, et al. '23; Melnikov, MT, et al. '25]:

- Case study $q\bar{q} \to X + Ng$: Color-correlated terms reorganized and cancelled in generality
- Case study $gq \to X + Ng + q$: Recombination of the collinear terms understood in general

REMARK 2: It is still challenging to achieve complete generality.

QUESTION: Where do the complications lie?

Parametrizing all processes

There are two main complications

- The cancellation of collinear singularities involves *all* partonic channels at once
- Bookkeeping is a huge issue: enumerate and parametrize all possible partonic channels that contribute to an arbitrary process

Focus on $n_{\mathrm{f}} = 1$ QCD and $X = Z$. All channels are parametrized by the baryonic charge

$$\begin{aligned} Q_{\mathcal{B}} \geq 0: \quad & \mathcal{B}_{N,n}^{Q_{\mathcal{B}}} = \left(\{g\}_{N-Q_{\mathcal{B}}-2n}, \{q\}_{n+Q_{\mathcal{B}}}, \{\bar{q}\}_{n}\right), \\ Q_{\mathcal{B}} < 0: \quad & \mathcal{B}_{N,n}^{Q_{\mathcal{B}}} = \left(\{g\}_{N-|Q_{\mathcal{B}}|-2n}, \{q\}_{n}, \{\bar{q}\}_{n+|Q_{\mathcal{B}}|}\right), \end{aligned} \qquad n \in \left[0, \left\lfloor \frac{N-|Q_{\mathcal{B}}|}{2} \right\rfloor\right]$$

The leading order cross section reads

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{LO}} = \left\langle F_{\mathrm{LM}}^{ab} \right\rangle = \sum_{n} \left\langle F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \right\rangle$$

REMARK: Soft and collinear singularities can now be calculated all at once.

General NLO formulas

At NLO it is straightforward to calculate the real contribution

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{R}} = \sum_n \Bigg\{ \Big\langle \mathcal{O}_{\mathsf{NLO}}^{(\mathfrak{m})} \Delta^{(\mathfrak{m})} \Big[F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}} | \mathfrak{m}_g] + F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}-1} | \mathfrak{m}_q] + F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}+1} | \mathfrak{m}_{\bar{q}}] \Big] \Big\rangle$$

$$+ \frac{\alpha_s}{2\pi} \Big[\big\langle \big[I_{\mathrm{S}}(\epsilon) + I_{\mathrm{C}}(\epsilon) \big] \cdot F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big\rangle + \frac{1}{\epsilon} \sum_x \Big\langle \mathcal{P}_{xa}^{\mathrm{gen}} \otimes F_{\mathrm{LM}}^{xb}[\mathcal{B}_{N,n}^{Q_{xb}}] + F_{\mathrm{LM}}^{ax}[\mathcal{B}_{N,n}^{Q_{ax}}] \otimes \mathcal{P}_{xb}^{\mathrm{gen}} \Big\rangle \Big] \Bigg\}$$

All poles are now known in full generality and the finite remainder can be obtained

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NLO}} = \big\langle F_{\mathrm{LV,fin}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big\rangle$$

$$+ \sum_n \Bigg\{ \Big\langle \mathcal{O}_{\mathsf{NLO}}^{(\mathfrak{m})} \Delta^{(\mathfrak{m})} \Big[F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}} | \mathfrak{m}_g] + F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}-1} | \mathfrak{m}_q] + F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}+1} | \mathfrak{m}_{\bar{q}}] \Big] \Big\rangle$$

$$+ \frac{\alpha_s}{2\pi} \Big[\big\langle I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big\rangle + \sum_x \Big\langle \mathcal{P}_{xa}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}}^{xb}[\mathcal{B}_{N,n}^{Q_{xb}}] + F_{\mathrm{LM}}^{ax}[\mathcal{B}_{N,n}^{Q_{ax}}] \otimes \mathcal{P}_{xb}^{\mathrm{NLO}} \Big\rangle \Big] \Bigg\}$$

General NNLO formulas (I)

The NNLO result is much more complicated

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{NNLO}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{FR}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{SU}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU}}$$

