

Soft-photon resummation for low-energy e^+e^- scattering

Coherent exclusive exponentiation (CEEX) for $\pi^+\pi^-\gamma$ and $\mu^+\mu^-\gamma$ at
KLOE

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01

Motivation



Why low-energy e^+e^- needs precision Monte Carlo

The $(g - 2)_\mu$ HVP puzzle

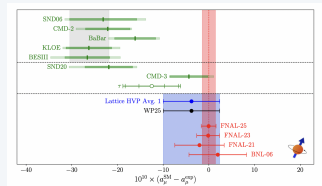
$$a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{3\pi^2} \int \frac{ds}{s} K(s) R(s): \pi^+\pi^- \text{ dominates}$$

- **Scan** (CMD-3, SND), **radiative return** (KLOE, BaBar, BESIII)
- **Tensions** between data sets [WP25]

Every analysis relies on an MC generator

Acceptances, luminosity, $\mu\mu\gamma$ normalisation
 $\Rightarrow$ **RadioMonteCarLow2**

radiomontecarlow2.gitlab.io [SciPost Phys. Comm. Rep. 9 (2025)]



$a_\mu^{\text{HVP,LO}}$ from e^+e^- data vs. lattice

[WP25: R. Aliberti et al., Phys. Rept. 1143 (2025) 1]

PHOKHARA: NLO for radiative return

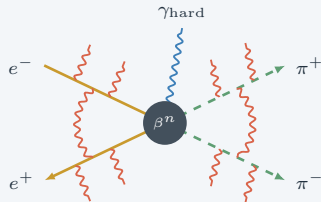
Used in radiative return, e.g. KLOE:

How can we make it better for the new, ongoing analyses?

- Radiative-correction uncertainties must sit well below the experimental precision.

Why soft-photon resummation?

- Each photon: $\alpha \times \log \frac{E_{\min}}{\sqrt{s}}$ (and $\log \frac{s}{m_e^2} \sim 15$ if collinear)
- $2 \rightarrow 3$ beyond NLO: a **5-point, two-loop, two-mass** amplitude — many scales!
- Soft amplitudes **factorise**: computed in the soft limit for any number of photons, real *and* virtual
- Independent handle on recently observed **tensions in fixed-order** codes [2601.19530]
- Built-in **IR subtraction**: a route towards NNLO

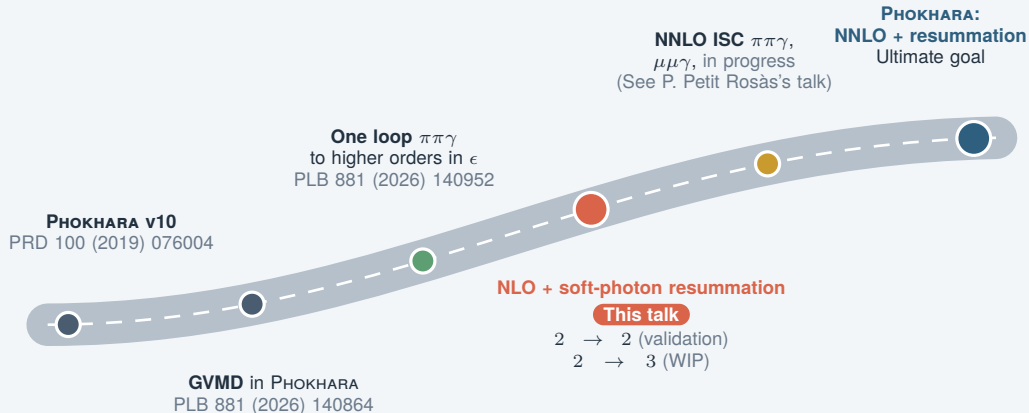


Hard process $\times$ real and virtual soft photons on every charged leg

	LO
	NLO (PHOKHARA), NLO+PS (BABAYAGA)
	+ Pion structure: $F \times$ sQED, GVMD (PHOKHARA)
	+ Resummation $\rightarrow$ this talk
	NNLO (in progress)

- Route chosen: YFS exponentiation, made *coherent* at amplitude level (CEEX).

Roadmap: the PHOKHARA plan



- CEEX is built as a resummation *module* to be incorporated into PHOKHARA.

