Quark mass effects in gradient flow observables

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Mass corrections

Aim:
Compute the quark mass effects of following gradient flow quantities to the three-loop level

- $S(t) = \langle \bar{\chi}(t)\chi(t)\rangle$,
- $R(t) = \langle \bar{\chi}(t) \overleftrightarrow{\not{D}} \chi(t)\rangle$,
- $E(t) = \frac{1}{4}\langle G_{\mu\nu}(t) G^{\mu\nu}(t)\rangle$.

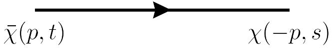
Motivation:
Mass effects of $S(t)$ and $R(t)$ can be used to supplement lattice data in precision determination of quark masses. [Takaura, Harlander & Lange 2025].

Gradient flow

- New flowed fields depend on a flowtime, t, e.g. for fermions

$$\partial_t \chi \;=\; \mathcal{D}^F_\mu \mathcal{D}^F_\mu \chi - \kappa \partial_\mu B^a_\mu T^a \chi.$$

- Ensure correct physics on boundary $\chi(t=0,x) = \psi(x)$.
- Uses: scale-setting, smearing, lattice-matching.
- Perturbation theory: new Feynman rules, e.g.

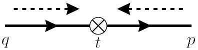


$$= \frac{-i\not{p} + m}{p^2 + m^2} e^{-(t+s)p^2}.$$

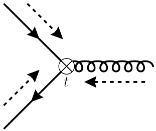
Flowed operators

Operator: $R_f(t) = \langle \bar{\chi}_f(t) \overleftrightarrow{\not{D}} \chi_f(t) \rangle$

Operator Feynman rules:



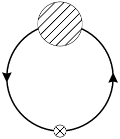
$= i(\not{p} - \not{q})$, and



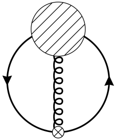
$= -2gT\gamma_\mu$,

The vacuum expectation value (vev) is calculated with

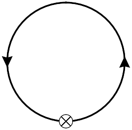
$R(t) =$



$+$



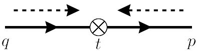
$=$



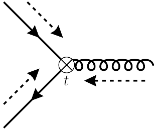
$+ \mathcal{O}(\alpha)$

Flowed operators

Operator: $R_f(t) = \langle \bar{\chi}_f(t) \overleftrightarrow{\not{D}} \chi_f(t) \rangle$
Operator Feynman rules:



$= i(\not{p} - \not{q}),$ and



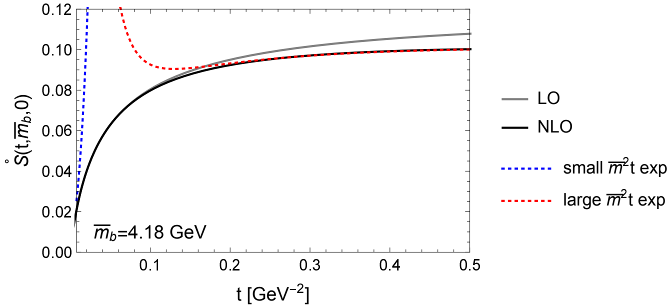
$= -2gT\gamma_\mu,$

The vacuum expectation value (vev) is calculated with

$$\begin{aligned} R(t) &= -2\frac{N_c}{(4\pi t)^2}\left[\int_p e^{-2tp^2} - m^2\int_p \frac{e^{-2tp^2}}{p^2+m^2}\right] + \mathcal{O}(\alpha) \\ &= -\frac{2N_c}{(4\pi t)^2} + \frac{2m^2 tN_c}{(4\pi t)^2}\left(1 - 2m^2 te^{2m^2 t}\Gamma(0, 2m^2 t)\right) + \mathcal{O}(\alpha). \end{aligned}$$

where $\Gamma(s, x)$ is the incomplete Γ-function.

