

Untangling the IBP Equations

Junhan William Liu

Based on the work done with Alexander Mitov in arXiv:2512.05923

October 2026



UNIVERSITY OF
CAMBRIDGE



Science and
Technology
Facilities Council



HARDING
Distinguished Postgraduate
SCHOLARS PROGRAMME

Integration-by-Parts (IBP) identities

- Scattering amplitudes can be decomposed into linear combination of scalar Feynman integrals
- Feynman integrals admit linear recurrence relations:

$$\int \prod_{r=1}^I d^d k_r \frac{\partial}{\partial k_i^\mu} \frac{q_{\text{IBP}}^\mu}{P_1^{\nu_1} P_2^{\nu_2} \cdots P_n^{\nu_n}} = 0,$$

where k_i takes any loop momentum and q_{IBP}^μ takes any loop or external momentum

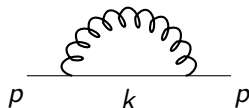
- It reduces all scalar integrals to a small set of basis (**Master Integrals**)

Solution of IBP equations

- Best known approach is proposed by Laporta (Laporta, 2000)
 - Substitute the abstract indices with explicit integer values
 - Produce an infinite system of equations
 - Use Gauss elimination to solve a relevant part of it
- Limitation for more complicated problems:
 - Time-consuming and RAM-demanding!
- Numeric solution only:
 - It solves $I_{1,-1,\dots}, I_{1,-2,\dots}$ but not $I_{\nu_1,\nu_2,\dots}$ for arbitrary $\nu_1,\nu_2,\dots$
- Recent development in recasting IBP into better form:
 - Syzygy (Gluza, Kajda & Kosower, 2011; Wu et al., 2025)
 - Block-triangular (Guan, Liu & Ma, 2020; Guan et al., 2025)
 - Symbolic reduction rules (Smith & Zeng, 2026; Feng et al., 2026)
- Very little improvement on solving the IBP identities analytically.

Why difficult?

Let's inspect one example:


$$I_{\nu_1, \nu_2} = \int d^d k \frac{1}{(k^2 - m^2)^{\nu_1} (p - k)^{2\nu_2}}$$

which gives the following IBP equations:

$$\begin{cases} 0 = -\nu_2 I_{\nu_1-1, \nu_2+1} + (d - 2\nu_1 - \nu_2) I_{\nu_1, \nu_2} - 2m^2 \nu_1 I_{\nu_1+1, \nu_2} \\ 0 = -\nu_2 I_{\nu_1-1, \nu_2+1} - (\nu_1 - \nu_2) I_{\nu_1, \nu_2} + \nu_1 I_{\nu_1+1, \nu_2-1} - 2m^2 \nu_1 I_{\nu_1+1, \nu_2} \end{cases}$$

- Multi-variable, linear, homogenous recurrence relations with rational coefficients
- Highly entangled: the integrals in each equation contain shifts on multiple indices
- Demands simultaneous solution of huge system of linear equations

Diagonalisation of IBP equations

- Diagonal form: changes one index at a time, keeping all other indices fixed (unshifted)

$$I_{\nu_1, \nu_2-1} = \frac{2m^2(d - \nu_1 - 2\nu_2)(1 + d - \nu_1 - 2\nu_2)}{(2 + d - 2\nu_1 - 2\nu_2)(d - \nu_1 - \nu_2)} I_{\nu_1, \nu_2}$$

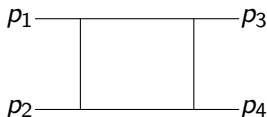
$$I_{\nu_1-1, 1} = \frac{2m^2(d - \nu_1 - 2)(\nu_1 - 1)}{(d - 2\nu_1)(d - 1 - \nu_1)} I_{\nu_1, 1}$$

- Efficient solution: no simultaneous solution of multiple equations!
- One can even solve them for abstract ν_1, ν_2 :

$$I_{\nu_1, \nu_2} = \frac{\Gamma(-\frac{d}{2} + \nu_1 + \nu_2)}{\Gamma(2 - \frac{d}{2}) \Gamma(\nu_1) \Gamma(1 - d + \nu_1 + 2\nu_2)} \times \\ \frac{\Gamma(1 - d + \nu_2) \Gamma(d - \nu_2)}{\Gamma(d - \nu_1 - \nu_2)} (d - 3)(m^2)^{2-\nu_1-\nu_2} I_{1,1}$$

What about multi-master cases?

