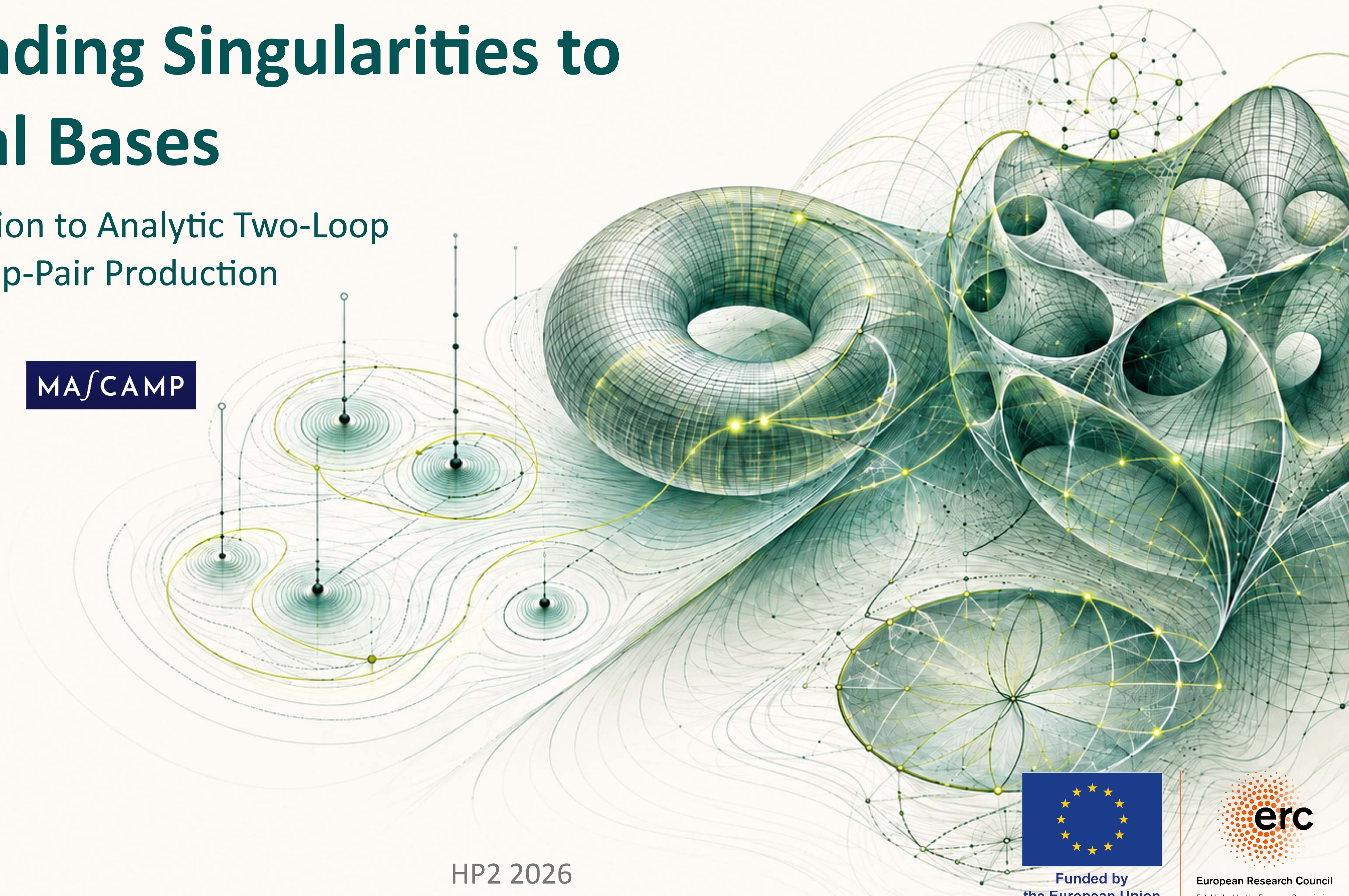


From Leading Singularities to Canonical Bases

With an Application to Analytic Two-Loop Corrections to Top-Pair Production

Christoph Nega

MA/CAMP



MAX PLANCK INSTITUTE
FOR GRAVITATIONAL PHYSICS
(ALBERT EINSTEIN INSTITUTE)

