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**PROJECT B1A (MELNIKOV/CZAKON): PRODUCTION OF COLOR SINGLET  
AT N3LO QCD AT THE LHC**

Kick-off meeting of the CRC 257, March 2019

There are plenty of reasons to be interested in the production of color singlet final states in hadron collisions:

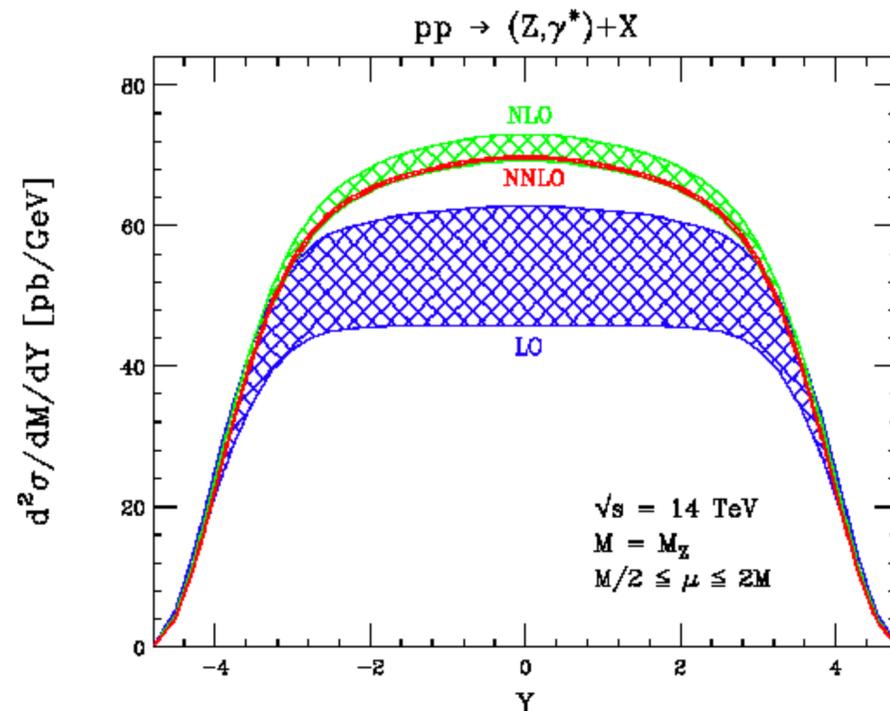
pp  $\rightarrow$  leptons: PDFs, W-mass, weak mixing angle;

pp  $\rightarrow$  Higgs: Higgs boson couplings;

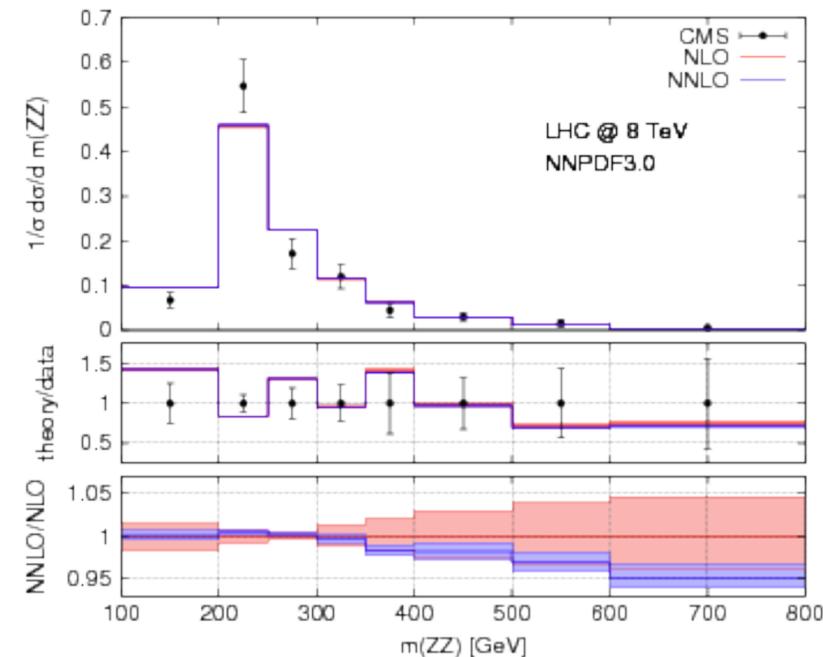
pp  $\rightarrow$  VH: Higgs boson couplings, Higgs decay to invisibles;

pp  $\rightarrow$   $V_1V_2$ : Higgs width, anomalous couplings, searches for new resonances etc.

All these processes are currently known through NNLO in perturbative QCD, fully differentially. Achieving this level of perturbative precision was the major undertaking in theoretical collider physics of the past 15 years.