02

Soft-photon resummation

The YFS route: exponentiate the IR

YFS: IR divergences of QED **factorise and exponentiate** to all orders [Annals Phys. 13 (1961) 379]

$$d\sigma = \sum_{n_\gamma=0}^{\infty} \frac{e^{Y(\Omega)}}{n_\gamma!} d\Phi_Q \prod_{i=1}^{n_\gamma} d\phi_i \tilde{S}(k_i) \Theta(k_i, \Omega) \left[\bar{\beta}_0 + \sum_j \frac{\bar{\beta}_1(k_j)}{\tilde{S}(k_j)} + \sum_{j < k} \frac{\bar{\beta}_2(k_j, k_k)}{\tilde{S}(k_j) \tilde{S}(k_k)} + \dots \right]$$

Ingredients

- $Y(\Omega)$: YFS form factor (virtual + unresolved real)
- $\tilde{S}(k)$: eikonal factor, at $|\mathcal{M}|^2$ level
- $\bar{\beta}_n$: IR-finite hard remainders $\rightarrow$ fixed-order accuracy

EEX in practice

- SHERPA-YFS [2203.10948], KKMC-EEX, ...
- ISR $\times$ FSR **interference** neglected
- The cost of the alternative: working at the amplitude level is tricky!

- No approximation in the master formula: accuracy is set by the $\bar{\beta}_n$ that are kept.

CEEX: coherent exclusive exponentiation

Exponentiate at the level of **helicity amplitudes** [Jadach, Ward, Was, hep-ph/0006359]

$$\sigma = \sum_{n_\gamma} \int_{\Omega} (d\text{Lips})_{n_\gamma+2} \underbrace{\frac{e^{\alpha Y(\tilde{\Omega})}}{n_\gamma!}}_{\text{YFS FF}} \sum_{\text{hel}} \left| \underbrace{\sum_{\phi \in [I,F]^{n_\gamma}} \prod_{i=1}^{n_\gamma} s_i^{\phi_i}}_{\text{coherent eikonals}} \underbrace{\left[\beta^0 + \sum_j \frac{\beta^1(k_j)}{s_j^{\phi_j}} + \dots \right]}_{\text{IR-safe amplitudes}} \right|^2$$

	YFS / EEX	CEEX (this work)
Exponentiation at the level of ISR–FSR interference	$ \mathcal{M} ^2$ Approx. / absent	Amplitudes, per helicity ✓ Exact, 2^{n_γ} partitions
Soft-photon spin correlations	Averaged	✓ Exact
Hard remainders	$\bar{\beta}_n$ from $ \mathcal{M} ^2$	β^n from Feynman amplitudes
Masses / divergences	—	All masses finite; IR & UV in dim. reg.
Truncation to fixed order	—	✓ LO, NLO truncation built in

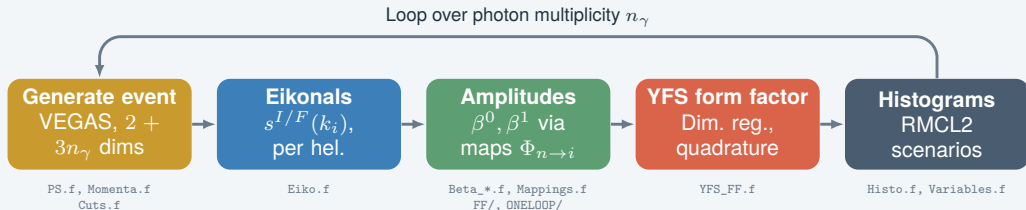
- This work: β^n from Feynman amplitudes, all masses finite, divergences in dim. reg.

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Ingredients for resummation at the amplitude level



Structure of the code



Implementation

Fortran, one executable (MPI/OpenMP)

$\pi\pi(\gamma)$: $F \times sQED$ $\mu\mu(\gamma)$: QED

$2 \rightarrow 2$: NLO and NLO+resummation included

To become a module of PHOKHARA

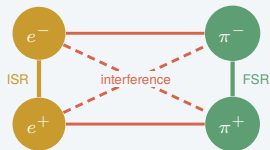
Built-in amplitude libraries from our theory group:
tensor decomposition $\rightarrow$ polarised amplitudes PLB 881
(2026) 140952

Checked against GoSAM at fixed order

YFS form factor

Sum over all **charged dipoles**: ISR, FSR and interference

$$Y(\Omega) = 2\alpha \sum_{i < j} \mathcal{R} \left(\underbrace{B(p_i, p_j)}_{\text{virtual}} + \underbrace{\tilde{B}(p_i, p_j, \Omega)}_{\text{real}} \right)$$



Resolved vs. unresolved

Real photons generated in Ω , FF integrated over $\tilde{\Omega}$;
 σ **independent** of $\tilde{\Omega}$; all masses kept

Lorentz-invariant unresolved domain

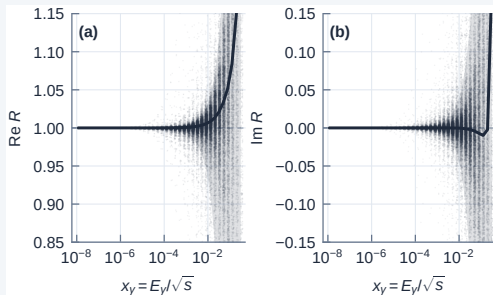
$$\tilde{\Omega} = \{ K \cdot k < \sqrt{s} \omega \}, K = p_{e^-} + p_{e^+} \quad (\text{lab: } E_\gamma < \omega)$$

$$Y(p_i, p_j, \tilde{\Omega}) = \frac{\alpha}{4\pi} \left[\underbrace{p_{ij} \log \frac{\omega^2}{s}}_{\text{universal}} + \underbrace{q_{ij}}_{\text{real: 1D}} + \underbrace{v_{ij}}_{\text{virtual}} \right]$$