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Collaborations

Joint work with:

Federico Buccioni, Claude Duhr, Felix Forner, Lennard Görge, Sara Maggio, Andreas von Manteuffel, Cesare C. Mella, Benjamin Sauer, Lorenzo Tancredi, and Fabian J. Wagner

Collaborations:

"Integrand Analysis, Leading Singularities and Canonical Bases beyond Polylogarithms" [1],

"Aspects of canonical differential equations for Calabi-Yau geometries and beyond" [2],

"On a procedure to derive ϵ -factorised differential equations beyond polylogarithms" [3],

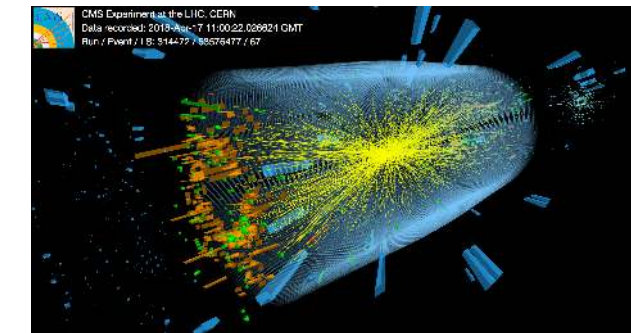
"Analytic two-loop corrections to top-quark pair production at hadron colliders" [4] (appearing soon)



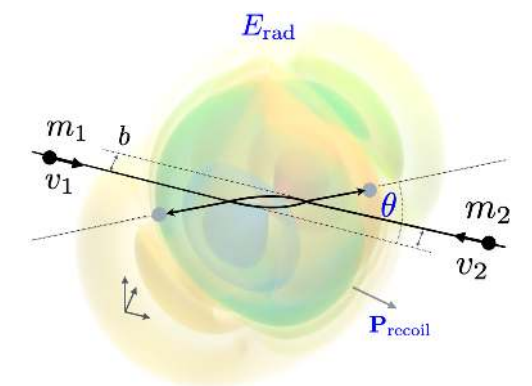
Why Canonical Integrals?

- Many **state-of-the-art scattering amplitude computations** use **canonical integrals**. These integrals make the **analytic properties manifest**.

Particle Physics



Gravity



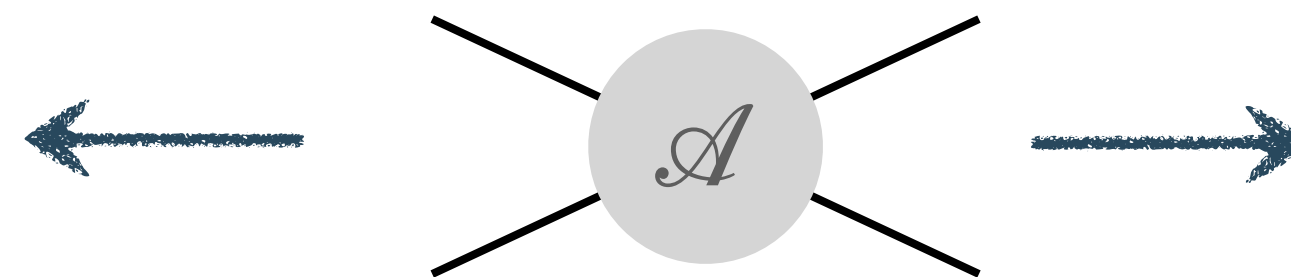
Special analytic properties

poles of $\mathcal{A}$
 $\sim \frac{1}{s}$

single particles go on-shell

branch cuts of $\mathcal{A}$
 $\sim \{\sqrt{s}, \log(s)\}$

multi-particle threshold



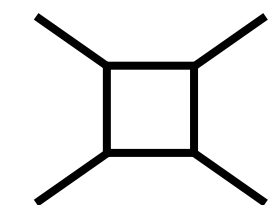
Decomposition through scalar Feynman integrals