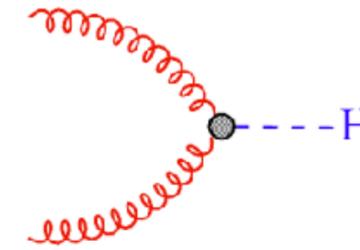
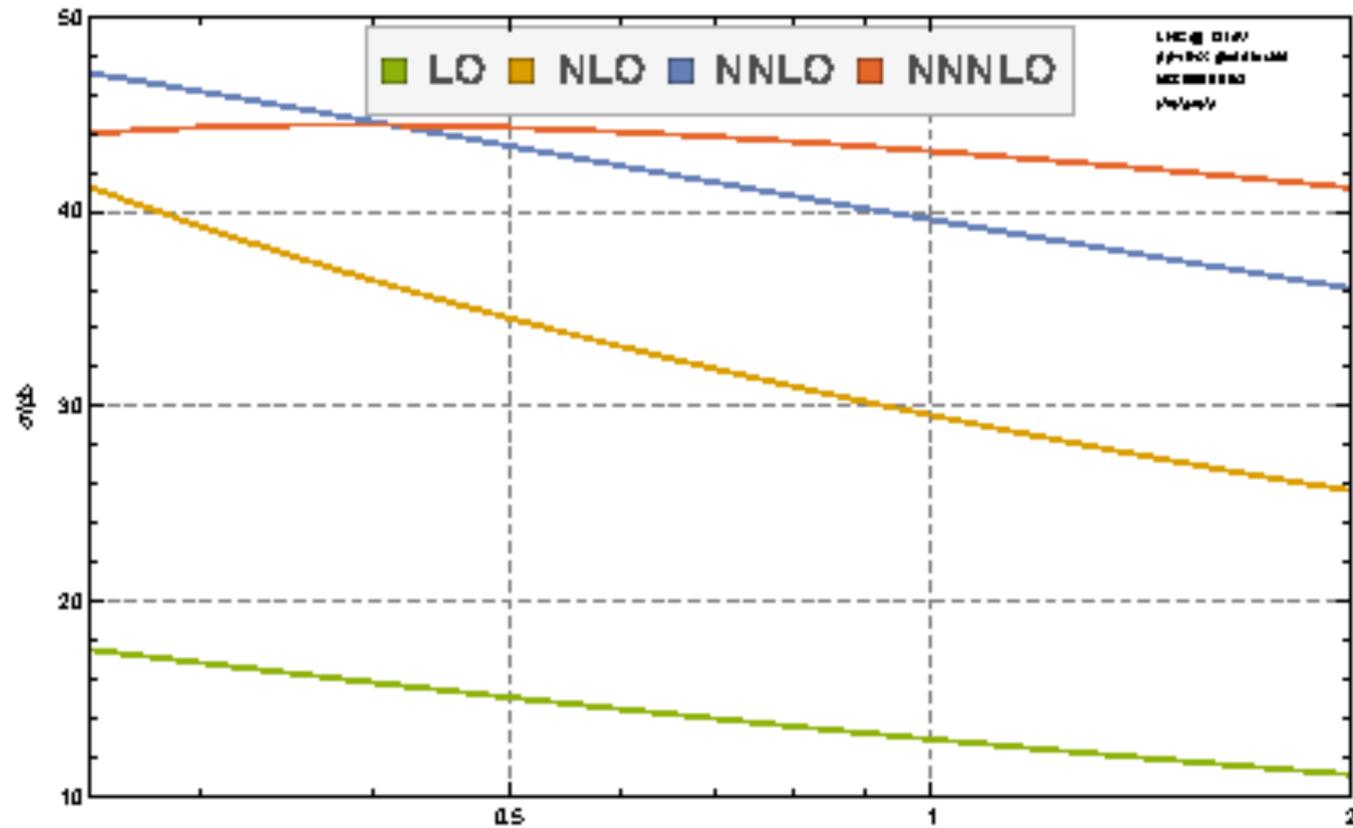
**C. Anastasioiu, L. Dixon, K. Melnikov and F. Petriello**



**M. Grazzini, S. Kallweit, P. Maierhoefer, D. Rathlev**

Although there are still some interesting things that need to be done at NNLO, attempts to extend the description of color-singlet production to N3LO have started.

Indeed, the Higgs production cross section has been computed through N3LO QCD a few years ago. The calculation employs [reverse unitarity](#) — a technical method that is particular to the total cross section or simple kinematic distributions.

