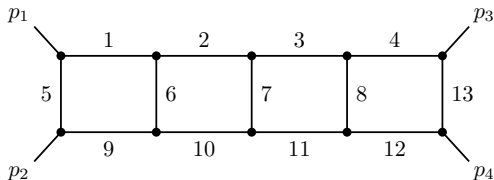


Differential Equations from Intersection Numbers

Giacomo Brunello

with **Vsevolod Chestnov** and **Ben Page**

in preparation



Challenge

Master integrals obey **differential equations**:

$$d_x \mathbf{I} = \Omega(x, \varepsilon) \mathbf{I}$$

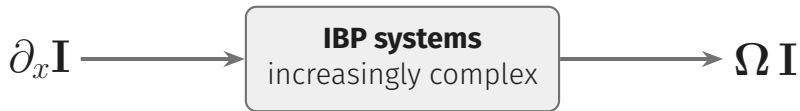

dim. reg.: $D = 4 - 2\varepsilon$
physics at $\varepsilon \rightarrow 0$

Challenge

Master integrals obey **differential equations**:

$$d_x \mathbf{I} = \Omega(x, \varepsilon) \mathbf{I}$$

dim. reg.: $D = 4 - 2\varepsilon$
physics at $\varepsilon \rightarrow 0$



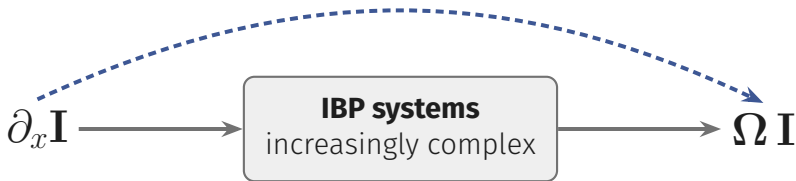
Challenge

Master integrals obey **differential equations**:

$$d_x \mathbf{I} = \Omega(x, \varepsilon) \mathbf{I}$$

dim. reg.: $D = 4 - 2\varepsilon$
physics at $\varepsilon \rightarrow 0$

bypass: **closed formulae**



Observation

Ω has a **polynomial** dependence on ε

$$\Omega = \varepsilon^p \Omega^{(p)} + \dots + \varepsilon \Omega^{(1)} + \Omega^{(0)}$$

Observation

Ω has a **polynomial** dependence on ε

$$\Omega = \varepsilon^p \Omega^{(p)} + \dots + \varepsilon \Omega^{(1)} + \Omega^{(0)} \xrightarrow{\text{canonical}} \Omega = \varepsilon \Omega^{(1)}$$

ε -factorised [Henn '13]

Observation

Ω has a **polynomial** dependence on ε

$$\Omega = \varepsilon^p \Omega^{(p)} + \dots + \varepsilon \Omega^{(1)} + \Omega^{(0)} \xrightarrow{\text{canonical}} \Omega = \varepsilon \Omega^{(1)}$$

ε -factorised [Henn '13]

physics, $D \rightarrow 4$



Observation

Ω has a **polynomial** dependence on ε

$$\Omega = \varepsilon^p \Omega^{(p)} + \dots + \varepsilon \Omega^{(1)} + \Omega^{(0)} \xrightarrow{\text{canonical}} \Omega = \varepsilon \Omega^{(1)}$$

ε -factorised [Henn '13]



Observation

Ω has a **polynomial** dependence on ε

$$\Omega = \varepsilon^p \boxed{\Omega^{(p)}} + \dots + \varepsilon \boxed{\Omega^{(1)}} + \boxed{\Omega^{(0)}} \xrightarrow{\text{canonical}} \Omega = \varepsilon \boxed{\Omega^{(1)}} \\ \varepsilon\text{-factorised [Henn '13]}$$



Closed formulae from Intersection Numbers

Outline

**Intersection
numbers**

project onto
master integrals



ε



∞



δ



0



**beautiful
formulae**

build differential
equations

localise on
critical points

factorise over
sectors

Feynman integrals as twisted period integrals

Lee–Pomeransky representation: [Lee, Pomeransky '13]

$$I_{a_1 \dots a_n} \propto \int_{\mathbb{R}_+^n} \prod_{i=1}^n \frac{dz_i z_i^{a_i-1}}{\Gamma(a_i)} \mathcal{G}^{\varepsilon-2}$$

$$D = 4 - 2\varepsilon$$

$$\mathcal{G} = \mathcal{U} + \mathcal{F}$$

Feynman integrals as twisted period integrals

Lee–Pomeransky representation: [Lee, Pomeransky '13]

$$I_{a_1 \dots a_n} \propto \int_{\mathbb{R}_+^n} \prod_{i=1}^n \frac{dz_i z_i^{a_i-1}}{\Gamma(a_i)} \mathcal{G}^{\varepsilon-2}$$

$$D = 4 - 2\varepsilon$$

$$\mathcal{G} = \mathcal{U} + \mathcal{F}$$

ill defined for $a_i \leq 0$

Feynman integrals as twisted period integrals

Lee–Pomeransky representation, with an **analytic regulator** δ : [Lee, Pomeransky '13]

$$I_{a_1 \dots a_n}^{(\delta)} \propto \int_{\mathbb{R}_+^n} \prod_{i=1}^n \frac{dz_i z_i^{a_i-1} + \varepsilon \delta}{\Gamma(a_i + \varepsilon \delta)} \mathcal{G}^{\varepsilon-2}$$

$$D = 4 - 2\varepsilon$$

$$\mathcal{G} = \mathcal{U} + \mathcal{F}$$

Feynman integrals as twisted period integrals

Lee–Pomeransky representation, with an **analytic regulator** δ : [Lee, Pomeransky '13]