$\mathcal{A} \sim T^{\mu\nu}$

tensor structure

$R(s)$

rational



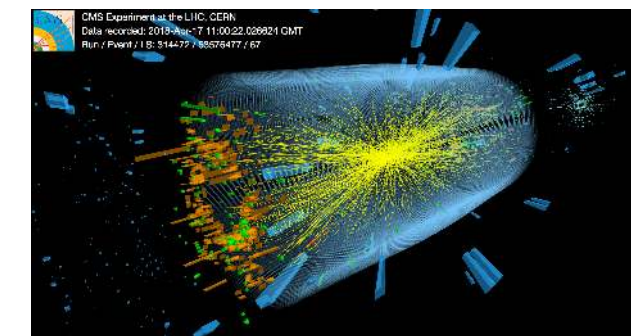
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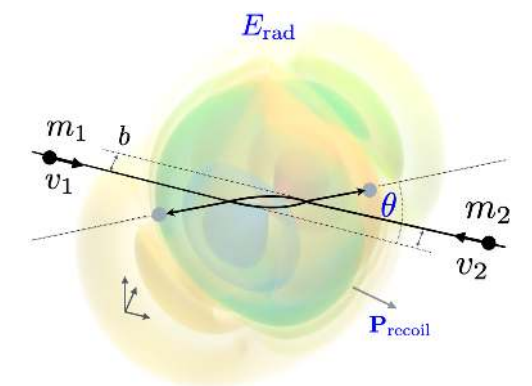
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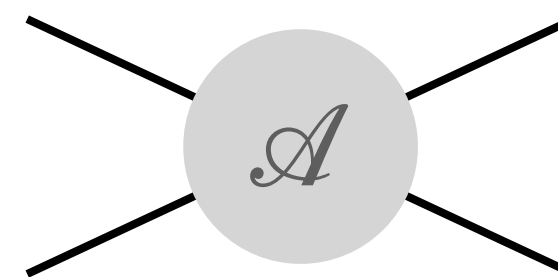
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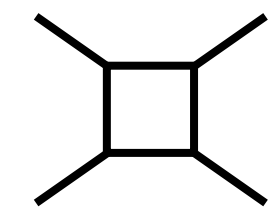
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scalar integrals

- Also, **canonical integrals** simplify the **differential equations** and allow for **pure & uniform transcendental** solutions.

$$\frac{\partial}{\partial z_i} \underline{I} = B(\underline{z}; \epsilon) \underline{I}$$

[Kotikov, Gehrmann and Remiddi]

canonical
integrals

$$\frac{\partial}{\partial z_i} \underline{J} = \epsilon A(\underline{z}) \underline{J} \quad [\text{Henn}]$$

ϵ -factorized

Pure & uniform transcendental
iterated integrals [Chen]



How to find them?

- For **polylogarithmic integrals**, it is well-known how to find canonical integrals. One approach is given by **studying the integrand** of candidate integrals.