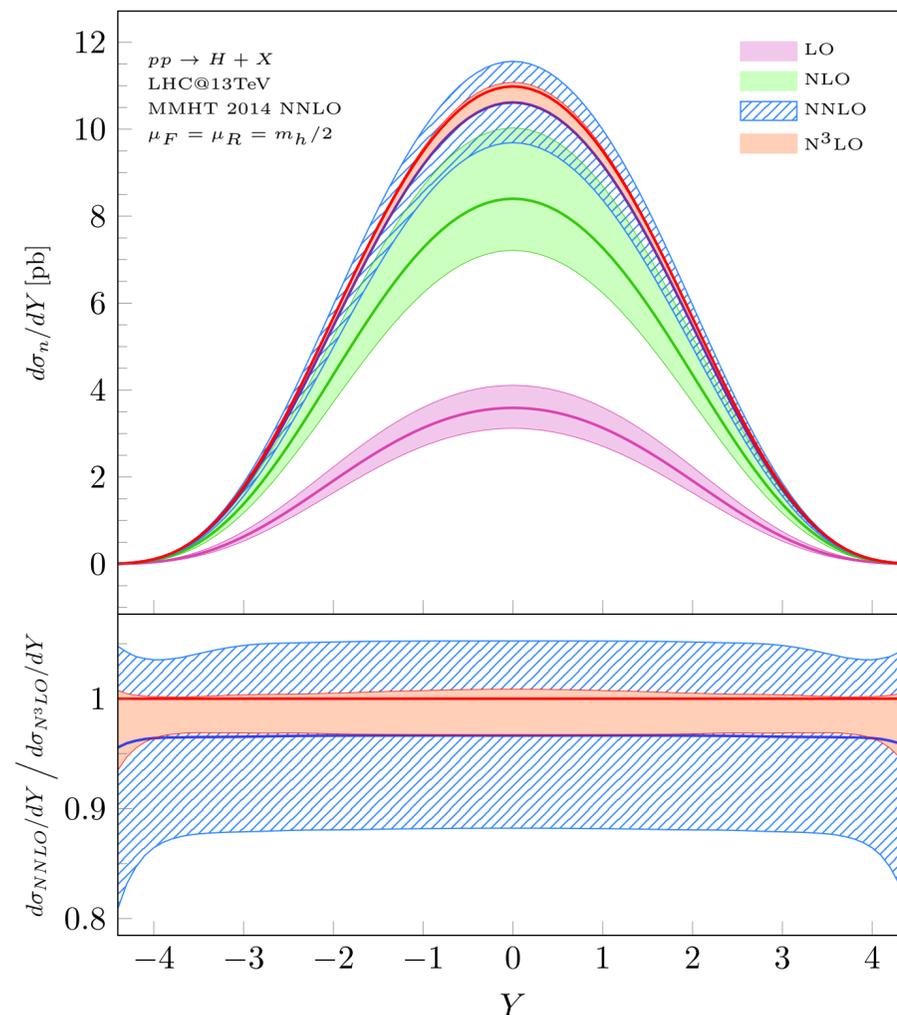


$\sigma/\text{pb}$	2 TeV	7 TeV	8 TeV	13 TeV	14 TeV
$\mu = \frac{m_H}{2}$	$0.99^{+0.43\%}_{-4.65\%}$	$15.31^{+0.31\%}_{-3.08\%}$	$19.47^{+0.32\%}_{-2.99\%}$	$44.31^{+0.31\%}_{-2.64\%}$	$49.87^{+0.32\%}_{-2.61\%}$
$\mu = m_H$	$0.94^{+4.87\%}_{-7.35\%}$	$14.84^{+3.18\%}_{-5.27\%}$	$18.90^{+3.08\%}_{-5.02\%}$	$43.14^{+2.71\%}_{-4.45\%}$	$48.57^{+2.68\%}_{-4.24\%}$

**Anastasiou, Duhr, Dulat, Furlan, Herzog, Gehrmann, Mistlberger etc.**

The scale dependence of the Higgs boson production cross section in gluon fusion in various orders of pQCD

Recently, this calculation was extended to cover Higgs rapidity distribution by [computing a few terms in the threshold expansion](#). By combining the N3LO prediction for the rapidity distribution and the NNLO prediction for H+j production, one can obtain fully differential predictions for Higgs boson production in gluon fusion.



Dulat, Mistlberger, Pelloni

Although, practically, this is a viable way to provide N3LO fully-differential description of Higgs boson production, it is not very appealing theoretically since we do not learn much from these computations.