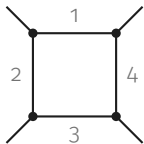
$$I_{a_1 \dots a_n}^{(\delta)} \propto \int_{\mathbb{R}_+^n} \prod_{i=1}^n \frac{dz_i z_i^{a_i-1} {}^{+\varepsilon\delta}}{\Gamma(a_i {}^{+\varepsilon\delta})} \mathcal{G}^{\varepsilon-2}$$

$$D = 4 - 2\varepsilon$$

$$\mathcal{G} = \mathcal{U} + \mathcal{F}$$

$a_i > 0$: denominator

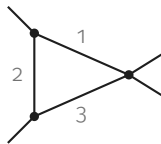
$a_i \leq 0$: numerator (subtopology)



sector

$$a_1, \dots, a_4 > 0$$

pinch 4



subsector

$$a_4 = 0$$

Feynman integrals as twisted period integrals

Lee–Pomeransky representation, with an **analytic regulator** δ : [Lee, Pomeransky '13]

$$I_{a_1 \dots a_n}^{(\delta)} \propto \int_{\mathbb{R}_+^n} \prod_{i=1}^n \frac{dz_i z_i^{a_i-1+\varepsilon\delta}}{\Gamma(a_i + \varepsilon\delta)} \mathcal{G}^{\varepsilon-2}$$

$$D = 4 - 2\varepsilon$$

$$\mathcal{G} = \mathcal{U} + \mathcal{F}$$

$$\mathcal{G}^\varepsilon \prod_{i=1}^n z_i^{\varepsilon\delta} = e^{\varepsilon W_\delta}$$

potential: $W_\delta = \log \mathcal{G} + \delta \sum_{i=1}^n \log z_i$

Feynman integrals as twisted period integrals

Lee–Pomeransky representation, with an **analytic regulator** δ : [Lee, Pomeransky '13]

$$I_{a_1 \dots a_n}^{(\delta)} \propto \int_{\mathbb{R}_+^n} \prod_{i=1}^n \frac{dz_i z_i^{a_i-1+\varepsilon\delta}}{\Gamma(a_i+\varepsilon\delta)} \mathcal{G}^{\varepsilon-2}$$

$$D = 4 - 2\varepsilon$$

$$\mathcal{G} = \mathcal{U} + \mathcal{F}$$

$$\mathcal{G}^\varepsilon \prod_{i=1}^n z_i^{\varepsilon\delta} = e^{\varepsilon W_\delta}$$

potential: $W_\delta = \log \mathcal{G} + \delta \sum_{i=1}^n \log z_i$

Twisted period integral: [Mastrolia, Mizera '18]

$$I^{(\delta)} \propto \int_{\mathbb{R}_+^n} e^{\varepsilon W_\delta} \varphi$$

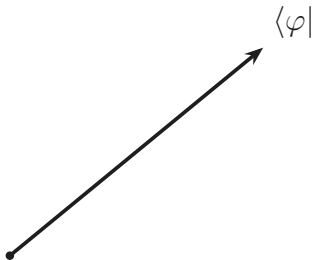
multivariate twist

differential form: $\varphi = \hat{\varphi} d^n z$

$$\hat{\varphi} \propto \frac{1}{\mathcal{G}^2} \prod_i z_i^{a_i-1}$$

Vector space of Feynman Integrals

Forms modulo IBPs form a **vector space**

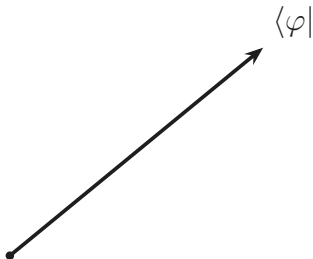


Vector space of Feynman Integrals

Forms modulo IBPs form a **vector space**

$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

finite dimension $\nu = \#$ of critical points of W_δ



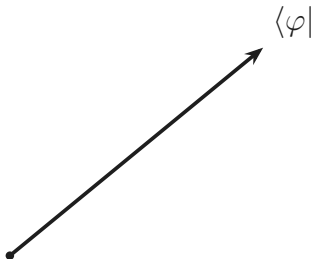
Vector space of Feynman Integrals

Forms modulo IBPs form a **vector space**

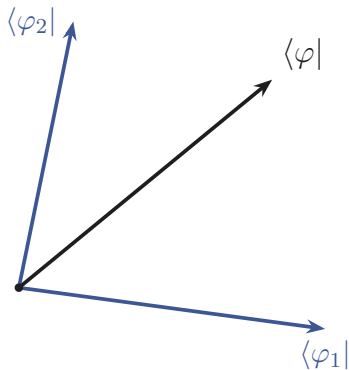
$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

finite dimension $\nu = \#$ of critical points of W_δ

$$\frac{\partial_i \mathcal{G}}{\mathcal{G}} + \frac{\delta}{z_i} = 0, \quad i = 1, \dots, n$$



Vector space of Feynman Integrals



Forms modulo IBPs form a **vector space**

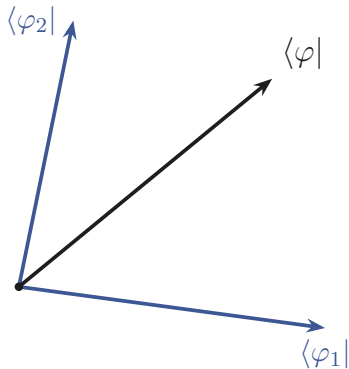
$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

finite dimension $\nu = \#$ of critical points of W_δ

$$\frac{\partial_i \mathcal{G}}{\mathcal{G}} + \frac{\delta}{z_i} = 0, \quad i = 1, \dots, n$$

decomposition $\langle \varphi | = c_1 \langle \varphi_1 | + \dots + c_\nu \langle \varphi_\nu |$