Approximations to full mass expansion



Graph from [Takaura, Harlander & Lange 2024]

Closed form solutions for massive contributions not known at the two-loop level. Employ simplifications:

- Small mass expansion, i.e. expand in $m^2 t$.
- Large mass expansion, i.e. expand in $1/m^2 t$.
- Solve numerically for a set of values $m_i^2 t$.

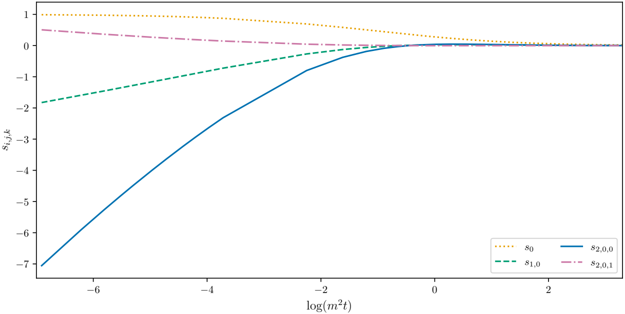
[Takaura, Harlander & Lange 2025]

Outline of calculations

Calculational setup:

- Choose an operator and calculate its Feynman rule: `frules` [Harlander & Geuskens (unpublished)].
- Consider diagrams contributing to the process and calculate their expansion in terms of gradient flow vacuum bubbles: `qgraf`, `tapir`, `exp` and `form`. [Nogueira 1991; Gerlach, Herren, Lang 2022; Harlander, Seidensticker, Steinhauser 1998; Seidensticker 1999; Vermaseren 1989].
- Numerically evaluate the integrals for each flow time: `ftint` [Harlander, Nellopoulos, Olsson & Wesle 2025].

$S_{i,j,k}$



Components of $S(t)$, $R(t)$ and $E(t)$ for 200 points in range of $m^2 t$

$$\frac{t}{m} S(t) = \frac{t}{m} \langle \bar{\chi}(t) \chi(t) \rangle \quad = \quad s_{i,j,k}(m^2 t) \alpha^i L_{\mu t}^j N_l^k$$

where $L_{mut} = \ln(2\mu^2 t) + \gamma_E$.

Measurable Ratios

Motivation:
Mass effects of $S(t) = \langle \bar{\chi}(t)\chi(t)\rangle$ and $R(t) = \langle \bar{\chi}_f(t) \overleftrightarrow{\not{D}} \chi_f(t)\rangle$ can be used to supplement lattice data in precision determination of quark masses.

[Takaura, Harlander & Lange 2025].

Renormalization:
In the gradient flow fields have multiplicative renormalization

$$\chi = \sqrt{Z_\chi}\chi_0, \quad \bar{\chi} = \sqrt{Z_\chi}\bar{\chi}_0,$$
$$B_\mu = \sqrt{Z_B}B_{0\mu}, \quad Z_B^{\overline{\text{MS}}} = 1,$$

but composite operators don't require additional renormalisation. Therefore

$$\mathring{S}(t) = \mathring{Z}_\chi S_0(t) \quad \text{and} \quad \mathring{R}(t) = \mathring{Z}_\chi R_0(t).$$

Measurable Ratios

Define "measurable" quantities

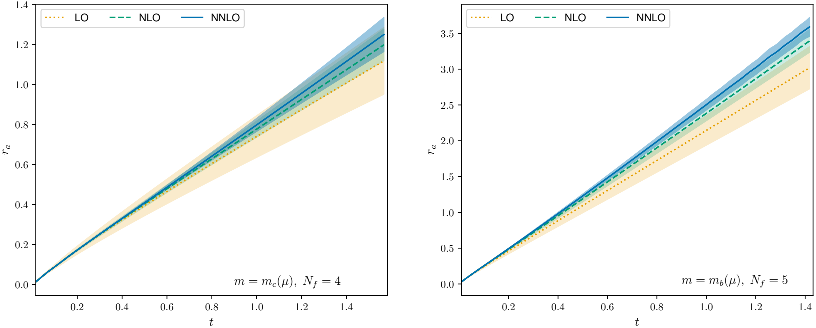
$$r_a(t) = \frac{\mathring{S}(t)}{\mathring{R}(t)} \quad = \quad \frac{S_0(t)}{R_0(t)},$$

$$r_b(t) = \frac{R(t)}{R(t)|_{m=0}} \propto \mathring{R}(t) \quad \text{and} \quad r_c(t) = m\frac{d}{dm}r_a(t),$$