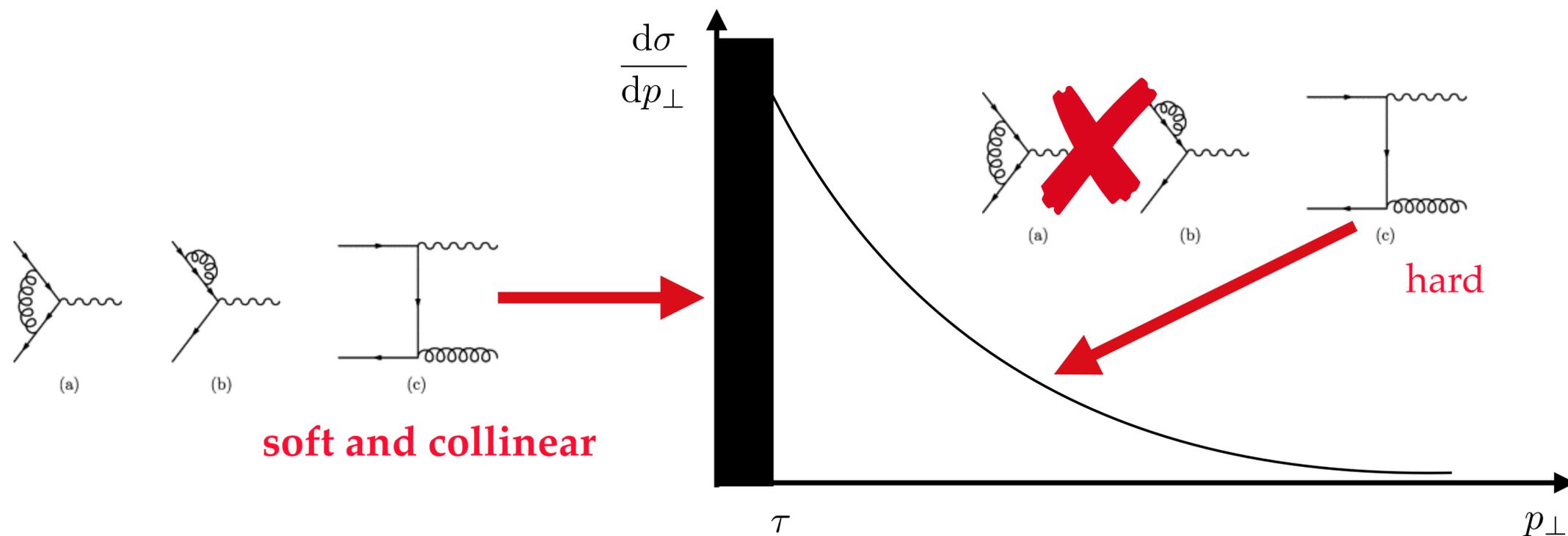
It would be interesting and instructive to develop alternative ways to perform fully-differential N3LO computations for generic color-singlet final states that make use of the fact that [the bulk of the calculation comes from H+j at NNLO and the only unknown contributions arise from soft and collinear dynamics](#).

For cases with more than one particle in the final state we will, eventually, need three-loop virtual amplitudes — that are not known — but this is a long shot anyway...

Suppose we want to know the Drell-Yan process  $qq \rightarrow V$  at N3LO in pQCD. We do not know how to do that but we do know how to compute the production of the V-boson in association with a jet at NNLO.

Interestingly, the two processes  $pp \rightarrow V+j$ @NNLO and  $pp \rightarrow V$ @N3LO are proportional to the same power in the strong coupling constant which means that the former is part of the latter.

However,  $pp \rightarrow V+j$ @NNLO and  $pp \rightarrow V$ @N3LO are physically different. The former requires a kinematic feature to distinguish it from the latter. Such a feature can e.g. be the transverse momentum of a V in the final state. It can be used to split the full N3LO phase-space into a V+j NNLO part and the genuine V+0 jets N3LO part. This is the idea behind the so-called slicing method.



Let us call the kinematic variable that provides such a separation  $\omega$ . Considering two distinct cases — when this variable is larger or smaller than an infinitesimal parameter  $\tau$ , we can write a fully differential cross section for the  $V$  production in the following way

$$d\sigma_{pp \rightarrow V}(\omega, \dots) = d\sigma_{pp \rightarrow V+j} \theta(\omega - \tau) + \delta(\omega) \int_0^\tau d\omega' \left[ \lim_{\omega' \rightarrow 0} [d\sigma_{pp \rightarrow V+j}] + \delta(\omega') d\sigma_{pp \rightarrow V} \right] + \mathcal{O}(\tau)$$

The integral provides non-vanishing contribution in the  $\tau \rightarrow 0$  limit if and only if there are terms in the integrand that are singular in the  $\omega \rightarrow 0$  limit. Hence, to find unresolved contributions we need to know the singular terms

$$\lim_{\omega \rightarrow 0} d\sigma_{pp \rightarrow V+j} \sim \omega^{-1+\#\epsilon}$$

If the limit of  $V+j$  for small values of the slicing variable is “simple enough to be computed”, we get a practical tool for extending a  $V+j$  NNLO computation to a fully differential N3LO computation for  $V$  production.

Simplifications in kinematic limits often (perhaps even always) imply the existence of [factorization theorems](#). For color-singlet production, these theorems are known for two kinematic variables: [the transverse momentum and the N-jettiness](#). A generic factorization theorem reads

$$\lim_{\omega \rightarrow 0} d\sigma_{pp \rightarrow V}(\omega, \dots) = B_\omega \otimes B_\omega \otimes S_\omega \otimes H \otimes d\sigma_V^{\text{LO}},$$

where B,S and H are the so-called beam, soft and hard functions, respectively.

In case of vector boson production, the hard function renormalizes the  $qQ \rightarrow V$  vertex (the form factor). It is proportional to  $\delta(\omega)$  and it is known to three loops.