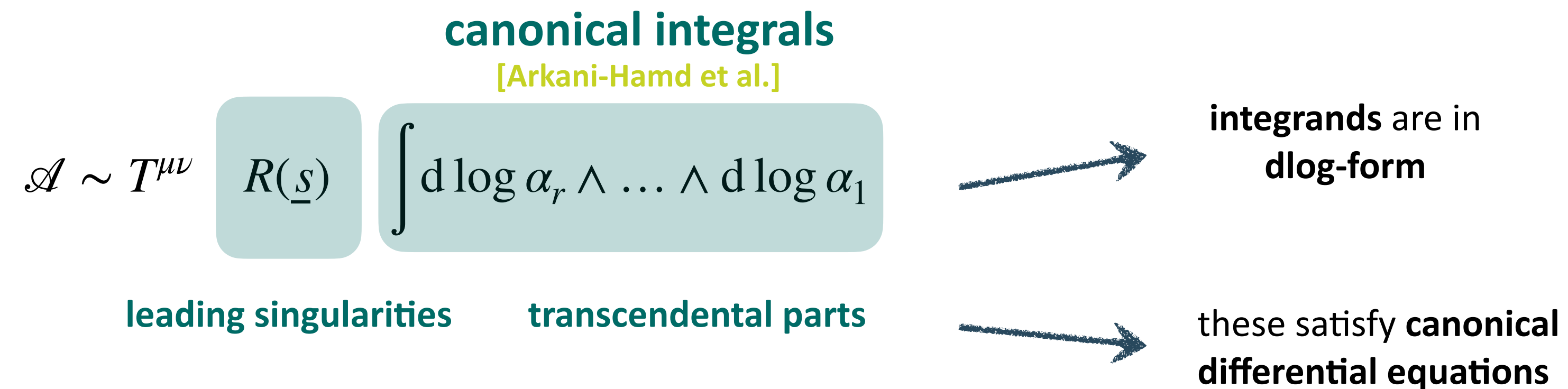
[Arkani-Ham et al.]

$$\mathcal{A} \sim T^{\mu\nu} R(\underline{s}) \int d \log \alpha_r \wedge \dots \wedge d \log \alpha_1 \longrightarrow \text{integrands are in dlog-form}$$



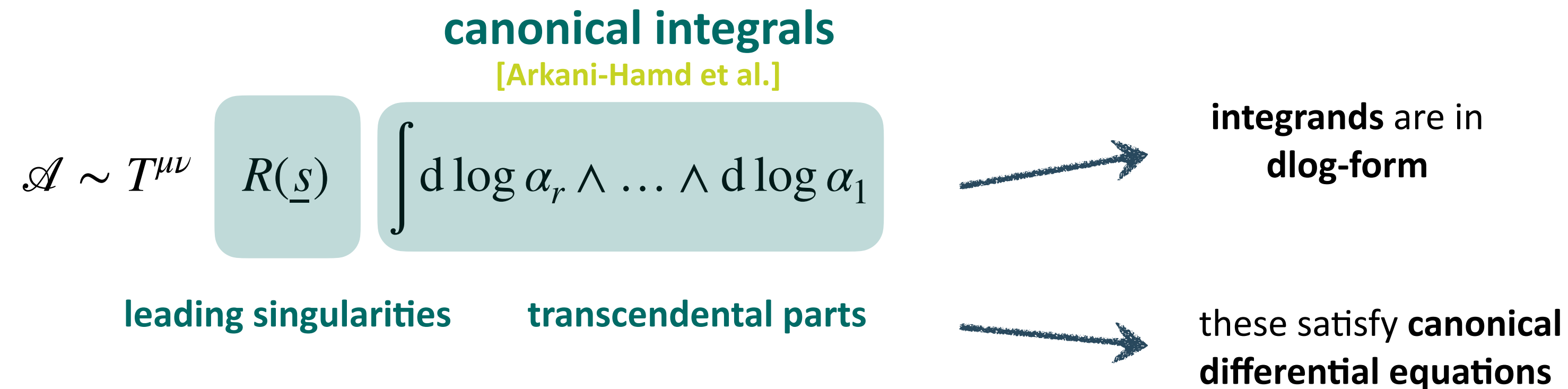
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Example:

$f(x) = \frac{1}{x} \log(x-1)$
not pure

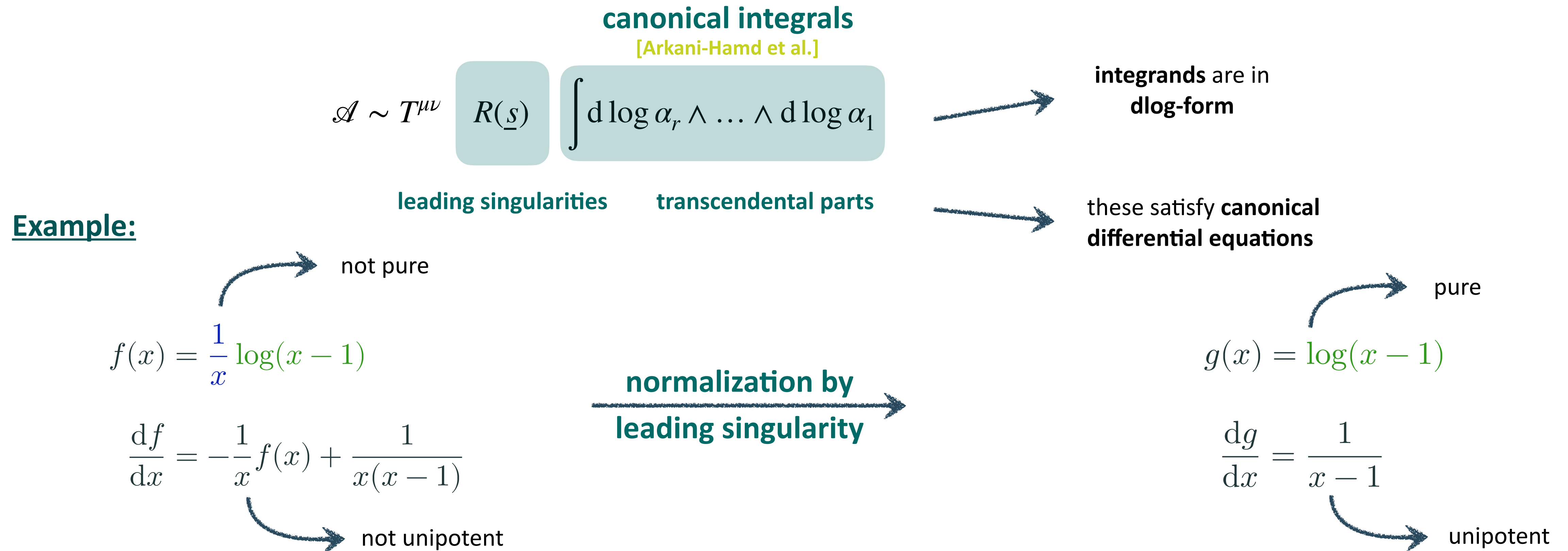
$$\frac{df}{dx} = -\frac{1}{x} f(x) + \frac{1}{x(x-1)}$$