Vector space of Feynman Integrals



Forms modulo IBPs form a **vector space**

$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

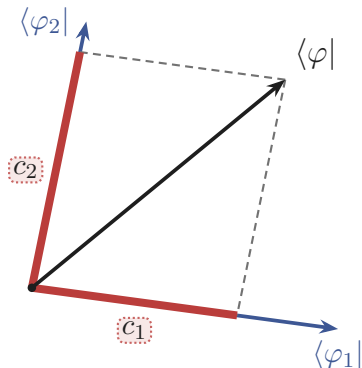
finite dimension $\nu = \#$ of critical points of W_δ

$$\frac{\partial_i \mathcal{G}}{\mathcal{G}} + \frac{\delta}{z_i} = 0, \quad i = 1, \dots, n$$

decomposition $\langle\varphi| = c_1 \langle\varphi_1| + \dots + c_\nu \langle\varphi_\nu|$

$\langle\varphi_a|$: **master forms** $\leftrightarrow$ master integrals

Vector space of Feynman Integrals



Forms modulo IBPs form a **vector space**

$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

finite dimension $\nu = \#$ of critical points of W_δ

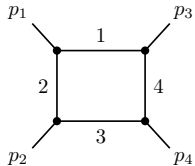
$$\frac{\partial_i \mathcal{G}}{\mathcal{G}} + \frac{\delta}{z_i} = 0, \quad i = 1, \dots, n$$

decomposition $\langle \varphi | = c_1 \langle \varphi_1 | + \dots + c_\nu \langle \varphi_\nu |$

$\langle \varphi_a |$: **master forms** $\leftrightarrow$ master integrals

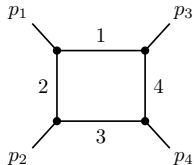
c_i : from IBPs or **Intersection Numbers**

Example: the one-loop box



$$y = t/s$$

Example: the one-loop box

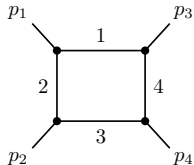


$$y = t/s$$

$$\mathcal{G} = z_1 + z_2 + z_3 + z_4 + z_1 z_3 + y z_2 z_4$$

$$W_\delta = \log \mathcal{G} + \delta \log(z_1 z_2 z_3 z_4)$$

Example: the one-loop box



$$y = t/s$$

$$\mathcal{G} = z_1 + z_2 + z_3 + z_4 + z_1 z_3 + y z_2 z_4$$

$$W_\delta = \log \mathcal{G} + \delta \log(z_1 z_2 z_3 z_4)$$

critical points $dW_\delta = 0$, to first order in δ

$$z_1 = z_3 = -1 - \delta$$

$$z_1 = z_3 = \delta/y$$

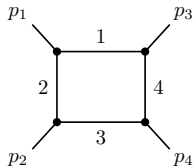
$$z_1 = z_3 = -1 - \frac{1+y}{y} \delta$$

$$z_2 = z_4 = \delta$$

$$z_2 = z_4 = -(1 + \delta)/y$$

$$z_2 = z_4 = -\frac{1}{y} - \frac{1+y}{y} \delta$$

Example: the one-loop box



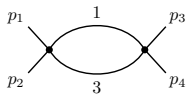
$$y = t/s$$

$$\mathcal{G} = z_1 + z_2 + z_3 + z_4 + z_1 z_3 + y z_2 z_4$$

$$W_\delta = \log \mathcal{G} + \delta \log(z_1 z_2 z_3 z_4)$$

$\nu = 3$ **master integrals**

critical points $dW_\delta = 0$, to first order in δ



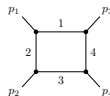
$$z_1 = z_3 = -1 - \delta$$

$$z_2 = z_4 = \delta$$



$$z_1 = z_3 = \delta/y$$

$$z_2 = z_4 = -(1 + \delta)/y$$



$$z_1 = z_3 = -1 - \frac{1+y}{y} \delta$$

$$z_2 = z_4 = -\frac{1}{y} - \frac{1+y}{y} \delta$$

Decomposition from Intersection numbers

forms

$$\langle \varphi | \leftrightarrow \int e^{+\varepsilon W_\delta} \varphi$$

dual forms

$$|\varphi^\vee\rangle \leftrightarrow \int e^{-\varepsilon W_\delta} \varphi^\vee$$

Decomposition from Intersection numbers

forms

$$\langle \varphi | \leftrightarrow \int e^{+\varepsilon W_\delta} \varphi$$

dual forms

$$|\varphi^\vee\rangle \leftrightarrow \int e^{-\varepsilon W_\delta} \varphi^\vee$$

Decomposition: $\langle \varphi | = \sum_{a=1}^\nu c_a \langle \varphi_a |$

$$c_a = \sum_{b=1}^\nu \langle \varphi | \varphi_b^\vee \rangle (\mathbf{C}^{-1})_{ba}$$

$$\mathbf{C}_{ab} = \langle \varphi_a | \varphi_b^\vee \rangle$$

Decomposition from Intersection numbers

forms

$$\langle \varphi | \leftrightarrow \int e^{+\varepsilon W_\delta} \varphi$$

dual forms

$$|\varphi^\vee\rangle \leftrightarrow \int e^{-\varepsilon W_\delta} \varphi^\vee$$