- Bare quark masses can be calculated on the lattice by comparing lattice computations to experiment for e.g. some pion mass.
- These bare quark masses can be renormalized in the continuum directly by calculating one of these ratios, e.g.

$$m_R = z(r_a)m_B$$

r_a scale variation

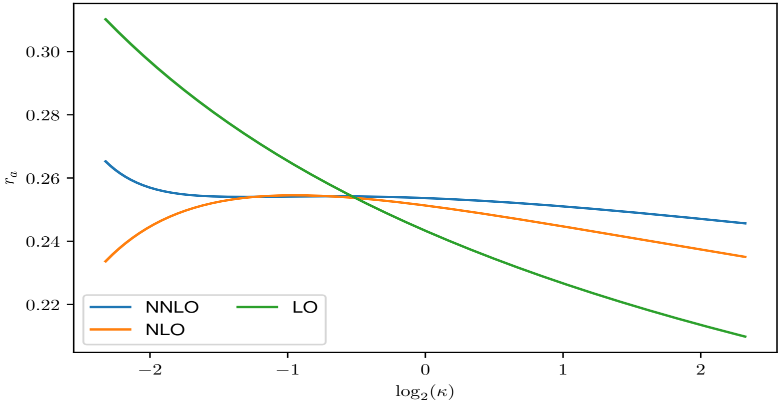


Running with RunDec: [K.G. Chetyrkin, J.H. Kühn, M. Steinhauser 2020, Navas et. al 2025]

$$m_b(m_b) = 4.183\text{GeV}, \quad m_c(m_c) = 1.273\text{GeV},$$

$$\mu = \kappa\mu_{\text{int}} = \kappa\sqrt{\frac{1}{2te^{\gamma_E}} + m^2}.$$

r_a scale variation



r_a plotted across a range of $\kappa = \mu/\mu_{\text{int}}$ for fixed values of t: 0.1GeV^{-2} with $m = m_b(\mu)$ $(r_b(t) = (S/R))$

Conclusions

We have computed the quark mass effects numerically of the following gradient flow quantities to the three-loop level

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along with measurable ratios.

Outlook:

- Have numerical evaluations of $r_a = S/R$, $r_b = R/R|_{m=0}$.
- Both should be comparable to lattice predictions of these quantities.
- $r_c = m\frac{d}{dm} r_a$ less sensitive to non-perturbative effects.
- Numerical differentiation introduces additional uncertainty.
- Precision analysis may require differentiation on integral level - evaluating new three-loop integrals.