Beam and soft functions address two distinct ways in which a V-boson with Born-like kinematics can be produced: the beam function describes effects of hard collinear emissions, the soft function — effects of soft emissions.

Both of these functions pick up a particular singular contributions of  $d\sigma_{pp \rightarrow V}$  in the  $\omega \rightarrow 0$  limit.

Beam and soft functions depend on the kinematic variable  $\omega$ . For example, in case of the transverse-momentum, all corrections to the soft function vanish and only the beam function needs to be computed. On the contrary, in case of N-jettiness, both the beam and the soft functions contribute and have to be calculated.

$$\lim_{\omega \rightarrow 0} d\sigma_{pp \rightarrow V}(\omega, \dots) = B_\omega \otimes B_\omega \otimes S_\omega \otimes H \otimes d\sigma_V^{\text{LO}}$$

**The transverse momentum slicing:** although the soft function does not receive quantum corrections, dimensional regularization is insufficient to make integrals that contribute to the beam function well-defined; **additional (analytic) regulator is required.**

**The N-jettiness slicing:** there are non-trivial corrections to the soft function but dimensional regularization is sufficient to make all the relevant integrals well-defined.

Both the transverse momentum and the N-jettiness have been used for NNLO computations; I am not aware of significant differences between the two slicing variables in terms of numerical performance (more comments below).

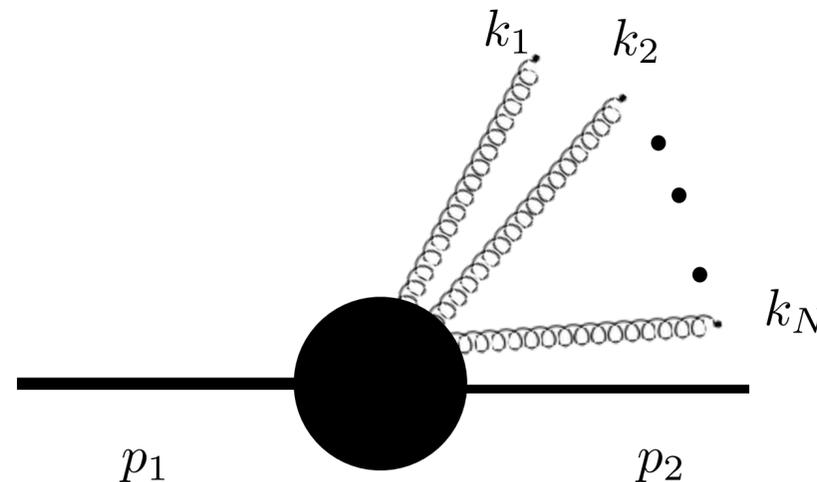
For the production of a color-singlet at N3LO, the relevant N-jettiness variables is the **zero-jettiness**. It is defined as follows

$$\mathcal{T} = \sum_j \min_{i \in \{1,2\}} \left[ \frac{2p_i \cdot k_j}{Q_i} \right]$$

The jettiness constraint simplifies in the collinear limit where we need to **compute the perturbative matching between beam functions and parton distribution functions**.

$$B_i(t, x, \mu) = \sum_j \int dx_1 dx_2 I_{ij}(t, x_1, \mu) f_j(x_2, \mu) \delta(x - x_1 x_2)$$

**In case of the soft function**, the zero-jettiness constraint does not simplify and the required analytic computations are quite challenging. However, numerical techniques for computing soft functions through NNLO have been developed and, perhaps, they can be extended to one order higher. Moreover, in case of zero-jettiness the soft function is just a collection of numbers and, once these numbers are known, they can be used universally.

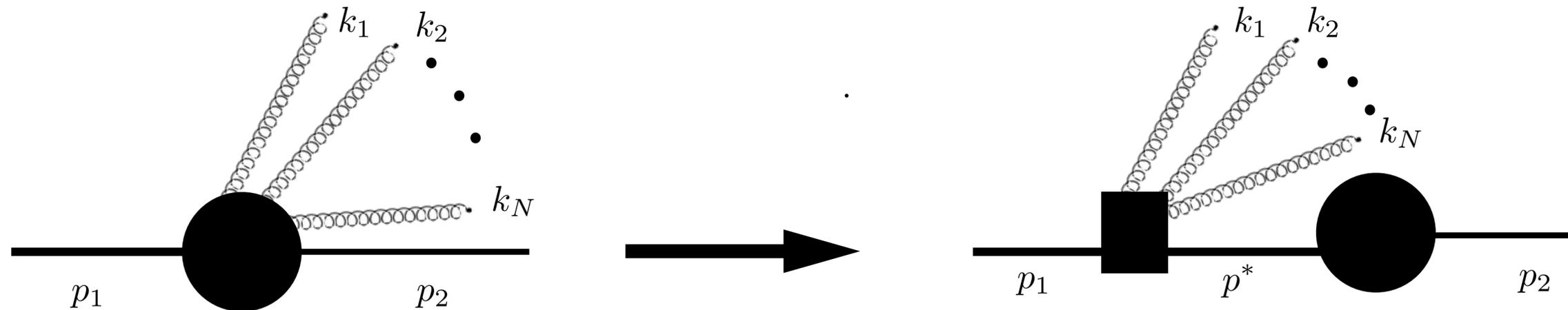


In the collinear limit, the zero-jettiness variable simplifies since one scalar product is always smaller than the other.