Decomposition: $\langle \varphi | = \sum_{a=1}^\nu c_a \langle \varphi_a |$

$$c_a = \sum_{b=1}^\nu \langle \varphi | \varphi_b^\vee \rangle (\mathbf{C}^{-1})_{ba}$$

$$\mathbf{C}_{ab} = \langle \varphi_a | \varphi_b^\vee \rangle$$

Intersection Numbers:

$$\langle \varphi | \varphi^\vee \rangle = \varepsilon^n \operatorname{Res}_{dW_\delta=0} \left[\hat{\varphi}^\vee \nabla_1^{-1} \cdots \nabla_n^{-1} \hat{\varphi} d^n z \right]$$

$$\nabla_i = \partial_i + \varepsilon \partial_i W_\delta$$

Decomposition from Intersection numbers

forms

$$\langle \varphi | \leftrightarrow \int e^{+\varepsilon W_\delta} \varphi$$

dual forms

$$|\varphi^\vee\rangle \leftrightarrow \int e^{-\varepsilon W_\delta} \varphi^\vee$$

Decomposition: $\langle \varphi | = \sum_{a=1}^{\nu} c_a \langle \varphi_a |$

$$c_a = \sum_{b=1}^{\nu} \langle \varphi | \varphi_b^\vee \rangle (\mathbf{C}^{-1})_{ba}$$

$$\mathbf{C}_{ab} = \langle \varphi_a | \varphi_b^\vee \rangle$$

Intersection Numbers:

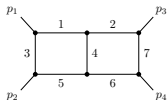
$$\langle \varphi | \varphi^\vee \rangle = \varepsilon^n \text{Res}_{dW_\delta=0} \left[\hat{\varphi}^\vee \nabla_1^{-1} \cdots \nabla_n^{-1} \hat{\varphi} d^n z \right]$$

$$\nabla_i = \partial_i + \varepsilon \partial_i W_\delta$$

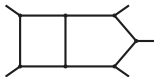
hard part: require solving differential equations

Evaluation of intersection numbers

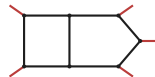
- **Projection onto masters** Mastrolia, Mizera '18
- **Multivariate fibration** Frellesvig, Gasparotto, Mandal, Mastrolia, Mattiazzi, Mizera '19
Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Mattiazzi, Mizera '20
- **Relative cohomology** Caron-Huot, Pokraka '21 (I, II)
- **Higher-order PDEs** Chestnov, Frellesvig, Gasparotto, Mandal, Mastrolia '22
- **Polynomial division** Fontana, Peraro '23
- **Polynomial division + relative cohomology** Brunello, Chestnov, Crisanti, Frellesvig, Mandal, Mastrolia '24
- **Companion matrices** Brunello, Chestnov, Mastrolia '24
- **LP relative cohomology** Lu, Wang, Yang '24
- **Fixed-branch integrals** Huang, Ma, Wang, Yang '26



polynomial division



companion matrices



FBI: $p_i^2 = m_i^2$

Intersection numbers as $\varepsilon \rightarrow \infty$

THE HIGHER RESIDUE
PAIRINGS $K_F^{(k)}$ FOR A FAMILY
OF HYPERSURFACE
SINGULAR POINTS

Kyoji Saito
Proc. Symp. Pure Math. 40 (1983)

$$\langle \varphi | \varphi^\vee \rangle = \langle \varphi | \varphi^\vee \rangle_{(0)} + \frac{1}{\varepsilon} \langle \varphi | \varphi^\vee \rangle_{(1)} + \dots$$

Saito's higher residue pairings

Intersection numbers as $\varepsilon \rightarrow \infty$

THE HIGHER RESIDUE
PAIRINGS $K_F^{(k)}$ FOR A FAMILY
OF HYPERSURFACE
SINGULAR POINTS

Kyoji Saito
Proc. Symp. Pure Math. 40 (1983)

$$\langle \varphi | \varphi^\vee \rangle = \langle \varphi | \varphi^\vee \rangle_{(0)} + \frac{1}{\varepsilon} \langle \varphi | \varphi^\vee \rangle_{(1)} + \dots$$

Saito's higher residue pairings

$$\varepsilon \rightarrow \infty: \quad \nabla_i^{-1} = \frac{1}{\varepsilon \partial_i W_\delta} - \frac{1}{\varepsilon \partial_i W_\delta} \partial_i \frac{1}{\varepsilon \partial_i W_\delta} + \mathcal{O}(\varepsilon^{-3})$$

algebraic order-by-order solution

Closed formulae at critical points

$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

$$\textbf{Hessian: } \Phi_\delta = \partial_i \partial_j W_\delta$$

Leading

$$\langle \varphi | \varphi^\vee \rangle_{(0)} = \sum_{dW_\delta=0} \frac{\hat{\varphi} \hat{\varphi}^\vee}{\det \Phi_\delta}$$

Closed formulae at critical points

$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

$$\textbf{Hessian: } \Phi_\delta = \partial_i \partial_j W_\delta$$

Leading

$$\langle \varphi | \varphi^\vee \rangle_{(0)} = \sum_{dW_\delta=0} \frac{\hat{\varphi} \hat{\varphi}^\vee}{\det \Phi_\delta}$$

Subleading

$$\langle \varphi | \varphi^\vee \rangle_{(1)} = \frac{1}{2} \sum_{dW_\delta=0} (\Phi_\delta^{-1})_{ij} \partial_j \left[\frac{\hat{\varphi} \partial_i \hat{\varphi}^\vee - \hat{\varphi}^\vee \partial_i \hat{\varphi}}{\det \Phi_\delta} \right]$$

Closed formulae at critical points

$$W_\delta = \log \mathcal{G} + \delta \sum_i \log z_i$$

$$\textbf{Hessian: } \Phi_\delta = \partial_i \partial_j W_\delta$$

Leading

$$\langle \varphi | \varphi^\vee \rangle_{(0)} = \sum_{dW_\delta=0} \frac{\hat{\varphi} \hat{\varphi}^\vee}{\det \Phi_\delta}$$