- What about system with multiple masters?
 - Higher order?
 - Anything alternative?
- Let's consider the following example:



$$I_{\nu_1, \nu_2, \nu_3, \nu_4} = \int d^d k \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4}}$$

- **Three** masters: two bubbles and a box

Matrix diagonal form

- One can derive diagonal equations with higher order, but it is found that these higher-order recurrences do not reduce all integrals fully to the masters (Mitov, 2005; Mitov & Moch, 2006).
- Essentially, when one derives diagonal equations for ν_1 , $\nu_1 = 1, 0, -1, \dots$ will be considered basis, and for ν_2 , $\nu_2 = 1, 0, -1, \dots$ will be considered basis.
- Therefore, to avoid this, we keep the system in **first order**, but consider recurrence in a **different object** (More in the backup).

$$\mathbf{V}_{\nu_1-1, \nu_2, \nu_3, \nu_4} = \mathbf{W}^{(4)}(\nu_1, \nu_2, \nu_3, \nu_4) \mathbf{V}_{\nu_1, \nu_2, \nu_3, \nu_4}$$
$$\begin{pmatrix} I_{\nu_1-1, \nu_2, \nu_3, \nu_4} \\ I_{\nu_1-2, \nu_2, \nu_3, \nu_4} \\ I_{\nu_1-1, \nu_2-1, \nu_3, \nu_4} \end{pmatrix} = \begin{pmatrix} W_{11}^{(4)} & W_{12}^{(4)} & W_{13}^{(4)} \\ W_{21}^{(4)} & W_{22}^{(4)} & W_{23}^{(4)} \\ W_{31}^{(4)} & W_{32}^{(4)} & W_{33}^{(4)} \end{pmatrix} \begin{pmatrix} I_{\nu_1, \nu_2, \nu_3, \nu_4} \\ I_{\nu_1-1, \nu_2, \nu_3, \nu_4} \\ I_{\nu_1, \nu_2-1, \nu_3, \nu_4} \end{pmatrix}$$

- Each line of these vector equations still contains four integrals: “third order” comes back as $\dim(\mathbf{V})$.

Matrix diagonal form

- Such equation can be derived for all four indices:

$$\mathbf{V}_{\nu_1-1, \nu_2, \nu_3, \nu_4} = \mathbf{W}^{(4)}(\nu_1, \nu_2, \nu_3, \nu_4) \mathbf{V}_{\nu_1, \nu_2, \nu_3, \nu_4}$$

$$\mathbf{V}_{1, \nu_2-1, \nu_3, \nu_4} = \mathbf{W}^{(3)}(\nu_2, \nu_3, \nu_4) \mathbf{V}_{1, \nu_2, \nu_3, \nu_4}$$

$$\mathbf{V}_{1, 1, \nu_3-1, \nu_4} = \mathbf{W}^{(2)}(\nu_3, \nu_4) \mathbf{V}_{1, 1, \nu_3, \nu_4}$$

$$\mathbf{V}_{1, 1, 1, \nu_4-1} = \mathbf{W}^{(1)}(\nu_4) \mathbf{V}_{1, 1, 1, \nu_4}$$

- The explicit solution of $\mathbf{V}_{\nu_1, \nu_2, \nu_3, \nu_4}$ can then be achieved by iterative matrix multiplication

$$\mathbf{V}_{\nu_1, \nu_2, \nu_3, \nu_4} = \left(\prod_{l=1}^{\nu_1+1} \mathbf{W}^{(4)}(l, \nu_2, \nu_3, \nu_4) \right) \left(\prod_{k=1}^{\nu_2+1} \mathbf{W}^{(3)}(k, \nu_3, \nu_4) \right) \times \\ \left(\prod_{n=1}^{\nu_3+1} \mathbf{W}^{(2)}(n, \nu_4) \right) \left(\prod_{m=1}^{\nu_4+1} \mathbf{W}^{(1)}(m) \right) \mathbf{V}_{1, 1, 1, 1}$$