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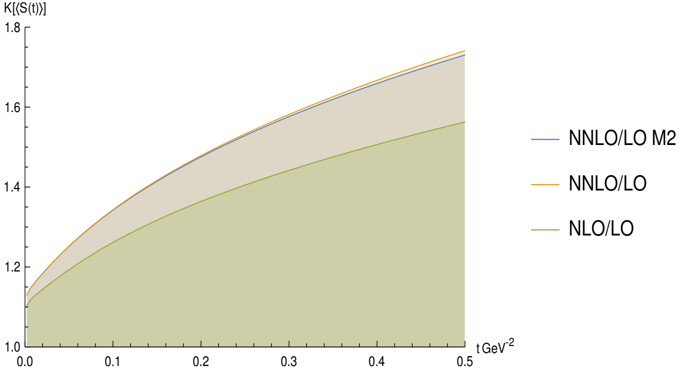
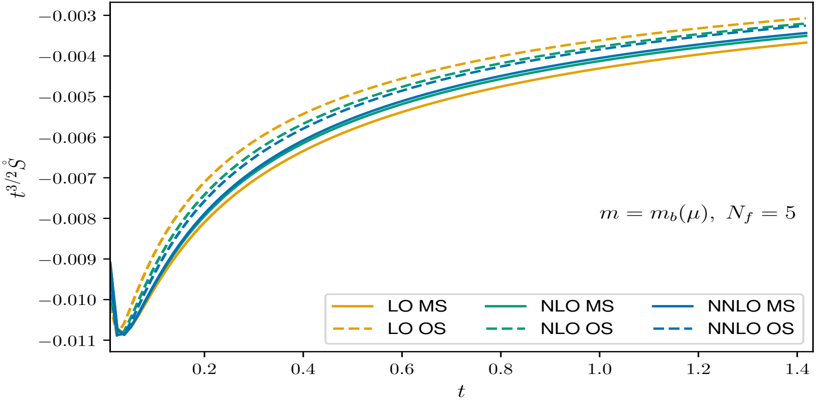
Any questions?

Backup: Second quark mass



Plot of $K[S(t)] = S(t)/S(t)|_{\text{LO}}$ for $N_f = 5$ with massive bottom and charm quarks in range $0 < t < 0.5\text{GeV}^{-2}$ $(S_f(t) = \langle \bar{\chi}_f(t)\chi_f(t) \rangle)$.

Backup: Uncertainty in $S(t)$



$$\mathring{S}^{\overline{\mathrm{MS}}}(t) = \mathring{S}^{\mathrm{OS}}(t),$$

$$\delta\mathring{S} \approx \frac{1}{2}\left(\mathring{S}(t)|_{\mu=2\mu_t} - \mathring{S}(t)|_{\mu=\mu_t/2}\right)$$

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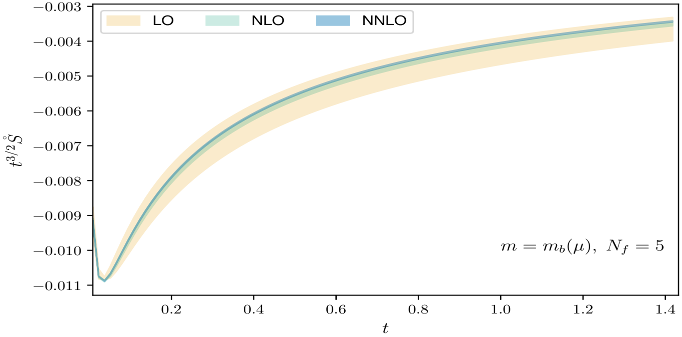
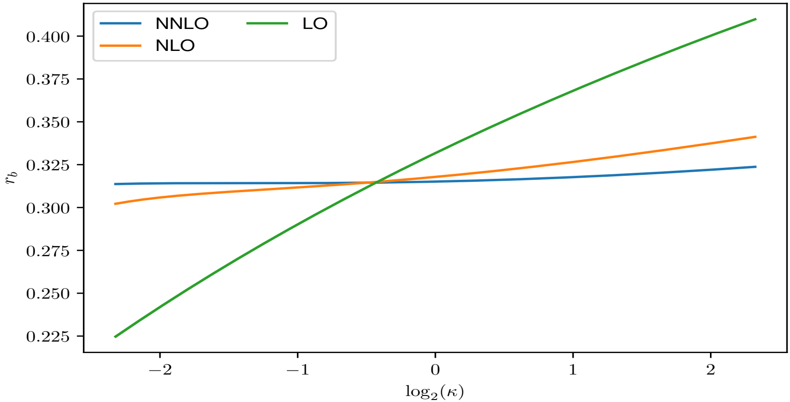