Subleading

$$\langle \varphi | \varphi^\vee \rangle_{(1)} = \frac{1}{2} \sum_{dW_\delta=0} (\Phi_\delta^{-1})_{ij} \partial_j \left[\frac{\hat{\varphi} \partial_i \hat{\varphi}^\vee - \hat{\varphi}^\vee \partial_i \hat{\varphi}}{\det \Phi_\delta} \right]$$

Only data at the critical points

Differential Equations from Intersection Numbers

$$\begin{aligned}\partial_x |\varphi_a\rangle &= \sum_{b=1}^{\nu} \Omega_{ab} |\varphi_b\rangle, & \Omega &= \mathbf{D} \mathbf{C}^{-1} \\ \mathbf{C}_{ab} &= \langle \varphi_a | \varphi_b^\vee \rangle, & \mathbf{D}_{ab} &= \langle \nabla_x \varphi_a | \varphi_b^\vee \rangle, & \nabla_x &= \partial_x + \varepsilon \partial_x W_\delta\end{aligned}$$

Differential Equations from Intersection Numbers

$$\begin{aligned}
 \partial_x |\varphi_a\rangle &= \sum_{b=1}^{\nu} \Omega_{ab} |\varphi_b\rangle, & \Omega &= \mathbf{D} \mathbf{C}^{-1} \\
 \mathbf{C}_{ab} &= \langle \varphi_a | \varphi_b^\vee \rangle, & \mathbf{D}_{ab} &= \langle \nabla_x \varphi_a | \varphi_b^\vee \rangle, & \nabla_x &= \partial_x + \varepsilon \partial_x W_\delta \\
 \underbrace{\varepsilon \boxed{\Omega^{(1)}} + \boxed{\Omega^{(0)}}}_{\text{linear in } \varepsilon} &= \underbrace{(\varepsilon \mathbf{D}^{(1)} + \mathbf{D}^{(0)} + \dots) (\mathbf{C}^{(0)} + \frac{1}{\varepsilon} \mathbf{C}^{(-1)} + \dots)^{-1}}_{\text{higher residue pairings}}
 \end{aligned}$$

Differential Equations from Intersection Numbers

$$\begin{aligned}\partial_x |\varphi_a\rangle &= \sum_{b=1}^{\nu} \Omega_{ab} |\varphi_b\rangle, & \Omega &= \mathbf{D} \mathbf{C}^{-1} \\ \mathbf{C}_{ab} &= \langle \varphi_a | \varphi_b^\vee \rangle, & \mathbf{D}_{ab} &= \langle \nabla_x \varphi_a | \varphi_b^\vee \rangle, & \nabla_x &= \partial_x + \varepsilon \partial_x W_\delta \\ \underbrace{\varepsilon \Omega^{(1)} + \Omega^{(0)}}_{\text{linear in } \varepsilon} &= \underbrace{(\varepsilon \mathbf{D}^{(1)} + \mathbf{D}^{(0)} + \dots) (\mathbf{C}^{(0)} + \frac{1}{\varepsilon} \mathbf{C}^{(-1)} + \dots)^{-1}}_{\text{higher residue pairings}}\end{aligned}$$

$$\varepsilon^1: \quad \Omega^{(1)} = \mathbf{D}^{(1)} (\mathbf{C}^{(0)})^{-1}$$

Leading

Differential Equations from Intersection Numbers

$$\partial_x |\varphi_a\rangle = \sum_{b=1}^{\nu} \Omega_{ab} |\varphi_b\rangle, \quad \Omega = \mathbf{D} \mathbf{C}^{-1}$$

$$\mathbf{C}_{ab} = \langle \varphi_a | \varphi_b^\vee \rangle, \quad \mathbf{D}_{ab} = \langle \nabla_x \varphi_a | \varphi_b^\vee \rangle, \quad \nabla_x = \partial_x + \varepsilon \partial_x W_\delta$$

$$\underbrace{\varepsilon \Omega^{(1)} + \Omega^{(0)}}_{\text{linear in } \varepsilon} = \underbrace{(\varepsilon \mathbf{D}^{(1)} + \mathbf{D}^{(0)} + \dots) (\mathbf{C}^{(0)} + \frac{1}{\varepsilon} \mathbf{C}^{(-1)} + \dots)^{-1}}_{\text{higher residue pairings}}$$

$$\varepsilon^1: \quad \Omega^{(1)} = \mathbf{D}^{(1)} (\mathbf{C}^{(0)})^{-1}$$

Leading

$$\varepsilon^0: \quad \Omega^{(0)} = (\mathbf{D}^{(0)} - \Omega^{(1)} \mathbf{C}^{(-1)}) (\mathbf{C}^{(0)})^{-1}$$

Leading + Subleading

Differential Equations from Intersection Numbers

$$\partial_x |\varphi_a\rangle = \sum_{b=1}^{\nu} \Omega_{ab} |\varphi_b\rangle, \quad \Omega = \mathbf{D} \mathbf{C}^{-1}$$

$$\mathbf{C}_{ab} = \langle \varphi_a | \varphi_b^\vee \rangle, \quad \mathbf{D}_{ab} = \langle \nabla_x \varphi_a | \varphi_b^\vee \rangle, \quad \nabla_x = \partial_x + \varepsilon \partial_x W_\delta$$

$$\underbrace{\varepsilon \Omega^{(1)} + \Omega^{(0)}}_{\text{linear in } \varepsilon} = \underbrace{(\varepsilon \mathbf{D}^{(1)} + \mathbf{D}^{(0)} + \dots) (\mathbf{C}^{(0)} + \frac{1}{\varepsilon} \mathbf{C}^{(-1)} + \dots)^{-1}}_{\text{higher residue pairings}}$$

$$\varepsilon^1: \quad \Omega^{(1)} = \mathbf{D}^{(1)} (\mathbf{C}^{(0)})^{-1}$$

Leading

$$\varepsilon^0: \quad \Omega^{(0)} = (\mathbf{D}^{(0)} - \Omega^{(1)} \mathbf{C}^{(-1)}) (\mathbf{C}^{(0)})^{-1}$$