$$\mathcal{T} = \sum_j \min_{i \in \{1,2\}} \left[ \frac{2p_i \cdot k_j}{Q_i} \right] \xrightarrow{\text{collinear emissions along } p_1} \min_{i \in \{1,2\}} [2p_i \cdot k_j] \Rightarrow 2p_1 \cdot k_j, \quad \text{for } \forall j$$

In the collinear limit, singular effects are only caused by emissions off the relevant external leg but only if physical gluon polarizations are used in the calculation.

$$p^* = zp_1 + \dots \qquad t = z \sum_{i=1}^N 2p_1 \cdot k_i \qquad s(1-z) = \sum_{i=1}^N 2p_2 \cdot k_i$$



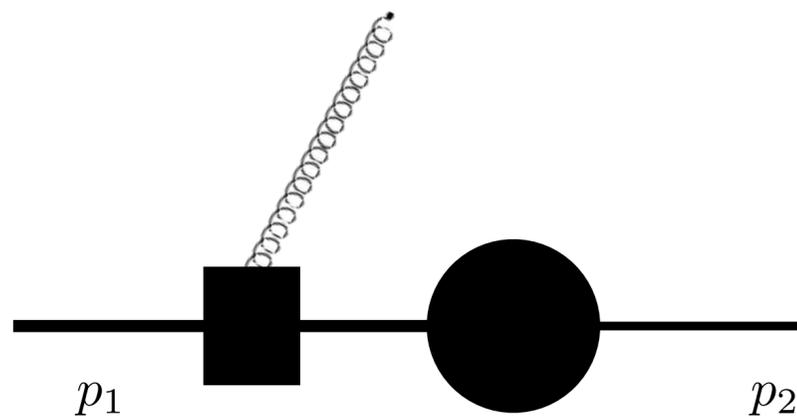
$$I_N(t, z) \sim \int \prod_{i=1}^N \frac{d^{d-1}k_i}{(2\pi)^{d-1} 2k_i^0} \delta \left( s(1-z) - \sum_{i=1}^N 2p_2 \cdot k_i \right) \delta \left( t - z \sum_{i=1}^N 2p_1 \cdot k_i \right) \frac{\hat{C}_p [\mathcal{M}(p_1, p_2, \{k_i\})^2]}{|\mathcal{M}_0(zp_1, p_2)|^2}$$

The matching coefficient is obtained after integrating the collinearly-projected matrix elements squared over constrained phase space. **N.B.:** Collinear projections typically give splitting functions, either tree-level or loop.

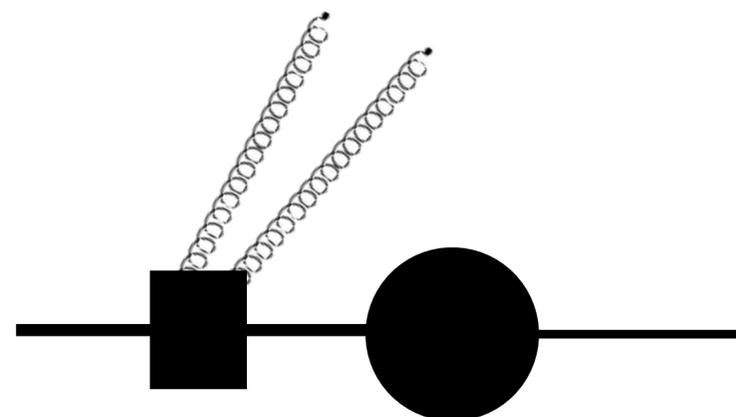
At N3LO, there are three contributions to the matching coefficient of the beam function. Since, in physical gauges, all the collinear-enhanced contributions factorize onto external lines, it may seem that loops (3 point 2-loop at most) are easy. However, the main challenge comes from the need to **use physical gauges for all gauge particles, both real and virtual**. When the physical gauge is used, additional “propagators” appear and lead to a large number of unconventional integrals that need to be calculated.