not unipotent



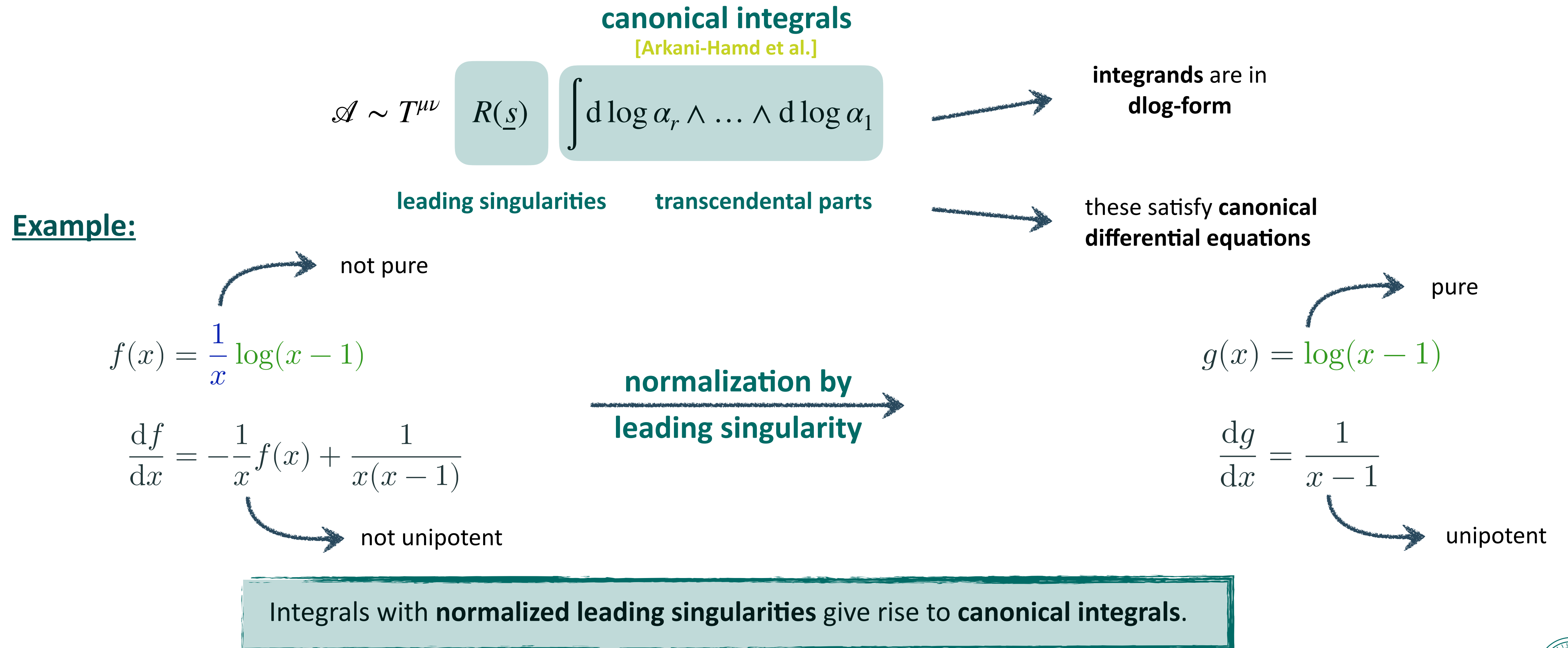
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- We have developed a **method [1-3]** to **compute canonical integrals beyond the polylogarithmic case**, which is divided into the following steps:

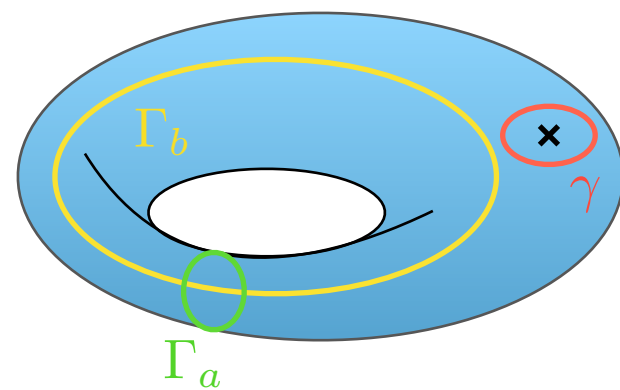


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- We have developed a **method [1-3]** to **compute canonical integrals beyond the polylogarithmic case**, which is divided into the following steps:

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Find a **start basis** that reflects the underlying geometry (**mixed Hodge structure**)



1st-kind
(no pole)

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The **derivative basis** is the reason why our approach terminates!

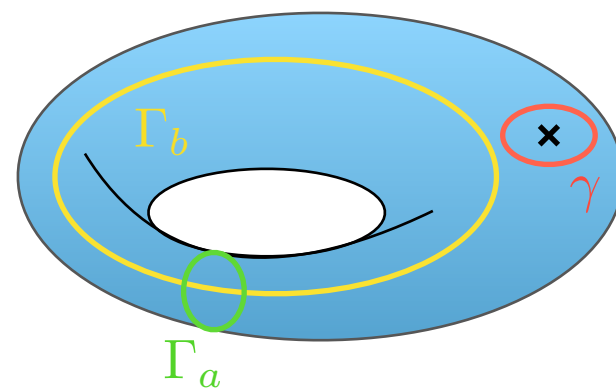


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$$W_n(z) = \underbrace{W^{\text{ss}}}_{\text{algebraic}} \cdot \underbrace{W^{\text{u}}}_{\text{logarithmic}}$$

We need a **basis of periods** of the underlying geometry.

Griffiths relations necessary for splitting.