Leading + Subleading

$\Rightarrow$ **Leading and subleading** pairings are all we need

One-loop box: the regulated results

$$\langle \text{box} | \chi \chi \rangle_{(0)} = \frac{\delta}{2(1+2\delta)^2}, \quad \langle \partial_y W \text{box} | \chi \chi \rangle_{(0)} = \frac{(1+\delta)(1+3\delta) - y\delta^2}{2y(1+y)(1+2\delta)^2}$$

One-loop box: the regulated results

$$\langle \text{box} | \text{box} \rangle_{(0)} = \frac{\delta}{2(1+2\delta)^2}, \quad \langle \partial_y W \text{box} | \text{box} \rangle_{(0)} = \frac{(1+\delta)(1+3\delta) - y\delta^2}{2y(1+y)(1+2\delta)^2}$$

$$\mathbf{C}^{(0)}(\delta) = \begin{pmatrix} -\frac{1+4\delta}{4} & \frac{\delta^2}{4(1+2\delta)} & \frac{\delta^3}{2(1+2\delta)^2} \\ \frac{\delta^2}{4(1+2\delta)} & -\frac{1+4\delta}{4} & \frac{\delta^3}{2(1+2\delta)^2} \\ \frac{\delta}{2(1+2\delta)^2} & \frac{\delta}{2(1+2\delta)^2} & \frac{1+6\delta+10\delta^2}{(1+2\delta)^3} \end{pmatrix}$$

One-loop box: the regulated results

$$\langle \text{box} | \text{box} \rangle_{(0)} = \frac{\delta}{2(1+2\delta)^2}, \quad \langle \partial_y W \text{box} | \text{box} \rangle_{(0)} = \frac{(1+\delta)(1+3\delta) - y\delta^2}{2y(1+y)(1+2\delta)^2}$$

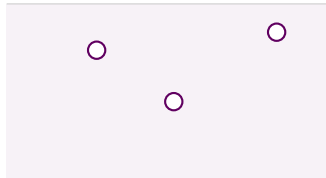
$$\mathbf{C}^{(0)}(\delta) = \begin{pmatrix} -\frac{1+4\delta}{4} & \frac{\delta^2}{4(1+2\delta)} & \frac{\delta^3}{2(1+2\delta)^2} \\ \frac{\delta^2}{4(1+2\delta)} & -\frac{1+4\delta}{4} & \frac{\delta^3}{2(1+2\delta)^2} \\ \frac{\delta}{2(1+2\delta)^2} & \frac{\delta}{2(1+2\delta)^2} & \frac{1+6\delta+10\delta^2}{(1+2\delta)^3} \end{pmatrix}$$

Dense matrices: DE recovered in the $\delta \rightarrow 0$ limit

$$\partial_y \begin{pmatrix} \text{box} \\ y \text{box} \\ \varepsilon y \text{box} \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{y} & 0 \\ \frac{2}{1+y} & -\frac{1}{y(1+y)} & -\frac{1}{y(1+y)} \end{pmatrix} \begin{pmatrix} \text{box} \\ y \text{box} \\ \varepsilon y \text{box} \end{pmatrix}$$

This work: $\delta \rightarrow 0$ expansion

the regulator is **not physical**

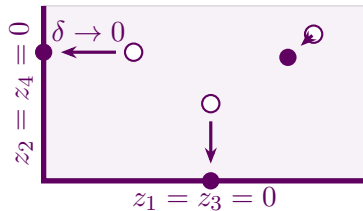


This work: $\delta \rightarrow 0$ expansion

the regulator is **not physical**

as $\delta \rightarrow 0$, critical points organise into **sectors**:

$$z_i \partial_i \mathcal{G} = 0 \quad \Rightarrow \quad \underbrace{z_i = 0}_{\text{pinched}} \quad \text{or} \quad \underbrace{\partial_i \mathcal{G} = 0}_{\text{cut}}$$



This work: $\delta \rightarrow 0$ expansion

the regulator is **not physical**

as $\delta \rightarrow 0$, critical points organise into **sectors**:

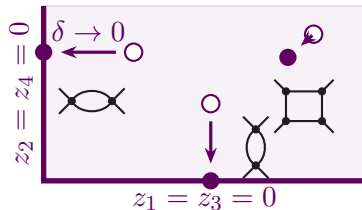
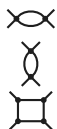
$$z_i \partial_i \mathcal{G} = 0 \quad \Rightarrow \quad \underbrace{z_i = 0}_{\text{pinched}} \quad \text{or} \quad \underbrace{\partial_i \mathcal{G} = 0}_{\text{cut}}$$

forms organise into **sectors**:

$$\hat{\varphi}_a = \underbrace{\Delta_a}_{\text{singular}} \underbrace{\psi_a}_{\text{regular}}$$

$$\Delta_a = \prod_{i \in \Gamma_a} \frac{\delta}{z_i}$$

Γ_a : pinched edges of φ_a



Δ_a	ψ_a
$\delta^2 / (z_2 z_4)$	$-z_3 / (2 \mathcal{G}^2)$
$\delta^2 / (z_1 z_3)$	$-y z_4 / (2 \mathcal{G}^2)$
1	$y / \mathcal{G}^2$