There are more to it (to make it simpler)

- Imagine if we constrain ourselves to $l_{\nu_1,1,\nu_3,1}$ i.e. we look for equation that takes the form:

$$\mathbf{V}_{\nu_1-1,1,\nu_3,1} = \mathbf{W}^{(2)}(\nu_1, \nu_3) \mathbf{V}_{\nu_1,1,\nu_3,1}$$
$$\begin{pmatrix} l_{\nu_1-1,1,\nu_3,1} \\ l_{\nu_1-2,1,\nu_3,1} \\ l_{\nu_1-1,0,\nu_3,1} \end{pmatrix} = \begin{pmatrix} W_{11}^{(2)} & W_{12}^{(2)} & W_{13}^{(2)} \\ W_{21}^{(2)} & W_{22}^{(2)} & W_{23}^{(2)} \\ W_{31}^{(2)} & W_{32}^{(2)} & W_{33}^{(2)} \end{pmatrix} \begin{pmatrix} l_{\nu_1,1,\nu_3,1} \\ l_{\nu_1-1,1,\nu_3,1} \\ l_{\nu_1,0,\nu_3,1} \end{pmatrix}$$

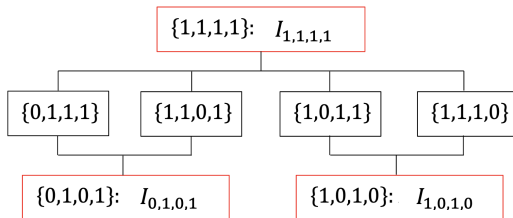
- It takes a very interesting form:

$$\begin{pmatrix} l_{\nu_1-1,1,\nu_3,1} \\ l_{\nu_1-2,1,\nu_3,1} \\ l_{\nu_1-1,0,\nu_3,1} \end{pmatrix} = \begin{pmatrix} W_{11}^{(2)} & W_{12}^{(2)} & W_{13}^{(2)} \\ W_{21}^{(2)} & W_{22}^{(2)} & W_{23}^{(2)} \\ 0 & 0 & W_{33}^{(2)} \end{pmatrix} \begin{pmatrix} l_{\nu_1,1,\nu_3,1} \\ l_{\nu_1-1,1,\nu_3,1} \\ l_{\nu_1,0,\nu_3,1} \end{pmatrix}$$

- Due to the 0s, the blue parts do not talk to the black parts
- Not a coincidence: **sector structure**

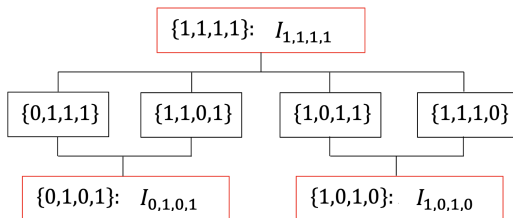
Sector structure

- Integrals with the same set of positive indices are in the same sector
 - e.g. $I_{1,-1,1,0}$ and $I_{1,0,2,-1}$
- Integrals with strictly more positive indices are in the higher sector
 - e.g. $I_{1,1,1,0}$ is in a higher sector than $I_{1,0,1,0}$ but not $I_{0,1,0,1}$
- Integrals are only related to the masters in their own sector or in their sub-sectors



Sector structure

- Therefore, when we look for the solution of $I_{0,1,0,1}$, the master $I_{1,0,1,0}$ is essentially irrelevant.
- We can temporarily ignore $I_{1,0,1,0}$ by setting $I_{\nu_1,0,\nu_3,1}$, $I_{\nu_1,1,\nu_3,0}$ etc. to zero: **the order is hence reduced!**
- The master $I_{1,1,1,1}$ cannot be ignored.