- IR poles cancel between B and $\tilde{B}$ in **dim. reg.**
- q_{ij} : 1D quadrature in the (p_i, p_j) rest frame; v_{ij} : logs & dilogs
- KKMC / SHERPA: non Lorentz-invariant parametrisation $\Rightarrow$ weight corrections [hep-ph/0006359]; here **no approximation**

Eikonals at amplitude level: soft-limit handling

$$\mathcal{M}(\pi\pi\gamma(k)) \rightarrow \mathcal{M}(\pi\pi)s(k), \quad s = s^{\text{ISR}} + s^{\text{FSR}} \quad (\text{coherent: one spinor convention, } \hat{p}_i = p_i - x_i p_\gamma)$$



Soft factorisation, per helicity

Factorisation is shown to agree **for each helicity separately**, at the **amplitude level**: $R \rightarrow 1$, phase included

$R = \mathcal{M}(\pi\pi\gamma(k)) / [\mathcal{M}(\pi\pi)s(k)]$, $\sqrt{s} = 0.7 \text{ GeV}$, 1000 points $\times$ 8 hel.

- Eikonals reproduce the soft limit per helicity, phase included, over seven decades.

Momentum mappings

Each β^i is an IR-subtracted amplitude with exactly i **photons**, evaluated on the mapped event $\Phi_{n \rightarrow i}$ ($\pi\pi\gamma_1 \cdots \gamma_n \rightarrow \pi\pi\gamma_1 \cdots \gamma_i$): [Schönherr, Krauss 0810.5071; Krauss, Price, Schönherr 2203.10948; Price, Krauss 2512.04959]

$$\sigma = \sum_n \int (\text{dLips})_{n+2} \frac{e^{\alpha Y(\tilde{\Omega})}}{n!} \sum_{\text{hel}} \left| \sum_{\phi} \prod_{i=1}^n s_i^{\phi_i}(k_i) \left[\beta^0(\Phi_{n \rightarrow 0}) + \sum_j \frac{\beta^1(\Phi_{n \rightarrow 1}^{(j)})}{s_j^{\phi_j}} + \sum_{j < l} \frac{\beta^2(\Phi_{n \rightarrow 2}^{(jl)})}{s_j^{\phi_j} s_l^{\phi_l}} + \dots \right] \right|^2$$

Eikonals $s_i(k_i)$ on the generated event; $\Phi_{n \rightarrow 1}^{(j)}$ keeps photon j , $\Phi_{n \rightarrow 2}^{(jl)}$ photons j, l

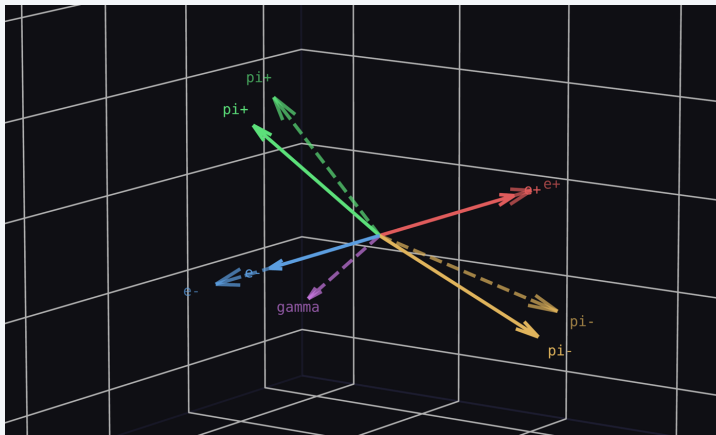
Requirements

- On-shell, momentum-conserving i -photon event
- Identity when the removed photons are soft
- Depends on the partition ϕ : each photon removed as ISR *or* FSR