- Fully-regulated: two regulated unresolved particles $\to \left\langle \overline{S}_{\mathfrak{mn}} \overline{S}_{\mathfrak{n}} \Omega_1 \Delta^{(\mathfrak{mn})} \Theta_{\mathfrak{mn}} F_{\mathrm{LM,RR}}^{ab} \right\rangle$
- Single-unresolved: one regulated unresolved particles $\to \left\langle \mathcal{O}_{\mathsf{NLO}}^{(\mathfrak{m})} \Delta^{(\mathfrak{m})} F_{\mathrm{LM,R}}^{ab} \right\rangle$
- Double-unresolved: no regulated unresolved particles $\to \left\langle F_{\mathrm{LM}}^{ab} \right\rangle$

Focus on the double-unresolved term

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU}} = \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU,db}} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU,sb},a} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU,sb},b} + \mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU,el}} \,.$$

- Double-boosted contribution

$$\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU,db}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \sum_n \sum_{x,y} \left\langle \mathcal{P}_{xa}^{\mathrm{NLO}} \otimes F_{\mathrm{LM}}^{xy}[\mathcal{B}_{N,n}^{Q_{xy}}] \otimes \mathcal{P}_{yb}^{\mathrm{NLO}} \right\rangle$$

General NNLO formulas (II)

- Single-boosted contribution

$$
\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU,sb},a} = \sum_n \sum_x \Bigg\{ \left(\frac{\alpha_s}{2\pi}\right)^2 \Big[\big\langle \mathcal{P}_{xa}^{\mathrm{NNLO}} \otimes F_{\mathrm{LM}}^{xb}[\mathcal{B}_{N,n}^{Q_{xb}}] \big\rangle + \big\langle \mathcal{P}_{xa}^{\mathrm{NLO}} \otimes \big[I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LM}}^{xb}[\mathcal{B}_{N,n}^{Q_{xb}}] \big] \big\rangle
$$

$$
+ \delta_{xa} \big\langle \mathcal{P}_{aa}^{\mathcal{W}} \otimes \big[\mathcal{W}_a^{a\|\mathfrak{n},\mathrm{fin}} \cdot F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big] \big\rangle \Big] + \frac{\alpha_s}{2\pi} \big\langle \mathcal{P}_{xa}^{\mathrm{NLO}} \otimes F_{\mathrm{LV,fin}}^{xb}[\mathcal{B}_{N,n}^{Q_{xb}}] \big\rangle \Bigg\}
$$

- Elastic contribution

$$
\mathrm{d}\hat{\sigma}_{ab}^{\mathrm{DU,el}} = \sum_n \Bigg\{ \left(\frac{\alpha_s}{2\pi}\right)^2 \big\langle \big[I_{\mathrm{cc}}^{\mathrm{fin}} + I_{\mathrm{ss}}^{\mathrm{fin}} + I_{\mathrm{tri}}^{\mathrm{fin}} + I_{\mathrm{unc}}^{\mathrm{fin}} \big] \cdot F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big\rangle
$$

$$
+ \left(\frac{\alpha_s}{2\pi}\right)^2 \sum_{i\in\mathcal{H}} \Big\langle \Big[\theta_{\mathcal{H}_\mathrm{f}} \gamma_{z,f_i\to f_i g}^{\mathcal{W}} \, \mathcal{W}_i^{i\|\mathfrak{n},\mathrm{fin}} + \delta^{(0)} \mathcal{W}_i^{\mathfrak{m}\|\mathfrak{n},\mathrm{fin}} + \delta^{\perp,(0)} \mathcal{W}_{\mathrm{r}}^{(i)} \Big] \cdot F_{\mathrm{LM}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \Big\rangle
$$

$$
+ \frac{\alpha_s}{2\pi} \big\langle I_{\mathrm{T}}^{(0)} \cdot F_{\mathrm{LV,fin}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big\rangle + \big\langle F_{\mathrm{LV^2,fin}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big\rangle + \big\langle F_{\mathrm{LVV,fin}}^{ab}[\mathcal{B}_{N,n}^{Q_{ab}}] \big\rangle \Bigg\}
$$

Conclusions and outlooks

To summarize

- A general NNLO infrared regularization is a very challenging open problem
- I presented a solution for $n_\mathrm{f} = 1$ massless QCD
- Already extended to massless QCD for arbitrary n_f and $X = Z, W^\pm$

Looking at the near future I see

- Extension to more complicated color-singlets, e.g. $X = W^+W^+$
- A general numerical implementation
- Application to Higgs boson production in weak boson fusion with an extra jet

Thank you!