Sector structure

- Hence, instead of considering all three masters simultaneously, we can divide and conquer
- We firstly consider just $l_{0,1,0,1}$ by setting $l_{1,0,1,0} = 0$.
 - 2D basis: $\mathbf{V}_{\nu_1, \nu_3} = (l_{\nu_1, 1, \nu_3, 1}, l_{\nu_1-1, 1, \nu_3, 1})^T$
 - Equation in ν_1 : $\mathbf{V}_{\nu_1-1, \nu_3} = \mathbf{W}^{(2)}(\nu_1, \nu_3) \mathbf{V}_{\nu_1, \nu_3}$ with

$$\mathbf{W}^{(2)} = \begin{pmatrix} 0 & 1 \\ \frac{(2-d+2\nu_1)(\nu_1-1)t}{2(2-d+\nu_1+\nu_3)(1-d+\nu_1+\nu_3)} & \frac{2-2\nu_1+2\nu_3-dt+2\nu_1t+2\nu_3t}{2(-1+d-\nu_1-\nu_3)} \end{pmatrix}$$

- Equation in ν_3 : $\mathbf{V}_{1, \nu_3-1} = \mathbf{W}^{(1)}(\nu_3) \mathbf{V}_{1, \nu_3}$ with

$$\mathbf{W}^{(1)} = \begin{pmatrix} \frac{1-\nu_3}{-3+d-\nu_3} & 1 \\ 0 & \frac{(2-d+2\nu_3)t}{2(-2+d-\nu_3)} \end{pmatrix}$$

- Proceed to the non-covered master: $l_{1,0,1,0}$ with $l_{1, \nu_2, 1, \nu_4}$
- Integrals with dotted propagators can be considered separately, since they normally belongs to simple topology.

To prove that I'm not lying...

...that diagonal equations do yield analytic solution.

Here's how.

$$\begin{aligned} I_{\nu_1,1,\nu_3,1} = & \frac{t^{-\nu_1-\nu_3}}{\Gamma(d-2-\nu_1-\nu_3)\Gamma(2-\frac{d}{2}+\nu_1+\nu_3)} \times \\ & {}_2F_1 \left[\begin{matrix} \nu_1 & \nu_3 \\ 2-\frac{d}{2}+\nu_1+\nu_3 \end{matrix} ; -\frac{1}{t} \right] I_a \\ & + \frac{\Gamma(3-d+\nu_1+\nu_3)\Gamma(1-\frac{d}{2}+\nu_1+\nu_3)}{\Gamma(\nu_1)\Gamma(2-\frac{d}{2}+\nu_1)\Gamma(\nu_3)\Gamma(2-\frac{d}{2}+\nu_3)} \times \\ & {}_2F_1 \left[\begin{matrix} 1-\nu_1 & 1-\nu_3 \\ \frac{d}{2}-\nu_1-\nu_3 \end{matrix} ; -\frac{1}{t} \right] I_b \end{aligned}$$

Triangular form

- Advantages of the diagonal form:
 - Fast solution to I_{-100} , if you want
 - Potential analytic continuation to non-integer values
- Disadvantages of diagonal form:
 - Large rational function in W
 - **Note:** a better choice of V may significantly simplify W
- To improve computational efficiency: **triangular form**
 - No diagonalisation any more
 - No simultaneous solution of multiple equations (full set of reduction rule)
 - Much simpler rational functions in the equations

Triangular form

Benchmarked against NeatIBP (syzygy) (Wu et al., 2025) using the program Kira (Lange, Usovitsch & Wu, 2025) parallelised across 24 cores

- 2-loop 5-point massless: planar (C_1) and non-planar (B_1)
- For B_1 , symbolic reduction of two spanning cuts
- All rank-5 integrals in the top-sector are solved
- We used two finite field libraries: FireFly (Klappert, Klein & Lange, 2021) and Ratracer (Magerya, 2022) (in the bracket and for C_1 only)

topology	Fermat	Finite field	
	triang.	syzygy	triang.
C_1	3.4 h	11.3 h (3.9 h)	8.0 h (3.8 h)
$B_1: \overline{M_6}$	2.31 d	3.21 d	5.41 d
$B_1: \overline{M_{10}}$	1.38 d	3.51 d	2.82 d
B_1 (rat. kin.)	41 s	590.5 s	622.2 s