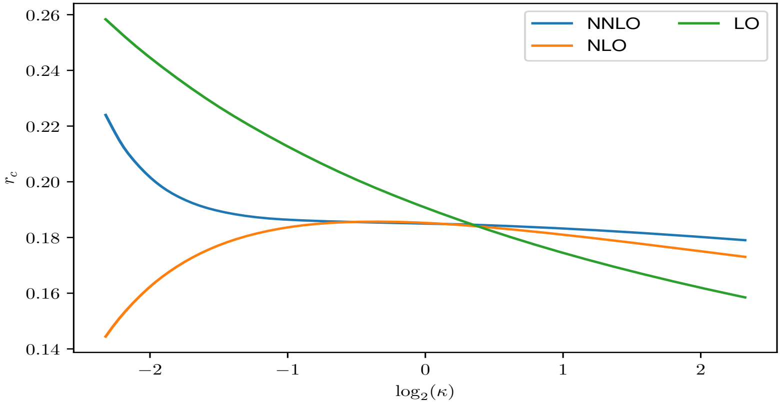
$$\mathring{S} \quad \propto \quad t^{-2}\frac{S_0}{R_0|_{m=0}} \Rightarrow \mu^2 \frac{d}{d\mu^2}\mathring{S} = 0,$$

$$\delta\mathring{S} \quad \approx \quad \tfrac{1}{2}\left(\mathring{S}(t)|_{\mu=2\mu_{\text{int}}} - \mathring{S}(t)|_{\mu=\mu_{\text{int}}/2}\right)$$

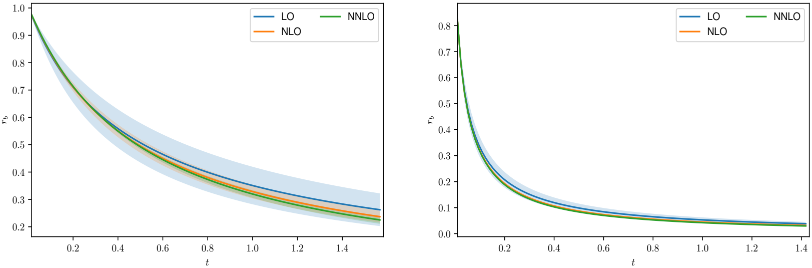


r_b plotted across a range of $\kappa = \mu/\mu_{\text{int}}$ for fixed values of t: 0.1GeV^{-2} with $m = m_b(\mu)$ $(r_b(t) = (R/R|_{m=0}))$

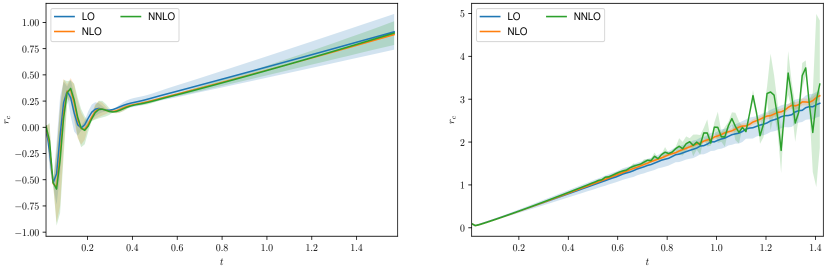
Backup: r_c scale variation



r_c plotted across a range of $\kappa = \mu/\mu_{\text{int}}$ for fixed values of t: 0.4GeV^{-2} with $m = m_b(\mu)$ ($r_c(t) = m\frac{d}{dm}(S/R)$)



r_b plotted against t in GeV^{-2} $N_f = 4$ (right) and $N_f = 5$ (left) with LO, NLO and NNLO plotted with an envelope error formed by scale variation.



r_a plotted against t in GeV^{-2} $N_f = 4$ (right) and $N_f = 5$ (left) with LO, NLO and NNLO plotted with an envelope error formed by scale variation.