Leading pairing as $\delta \rightarrow 0$

$$\langle \varphi_a | \varphi_b^\vee \rangle_{(0)} = \sum_{\Gamma_a \subseteq \Gamma \subseteq \Gamma_b} \sum_{z^* \in \Gamma} \frac{\psi_a \psi_b}{\det \Phi_{\bar{\Gamma}}} R_{ab}^\Gamma$$


sectors
 Γ
critical points
cut variables
pinched variables

Leading pairing as $\delta \rightarrow 0$

$$\langle \varphi_a | \varphi_b^\vee \rangle_{(0)} = \sum_{\Gamma_a \subseteq \Gamma \subseteq \Gamma_b} \sum_{z^* \in \Gamma} \frac{\psi_a \psi_b}{\det \Phi_{\bar{\Gamma}}} R_{ab}^\Gamma$$

The diagram illustrates the factorization of the pairing into four distinct components. Each component is represented by a mathematical expression above a downward-pointing arrow, which then points to a descriptive label. The components are: 1. Sectors, represented by the sum over $\Gamma_a \subseteq \Gamma \subseteq \Gamma_b$, with the label 'sectors' in purple and Γ below it. 2. Critical points, represented by the sum over $z^* \in \Gamma$, with the label 'critical points' in red. 3. Cut variables, represented by the fraction $\frac{\psi_a \psi_b}{\det \Phi_{\bar{\Gamma}}}$, with the label 'cut variables' in grey. 4. Pinched variables, represented by R_{ab}^Γ , with the label 'pinched variables' in grey.

sectors
 Γ

critical points

cut variables

pinched variables

The pairing **factorises over the sectors**

Leading pairing as $\delta \rightarrow 0$

$$\langle \varphi_a | \varphi_b^\vee \rangle_{(0)} = \sum_{\Gamma_a \subseteq \Gamma \subseteq \Gamma_b} \sum_{z^* \in \Gamma} \frac{\psi_a \psi_b}{\det \Phi_{\bar{\Gamma}}} R_{ab}^\Gamma$$

The diagram illustrates the factorization of the pairing into four components, each with a downward arrow from the corresponding term in the equation above:

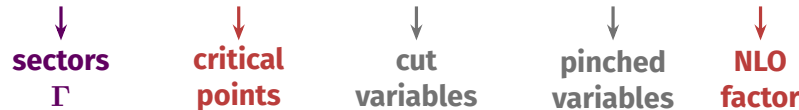
- sectors** Γ (purple text, purple arrow)
- critical points** (red text, red arrow)
- cut variables** (grey text, grey arrow)
- pinched variables** (grey text, grey arrow)

The pairing **factorises over the sectors**

finite fields and **companion matrix** technology [FiniteFlow] [SPQR]

Sub-leading pairing as $\delta \rightarrow 0$

$$\langle \varphi_a | \varphi_b^\vee \rangle_{(1)} = \sum_{\Gamma_a \subseteq \Gamma \subseteq \Gamma_b} \sum_{z^* \in \Gamma} \frac{\psi_a \psi_b}{\det \Phi_{\bar{\Gamma}}} R_{ab}^\Gamma \frac{T_{ab}}{2}$$


sectors **critical points** **cut variables** **pinched variables** **NLO factor**

The pairing **factorises over the sectors**

finite fields and **companion matrix** technology [FiniteFlow] [SPQR]

One-loop differential equation

$$\langle \text{box} | \text{cross} \rangle_{(0)} = \underbrace{\frac{1}{2y}}_{\text{box}} - \underbrace{\frac{1}{2y}}_{\text{cross}} = 0, \quad \langle \partial_y W \text{box} | \text{cross} \rangle_{(0)} = \frac{1}{2y(1+y)}$$

One-loop differential equation

$$\langle \text{box} | \text{cross} \rangle_{(0)} = \underbrace{\frac{1}{2y}}_{\text{box}} - \underbrace{\frac{1}{2y}}_{\text{cross}} = 0, \quad \langle \partial_y W \text{box} | \text{cross} \rangle_{(0)} = \frac{1}{2y(1+y)}$$

Block-triangular structure: no regulator left

$$\mathbf{C}^{(0)} = \begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4y} & 0 \\ -\frac{1}{2(1+y)} & \frac{1}{2y(1+y)} & -\frac{1}{y(1+y)} \end{pmatrix}$$

One-loop differential equation

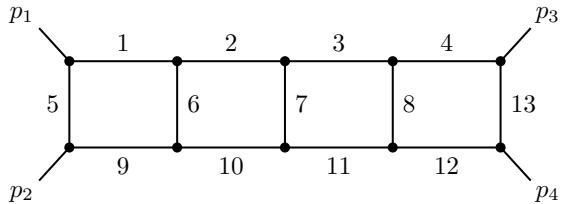
$$\langle \text{box} | \text{cross} \rangle_{(0)} = \underbrace{\frac{1}{2y}}_{\text{box}} - \underbrace{\frac{1}{2y}}_{\text{cross}} = 0, \quad \langle \partial_y W \text{box} | \text{cross} \rangle_{(0)} = \frac{1}{2y(1+y)}$$

Block-triangular structure: no regulator left

$$\mathbf{C}^{(0)} = \begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4y} & 0 \\ -\frac{1}{2(1+y)} & \frac{1}{2y(1+y)} & -\frac{1}{y(1+y)} \end{pmatrix}$$

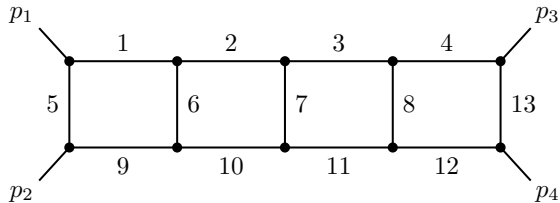
$$\partial_y \begin{pmatrix} \text{box with dot} \\ y \text{cross} \\ \varepsilon y \text{box} \end{pmatrix} = \varepsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{y} & 0 \\ \frac{2}{1+y} & -\frac{2}{y(1+y)} & -\frac{1}{y(1+y)} \end{pmatrix} \begin{pmatrix} \text{box with dot} \\ y \text{cross} \\ \varepsilon y \text{box} \end{pmatrix}$$