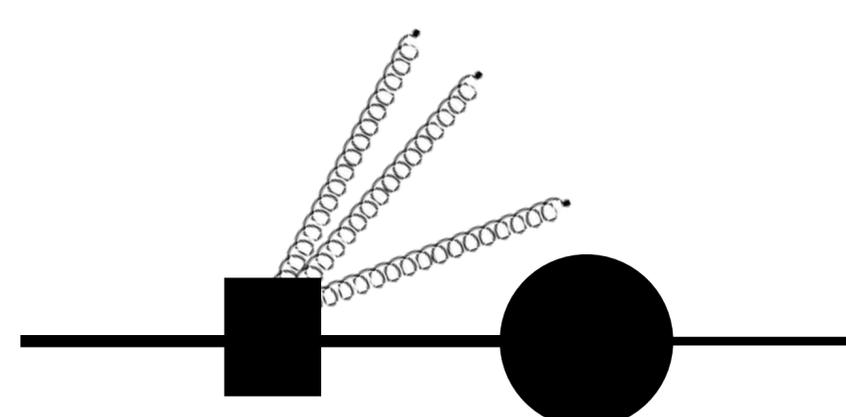
$$P_{\mu\nu}^{ab} = \frac{-i\delta^{ab}}{k^2 + i0} \left( -g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} \right)$$



2-loops, 1 emission



1-loop, 2 emissions



0-loops, 3 emissions

All the different contributions can be calculated by 1) mapping phase-space integrals onto loop integrals (reverse unitarity); 2) using integration-by-parts identities to reduce them to the smallest set that needs to be computed and 3) by solving differential equations that these master integrals satisfy.

The differential equations for master integrals are **ordinary differential equations** (in  $z$ ) since the dependence on the off-shellness/jettiness parameter factorizes. These equations can be put into a canonical Fuchsian (epsilon) form. **It is interesting that the natural variable, which makes differential equations rational, is different from  $z$ .**

$$z = \frac{(1+x)^2}{x} \quad \sqrt{z(z-4)} \rightarrow \frac{(1+x)(1-x)}{x}$$

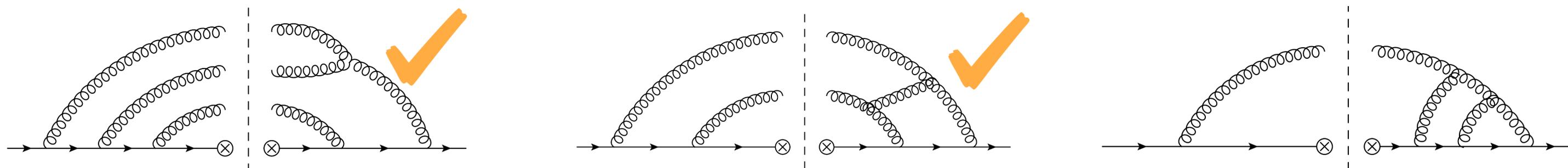
For  $z$  between 0 and 1,  $x$  is complex-valued.

The boundary conditions for systems of differential equations are obtained by considering the soft,  $z \rightarrow 1$ , limit. Plenty of non-trivial boundary integrals have to be computed; this is done using a variety of methods.

We have so far analytically computed the triple-real (almost all) and the double-real single-virtual integrals that are needed to determine the quark beam function at N3LO QCD.

**K. M., R. Rietkerk, L. Tancredi, Ch. Wever**

The remaining triple-real integrals that are unknown are related to  $q \rightarrow qqq^*$  splittings (identical quarks and are subleading in  $N_c$ ). Calculation of remaining triple-real integrals — that will also be required for the gluon beam function — remains difficult but most likely can be tackled using the already developed methods.



A calculation of the zero-jettiness soft function presents an interesting problem. According to the definition of zero-jettiness, the observable depends on the relative magnitude of gluon momenta components along the momenta of incoming quark and anti-quark.

$$\mathcal{T} = \sum_j \min_{i \in \{1,2\}} \left[ \frac{2p_i \cdot k_j}{Q_i} \right] \quad k_i = \alpha_i p_1 + \beta_i p_2 + k_{i,\perp} \quad \mathcal{T} = \sum_{j=1}^3 \min(\alpha_j, \beta_j)$$

$$\begin{aligned} \delta(\mathcal{T} - \tau) &= \theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)\delta(\beta_1 + \beta_2 + \beta_3 - \tau) \\ &\quad + \theta(\beta_1 - \alpha_1)\theta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)\delta(\alpha_1 + \beta_2 + \beta_3 - \tau) + \dots \end{aligned}$$

$$S_{RRR}(\tau) = \int \prod_{i=1}^3 \frac{d^d k_i}{(2\pi)^d} \delta(k_i^2) \delta(\mathcal{T} - \tau) \text{Eik}(\{k_i\}, p_1, p_2)$$