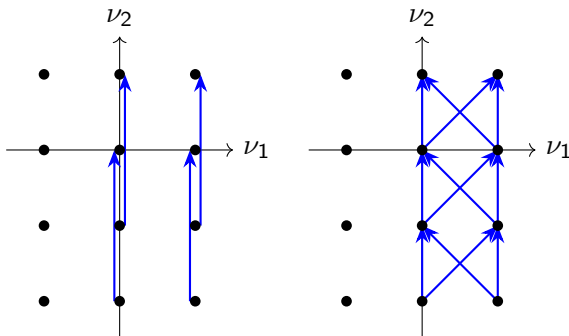
- When triangular forms are used with finite field reconstruction, > 90% goes into coefficient evaluation

Conclusions

- We proposed an automated algorithm that achieved diagonalisation of the IBP system (**never achieved before**)
 - Efficient reduction for problem demanding high numerator power (gravity)
 - Direct pathway to find analytical solution to the IBP identities (ongoing) or other multi-variable recurrence relations in general (More in the backup)
- As a by-product, we presented the triangular approach
 - QCD-pheno-efficiency-oriented
 - A factor of 2 improvement in efficiency compared to the state-of-the-art algorithm
 - More state-of-the-art computation...

Backup: Matrix diagonal form

- Difference between higher-order diagonal (left) and matrix diagonal (right)



- Each point (ν_1, ν_2) is an integral I_{ν_1, ν_2}
- The arrow represents how the mappings are done with each equations
- Not magic that vector recurrence gives full reduction to just the master integrals, but because it is fundamentally different

Backup: Something else to do with Gauss

- Contiguous relations of ${}_2F_1$ -hypergeometric function

$$(c-a-1)F(a, b, c) + aF(a+1, b, c) - (c-1)F(a, b, c-1) = 0, \dots$$

- Through standard IBP pipeline, one can find
 - Finite solution
 - Two boundary conditions

$$F(1, 1, 1) = \frac{1}{1-z} \quad F(1, 1, 2) = -\frac{\log(1-z)}{z}$$

- Same matrix diagonal form can be derived

$$\begin{pmatrix} F(a+2, b, c) \\ F(a+1, b, c) \end{pmatrix} = \begin{pmatrix} \frac{a+bz-cz}{a(1-z)} & \frac{(a-c)(b-c)z}{ac(1-z)} \\ \frac{c}{a} & \frac{a-c}{a} \end{pmatrix} \begin{pmatrix} F(a+1, b, c) \\ F(a, b, c) \end{pmatrix}$$

- Vanishing second column at $a = c$:
- decoupling from the $\log(1-z)$ (textbook results)

Like Feynman integral, the Gauss hypergeometric function itself also satisfy:

- recurrence relations, namely the contiguous relations, e.g.

$$(c-a-1)F(a, b, c; z) + aF(a+1, b, c; z) - (c-1)F(a, b, c-1; z) = 0$$

- differential equation (with well known solution):

$$z(1-z)\frac{d^2F}{dz^2} + [c - (a+b+1)z]\frac{dF}{dz} - abF = 0$$

Again, like Feynman integral, these relations are not completely independent.

Backup: Carl Friedrich Gauss

- Solution of differential equation (the famous well known result):

$$A_{a,b,c} {}_2F_1 \left[\begin{matrix} a & b \\ c \end{matrix} ; z \right] + B_{a,b,c} z^{1-c} (1-z)^{c-a-b} {}_2F_1 \left[\begin{matrix} 1-a & 1-b \\ 2-c \end{matrix} ; z \right]$$

which provides the z -dependence.

- Solution of recurrence relations (via the diagonal approach):

$$f_1(z) (1-z) {}_2F_1 \left[\begin{matrix} a & b \\ c \end{matrix} ; z \right] + f_2(z) (-1)^c \frac{(1-c)_a (1-c)_b}{(c-1)\Gamma(a)\Gamma(b)} \times z^{1-c} (1-z)^{c-a-b} {}_2F_1 \left[\begin{matrix} 1-a & 1-b \\ 2-c \end{matrix} ; z \right]$$

which provides the index dependence