Which maps where

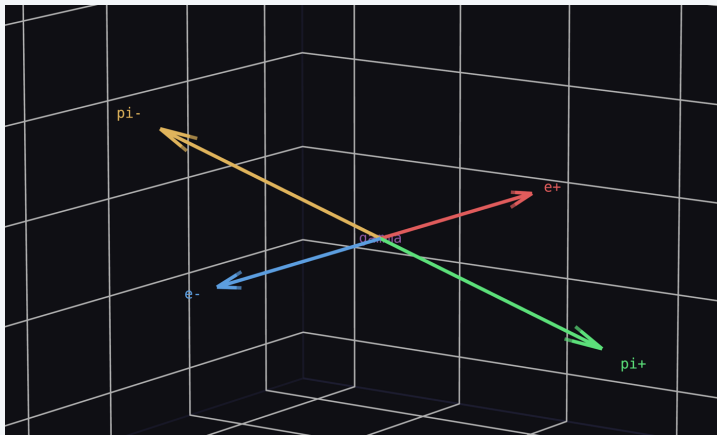
	LO+resum	NLO+resum
$2 \rightarrow 2$	$\Phi_{n \rightarrow 0}$	$+ \Phi_{n \rightarrow 1}$
$2 \rightarrow 3$	$\Phi_{n \rightarrow 0}, \Phi_{n \rightarrow 1}$	$+ \Phi_{n \rightarrow 2}$

Momentum mapping $\Phi_{1 \rightarrow 0}$: hard photon



Dashed: before, solid: after — the pions recoil to absorb the removed photon

Momentum mapping $\Phi_{1 \rightarrow 0}$: soft photon



Dashed: before, solid: after — the mapping reduces to the identity

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Resummation at LO



Setups: RMCL2 scenarios

CMD ($2 \rightarrow 2$), 0.7 GeV

$$e^+e^- \rightarrow \pi^+\pi^-, \mu^+\mu^-$$

$1 \leq \theta_{\text{avg}} \leq \pi - 1$ rad: central tracks

$p_{\pm} > 0.45 \sqrt{s}/2$: energetic tracks, no hard photon

$|\delta\phi| < 0.15$, $|\xi| < 0.25$ rad: back-to-back pair

KLOE-SA ($2 \rightarrow 3$), 1.02 GeV

$$e^+e^- \rightarrow \pi^+\pi^-\gamma, \mu^+\mu^-\gamma, \text{ photon untagged}$$

$\theta_{\gamma} < 15^\circ$ or $> 165^\circ$: photon along the beam, ISR dominated

$0.35 \leq M_{XX}^2 \leq 0.95 \text{ GeV}^2$: ρ region

Tracks at 50° – 130° , $|p_z| > 90$ or

$p_{\perp} > 160 \text{ MeV}$

KLOE-LA ($2 \rightarrow 3$), 1.02 GeV

$$e^+e^- \rightarrow \pi^+\pi^-\gamma, \mu^+\mu^-\gamma, \text{ photon tagged}$$

$50^\circ \leq \theta_{\gamma} \leq 130^\circ$, $E_{\gamma} > 20 \text{ MeV}$: photon detected, more FSR

$0.1 \leq M_{XX}^2 \leq 0.85 \text{ GeV}^2$: down to threshold

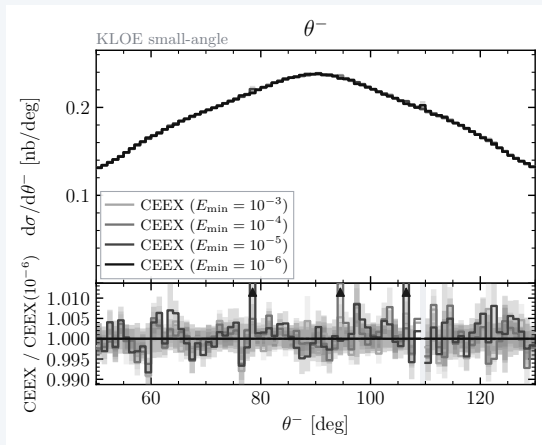
Same track cuts as SA

References: BABAYAGA LO, NLO, NLO+PS and McMule (CMD); PHOKHARA LO, NLO and BABAYAGA@NLO (KLOE); same inputs, no vacuum polarisation.

CEEX shown: NLO for all channels ($\pi\pi\gamma$: GoSAM and library, $\mu\mu\gamma$: GoSAM); LO+resum for $\pi\pi\gamma$; NLO+resum for $2 \rightarrow 2$.