Four-loop ladder



13 propagators + 9 numerators

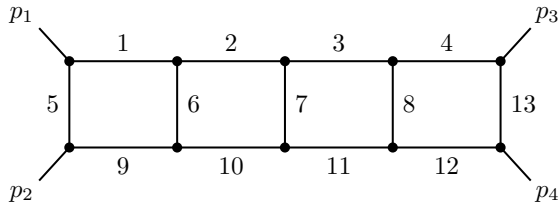
Four-loop ladder



13 propagators + 9 numerators

159 critical sectors $\rightarrow$ **246** forms

Four-loop ladder

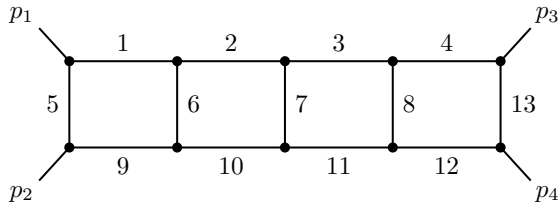


13 propagators + 9 numerators

159 critical sectors $\rightarrow$ **246** forms

symmetries $\rightarrow$ **88** masters

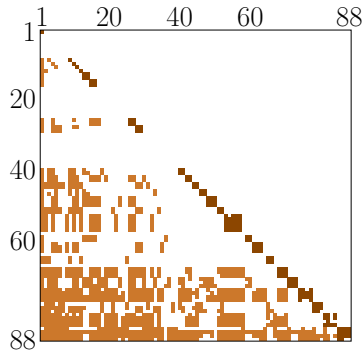
Four-loop ladder



13 propagators + 9 numerators

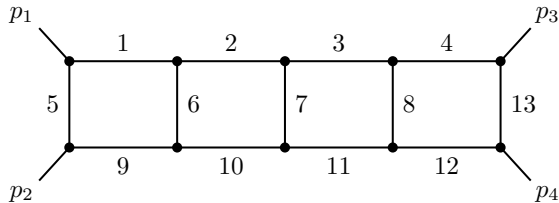
159 critical sectors $\rightarrow$ **246** forms

symmetries $\rightarrow$ **88** masters



$$d\mathbf{I} = \varepsilon \left(\mathbf{A}_0 d \log y + \mathbf{A}_1 d \log(1 + y) \right) \mathbf{I}, \quad y = t/s$$

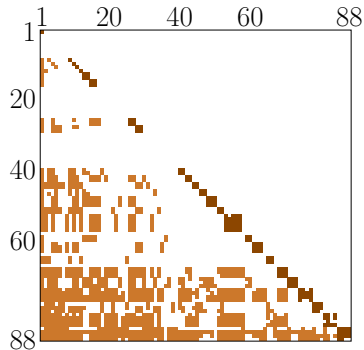
Four-loop ladder



13 propagators + 9 numerators

159 critical sectors $\rightarrow$ **246** forms

symmetries $\rightarrow$ **88** masters



$$d\mathbf{I} = \varepsilon \left(\mathbf{A}_0 d \log y + \mathbf{A}_1 d \log(1 + y) \right) \mathbf{I}, \quad y = t/s$$

Now a laptop computation!

Conclusions

Differential Equations from Intersection Numbers:

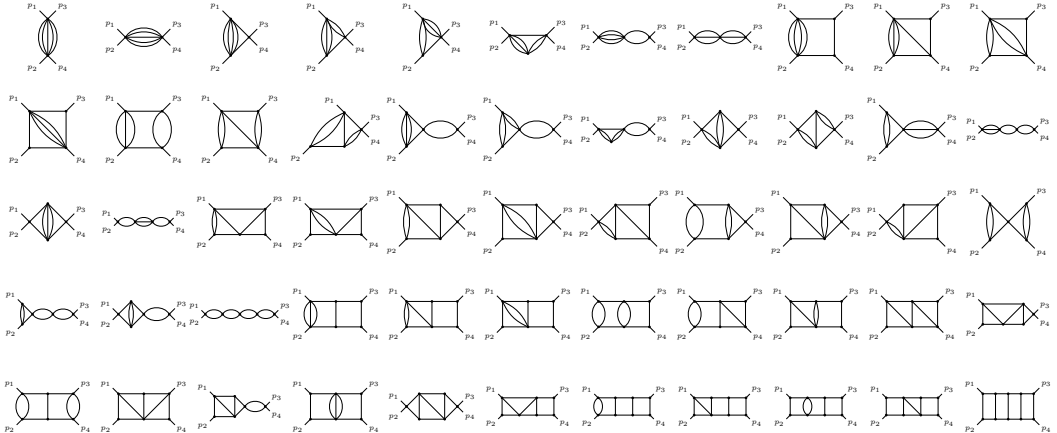
- $\epsilon \rightarrow \infty$: factorisation over critical points, LO and NLO in closed form
- $\delta \rightarrow 0$: critical points organise into sectors, no regulator left
- **canonical differential equations** for the **four-loop massless ladder**

Outlook:

- DEs at arbitrary order in ϵ
- amplitude decomposition from intersection numbers

The activity was carried out within the project:
NOTIMEFORCOSMO “No time for cosmology: Decoding
dynamics from static cosmological correlations”,
Grant Agreement 101126304, CUP E53C23002380006

Four-loop ladder: critical sectors



the 55 inequivalent critical sectors: 88 master integrals