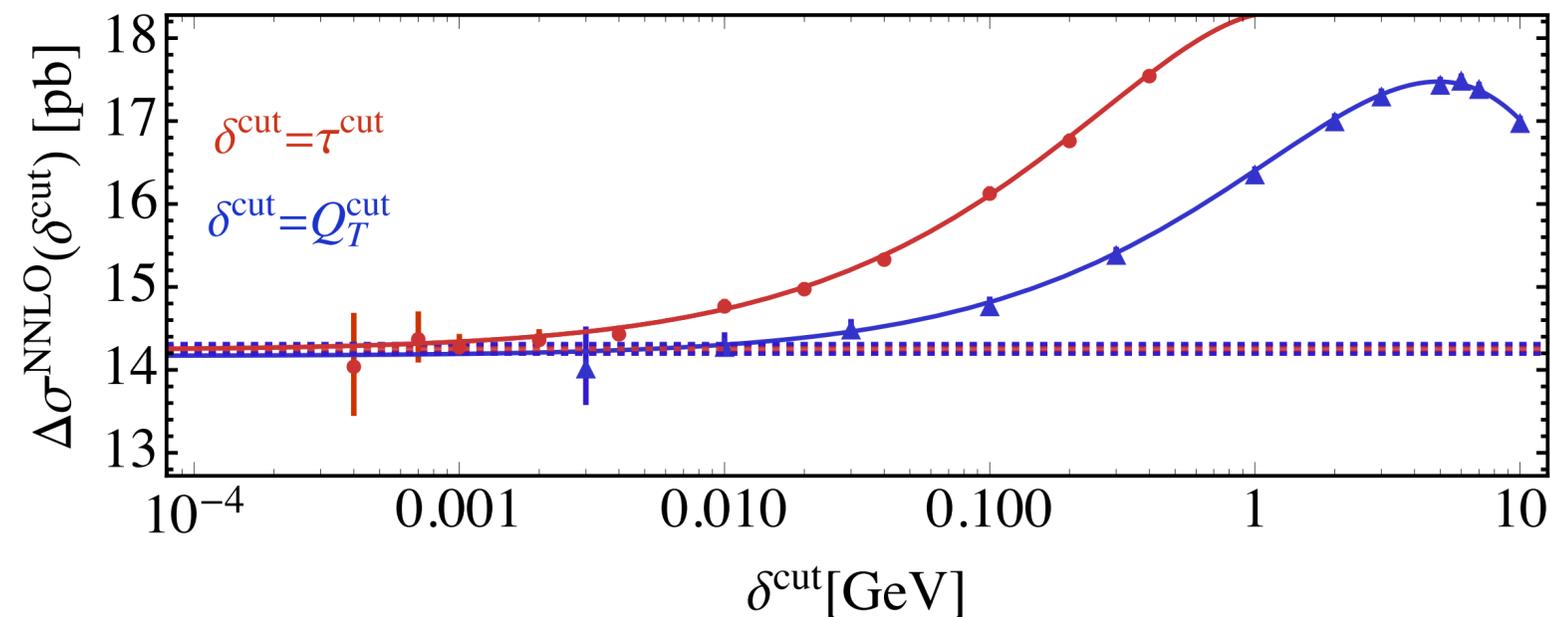
A standard way to compute constrained phase-space integrals is to use reverse unitarity which consists in replacing delta-functions in the integrand with “propagator-like” quantities and using integration-by-part identities for reducing the original integrals to a few master integrals and for deriving differential equations that these integrals satisfy (where appropriate).

$$\delta(F(\{k_i, p_i\})) \rightarrow \frac{1}{F(\{k_i, p_i\})}$$

However, in case of the soft function, we have to deal with the [theta-functions in the integrand](#) and it is an interesting (open) question what to do in that case.

The last ingredient that is required to calculate the cross section for  $pp \rightarrow V$  at N3LO is the fully-differential NNLO QCD computation for  $pp \rightarrow V+j$ . Such calculations have already been done by several groups for  $V=Z,W$ , photon and also for  $pp \rightarrow H+j$ ; so, in principle, we know how to perform them.

However, we will require these NNLO computations for very small values of the slicing parameter because of the relatively large contribution of power-suppressed terms. This implies that, to have a chance to succeed, the NNLO computation of the  $V+j$  process should be made very efficient and numerically stable. Achieving this is not trivial and will require a separate effort.



Campbell, Ellis, Li, Williams

The dependence of the NNLO contribution to  $\sigma(pp \rightarrow \gamma\gamma)$  production cross section at the 14 TeV LHC on the slicing parameter. In this case, the NNLO QCD corrections are large and need to be known precisely.

The goal of the project B1a is to push theoretical description of the color-singlet production at the LHC to N3LO in perturbative QCD, fully differentially. Without a valid subtraction scheme at N3LO, one can resort to slicing to simplify computations in singular kinematic limits.

Singular limits are described in terms of beam and soft functions that need to be computed through N3LO. They are later combined with fully-differential computations of  $X+j$  at NNLO. We are in the process on pushing our way through the different ingredients required for the computation of the matching coefficient of quark beam function.

Extension of these computations to the gluon beam function and the soft functions will require significant further effort and, in case of the soft function, perhaps even new insights.

Knowledge of beam and soft functions at higher orders in perturbation theory is also interesting in its own right since factorization theorems can be used to improve resummed predictions for relevant processes; I did not talk about this but it is a clear possibility.

$$\lim_{\omega \rightarrow 0} d\sigma_{pp \rightarrow V}(\omega, \dots) = B_\omega \otimes B_\omega \otimes S_\omega \otimes H \otimes d\sigma_V^{\text{LO}}$$