$E_{\min}$ independence: a strong check of the implementation

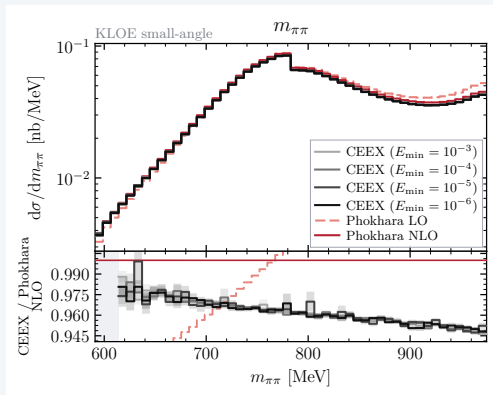
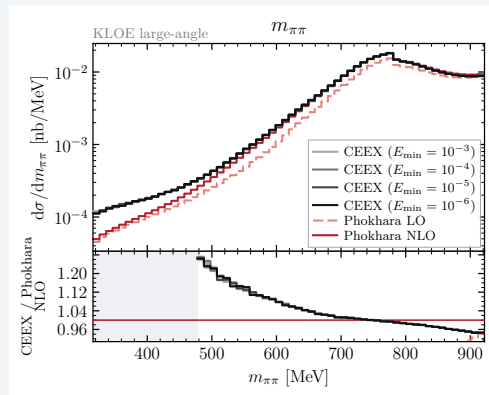
- $E_{\min}$ is a **generation cut only**
- $E_{\gamma} > E_{\min}$: photons generated explicitly
- $E_{\gamma} < E_{\min}$: integrated in the YFS form factor
- $\Rightarrow \sigma$ and shapes **independent of $E_{\min}$** , if eikonals, mappings and form factor are consistent
- Test: same run at $E_{\min} = 10^{-3} \dots 10^{-6}$ GeV



KLOE-SA θ^- , LO+resum, ratio to $E_{\min} = 10^{-6}$ GeV (campaign of 6 Oct)

LO+resummation: KLOE-SA and LA, $\pi\pi\gamma$

PRELIMINARY

KLOE-SA $M_{\pi\pi}$ vs. PHOKHARA LO / NLOKLOE-LA $M_{\pi\pi}$ vs. PHOKHARA LO / NLO

- CEEX vs. PHOKHARA NLO: SA 15.43 vs. 16.06 nb (-3.9%), LA 3.217 vs. 3.228 nb (-0.3%).

05

One loop and fixed-order NLO



Tensor decomposition: one-loop $\pi\pi\gamma$ [PLB 881 (2026) 140952]

4D tensor decomposition

$\mathcal{A}^{(L)} = \sum_{i=1}^8 \mathcal{F}_i^{(L)}(s_{ij}, m^2) \mathcal{T}_i$, 8 tensors at *any* loop order:

$$\begin{aligned} \mathcal{T}_1 &= \bar{v}_2 \not{p}_3 u_1 (p_1 \cdot \varepsilon_4) & \mathcal{T}_2 &= \bar{v}_2 \not{p}_3 u_1 (p_3 \cdot \varepsilon_4) \\ \mathcal{T}_3 &= \bar{v}_2 \not{p}_5 u_1 (p_1 \cdot \varepsilon_4) & \mathcal{T}_4 &= \bar{v}_2 \not{p}_5 u_1 (p_3 \cdot \varepsilon_4) \\ \mathcal{T}_5 &= \bar{v}_2 u_1 (p_1 \cdot \varepsilon_4) & \mathcal{T}_6 &= \bar{v}_2 u_1 (p_3 \cdot \varepsilon_4) \\ \mathcal{T}_7 &= \bar{v}_2 \not{p}_3 \not{p}_5 u_1 (p_1 \cdot \varepsilon_4) & \mathcal{T}_8 &= \bar{v}_2 \not{p}_3 \not{p}_5 u_1 (p_3 \cdot \varepsilon_4) \end{aligned}$$

$e^-(p_1) e^+(p_2) \rightarrow \pi^-(p_3) \gamma(p_4) \pi^+(p_5)$, $\varepsilon_4 \cdot p_2 = 0$;
F $\times$ SQED: 81 diagrams, 2 families

Polarised amplitudes

$\mathcal{A}_h^{(L)} = \Phi_h \sum_i (\mathbb{H}_{hi}^{(0)} + \text{tr}_5 \mathbb{H}_{hi}^{(1)}) \mathcal{F}_i^{(L)}$, 8 helicities

Φ_h : phases with $p_i^b = p_i - x_i p_\gamma$ (as eikonals)

Spurious Gram poles **cancel** in the polarised amplitudes

New: $\mathcal{O}(\epsilon)$ and $\mathcal{O}(\epsilon^2)$

Six ISR/FSR groups up to ϵ^2 — **not computed before**, needed for NNLO; checked vs. Catani IR poles, GoSAM, DIFFEXP (10–12 digits)

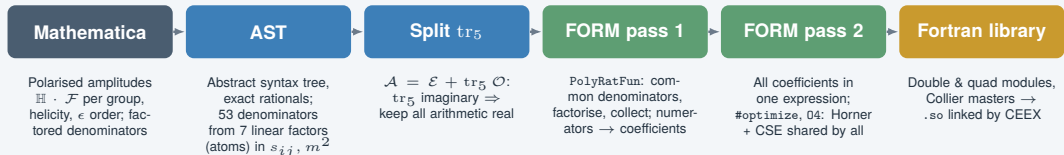
Master integrals

38 masters (box & pentagon); canonical DEs
 $d\vec{W} = \epsilon \sum_i \mathbb{A}_i d \log \beta_i \vec{W}$: boundary values from AMFlow, evolved numerically to each point, up to weight 4

Output

$\mathcal{A}_h^{(1)}$ per point: 8 helicities, six groups, $\epsilon^{-1} \dots \epsilon^2$
Fortran library (double & quad) $\rightarrow \beta_1^{(1)}$, NLO in CEEX

Optimising the polarised amplitudes with FORM



Size: ϵ^{-1} and ϵ^0 , six groups

Mathematica input	52 MB
Before: via form factors $\mathcal{F}_i$	1.7 GB Fortran
After: polarised amplitudes	90 MB Fortran
Compiled library	85 MB

Polarised amplitudes, not form factors

We optimise the polarised amplitudes $\mathcal{A}_h$ directly: easier to deal with than the $\mathcal{F}_i$ — no Gram denominators, stable in the collinear region, $\sim 20\times$ less code
Each step checked exactly over $\text{GF}(2^{61}-1)$

- In CEE X: ~ 3 ms/point, $1.1\text{--}1.5\times$ faster than GoSAM.

NLO truncation: YFS as a slicing scheme

$e^+e^- \rightarrow \pi^+\pi^-$ ($2 \rightarrow 2$): expand the master formula in α (FF, eikonals, β 's), keep $\mathcal{O}(\alpha)$:

$$\sigma_{\text{NLO}}^{\text{trunc.}} = \int (d\text{PS})_2 \left[\underbrace{|\mathcal{M}_{\text{LO}}|^2 (1 + \alpha Y_V + \alpha Y_R) + 2\mathcal{R}\left(\mathcal{M}_{\text{LO}}^\dagger (\mathcal{M}_{1L}^{\text{ren}} - \tfrac{1}{2}\alpha Y_V \mathcal{M}_{\text{LO}})\right)}_{\text{finite}} \right] + \underbrace{\int_{\Omega} (d\text{PS})_3 |\mathcal{M}_R|^2}_{\text{finite}}$$

A phase-space slicing

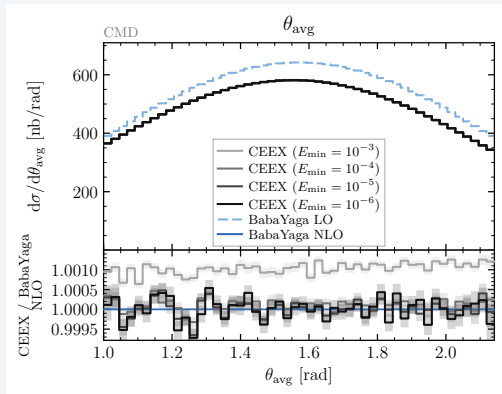
$E_\gamma < E_{\min}$: *analytically*, in $\alpha Y_R |\mathcal{M}_{\text{LO}}|^2$
 $E_\gamma > E_{\min}$ (in Ω): $|\mathcal{M}_R|^2$ *numerically*, no counterterm
 Exact as $E_{\min} \rightarrow 0$

One loop, IR-safe

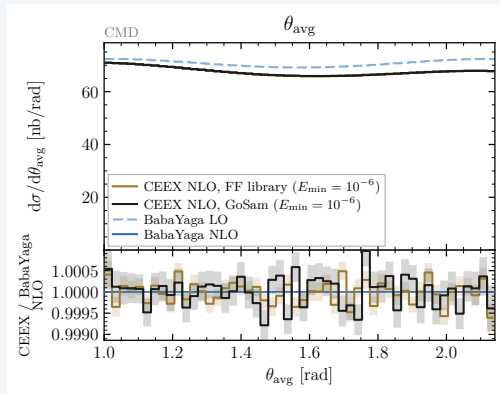
$\beta_0^{(1)} = \mathcal{M}_{1L}^{\text{ren}} - \tfrac{1}{2}\alpha Y_V \mathcal{M}_{\text{LO}}$
 $\mathcal{M}_{1L}^{\text{ren}} = a_0 + \frac{a_{-1}^{\text{IR}}}{\epsilon_{\text{IR}}} + \frac{a_{-1}^{\text{UV}}}{\epsilon_{\text{UV}}} + \frac{\text{ct}}{\epsilon_{\text{UV}}}$
 IR pole of $\mathcal{M}_{1L} = \text{YFS virtual pole}$
 [hep-ph/0011222]: $\beta_0^{(1)}$ finite, checked **point by point**

- Same code gives exact NLO (truncated) and resummed predictions: a built-in cross-check.

NLO $2 \rightarrow 2$: CMD at 0.7 GeV, $\pi\pi$ and $\mu\mu$



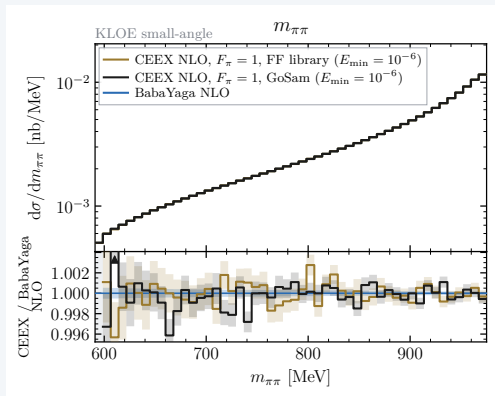
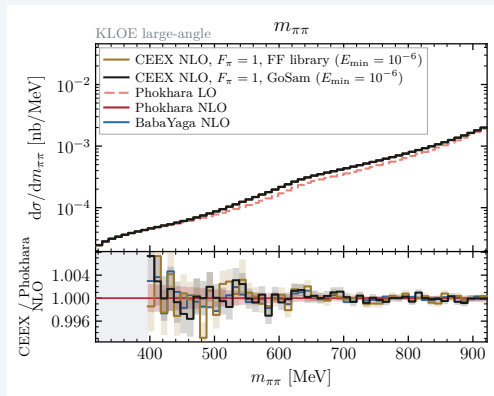
$e^+e^- \rightarrow \pi^+\pi^-: \theta_{\text{avg}}$ vs. BABAYAGA NLO



$e^+e^- \rightarrow \mu^+\mu^-: \theta_{\text{avg}}$ vs. BABAYAGA NLO

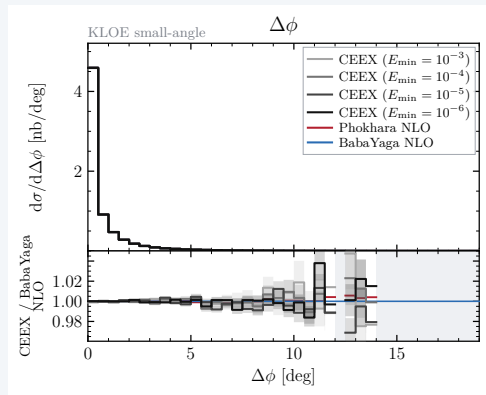
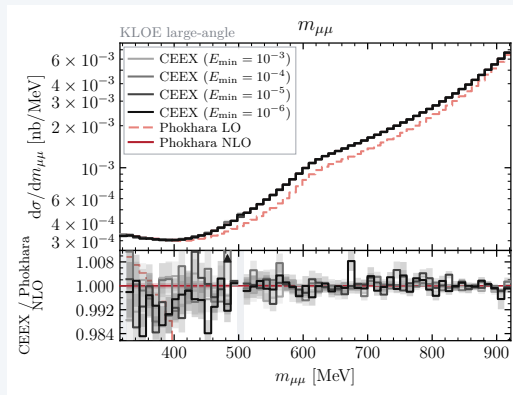
- As $E_{\text{min}} \rightarrow 0$, the CEEX curves converge to the NLO reference.

NLO $\pi\pi\gamma$: KLOE small and large angle (GoSAM and library)

KLOE-SA $M_{\pi\pi}$ vs. BABA YAGA NLO ($F_\pi = 1$)KLOE-LA $M_{\pi\pi}$ vs. BABA YAGA NLO ($F_\pi = 1$)

- GoSAM / lib. vs. BABA YAGA: SA -0.010(9)% / -0.001(12)%, LA -0.005(9)% / +0.001(8)%.

NLO $\mu\mu\gamma$: KLOE small and large angle (GoSAM)

KLOE-SA $\Delta\phi$ vs. BABA YAGA NLOKLOE-LA $M_{\mu\mu}$ vs. PHOKHARA NLO

- High-stat. CEEEX/BABA YAGA -1: $+0.003(5)\%$ (SA), $-0.014(9)\%$ (LA); plots at campaign stat.

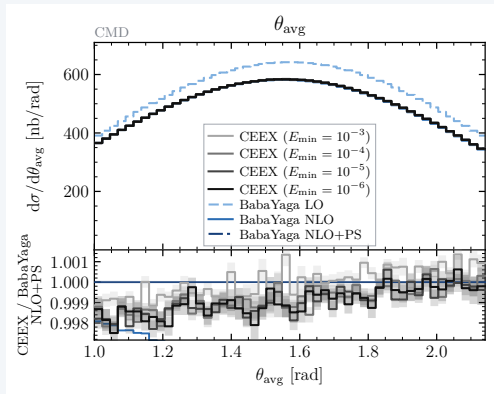
06

NLO + resummation

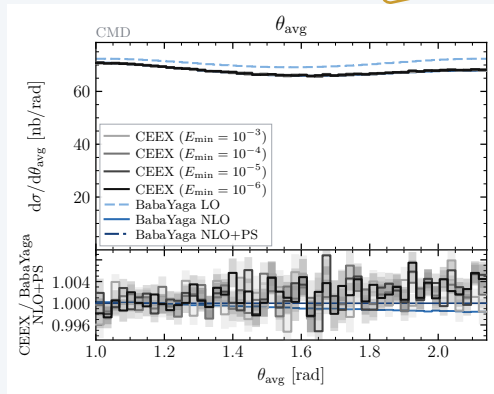


NLO+resummation $2 \rightarrow 2$: CMD at 0.7 GeV

PRELIMINARY



$e^+e^- \rightarrow \pi^+\pi^-: \theta_{\text{avg}}, \text{ratio to BABA YAGA NLO+PS}$



$e^+e^- \rightarrow \mu^+\mu^-: \theta_{\text{avg}}, \text{ratio to BABA YAGA NLO+PS}$

- CEEX NLO+resum / BABA YAGA NLO+PS −1: -0.094(3)% ($\pi\pi$), +0.153(15)% ($\mu\mu$, validation).

Resummed vs. fixed order: summary

PRELIMINARY

σ [nb]	Reference		CEEX		CEEX / ref. -1 [%]	
	NLO	NLO+PS	LO+resum	NLO+resum	$\frac{\text{LO+resum}}{\text{NLO}}$	$\frac{\text{NLO+resum}}{\text{NLO+PS}}$
$2 \rightarrow 2$: CMD, $\sqrt{s} = 0.7$ GeV, reference BABAYAGA						
$e^+e^- \rightarrow \pi^+\pi^-$	567.357	570.043	547.048(8)	569.509(17)	-3.6	-0.094(3)
$e^+e^- \rightarrow \mu^+\mu^-$	77.1668	77.2256	74.3507(28)	77.343(12)	-3.6	+0.153(15)
$2 \rightarrow 3$: KLOE, $e^+e^- \rightarrow \pi^+\pi^-\gamma$, $\sqrt{s} = 1.02$ GeV, reference PHOKHARA						
KLOE-SA	16.0568	—	15.427(4)	WIP	-3.9	—
KLOE-LA	3.22754	—	3.2175(3)	WIP	-0.3	—

Reading the numbers: LO+resum $\neq$ NLO, the hard virtual $\beta_0^{(1)}$ enters only at NLO+resum; for $2 \rightarrow 2$, NLO+resum vs. NLO+PS is the like-for-like comparison. **Status:** $2 \rightarrow 2$ NLO+resum in validation ($\mu\mu$: no like-for-like reference yet); $2 \rightarrow 3$ NLO+resum WIP (β^2 , $\Phi_{n \rightarrow 2}$).

- $2 \rightarrow 2$: NLO+resum within 0.15% of BABAYAGA NLO+PS; $2 \rightarrow 3$ at LO+resum.

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Conclusions



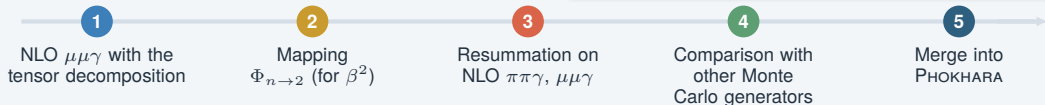
Road to NLO+resummation: showstoppers & next steps

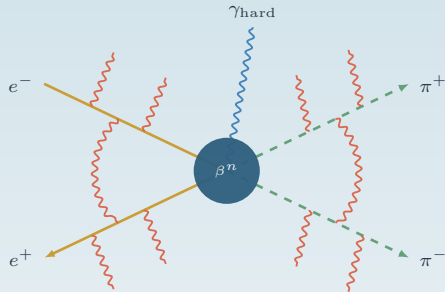
$2 \rightarrow 2: \pi\pi$ **Validation** $\mu\mu$ **Validation**

- Ingredients in place: $\beta_0^{(1)}$ (one loop), β^1 (tree + γ), $\Phi_{n \rightarrow 0}, \Phi_{n \rightarrow 1}$
- $E_{\min}$ **independence** at NLO+resum
- YFS form factor calculated in dim. reg. and validated
- Needs validation against other resummation codes

$2 \rightarrow 3: \pi\pi\gamma, \mu\mu\gamma$ **WIP**

- Have: $\beta_1^{(1)}$ = one-loop $\pi\pi\gamma$ (library), tree $\pi\pi\gamma\gamma$
- Need: β^2 (tree $\gamma\gamma$ — eikonals), mapping $\Phi_{n \rightarrow 2}$
- **CPU**: one loop per event and partition, quad in collinear corners
- $\mu\mu\gamma$: fixed-order NLO validated vs. BABAYAGA, $< 10^{-4}$ (high stat.)
- $\mu\mu\gamma$: **polarised amplitudes from the form factors** still to do — tensor structure more complicated





Thank you

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Stay tuned for the next release of PHOKHARA 📩

NNLO + coherent soft-photon resummation