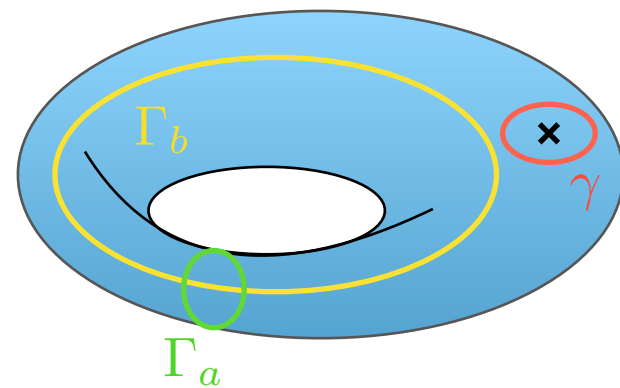


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Transcendental functions related to **periods** of geometry.

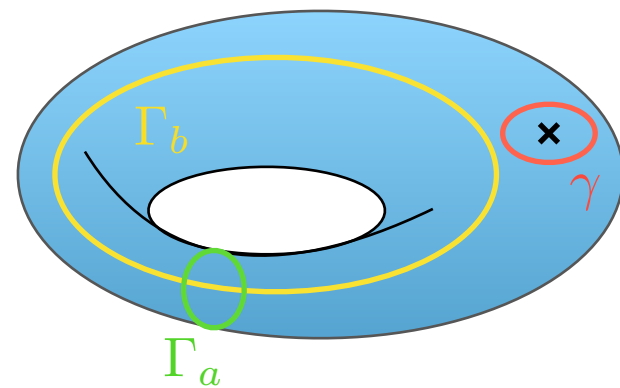


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- ⦿ In our **recent paper [1]**, we have given these **three steps** an **interpretation** in terms of **integrand analysis**.



Leading Singularities beyond Polylogs

- To argue that our procedure [2-3] indeed produces integrals with normalized leading singularities satisfying canonical ϵ -factorized differential equations, we first have to **generalize** the **concept of Leading Singularities**:

Leading singularities are all functions obtained by iteratively evaluating the integrand close to $\epsilon = 0$ on a basis of closed independent contours.



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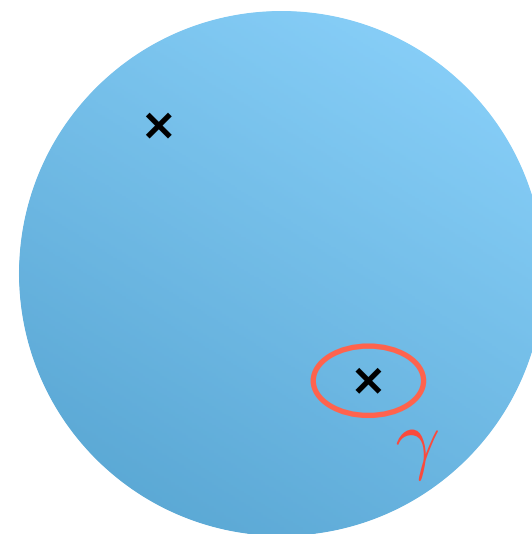
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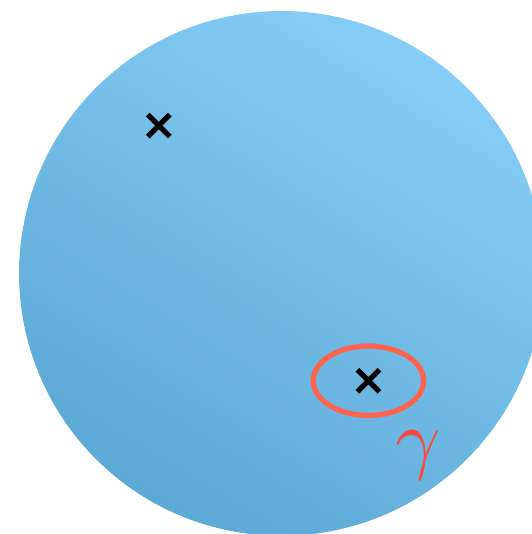
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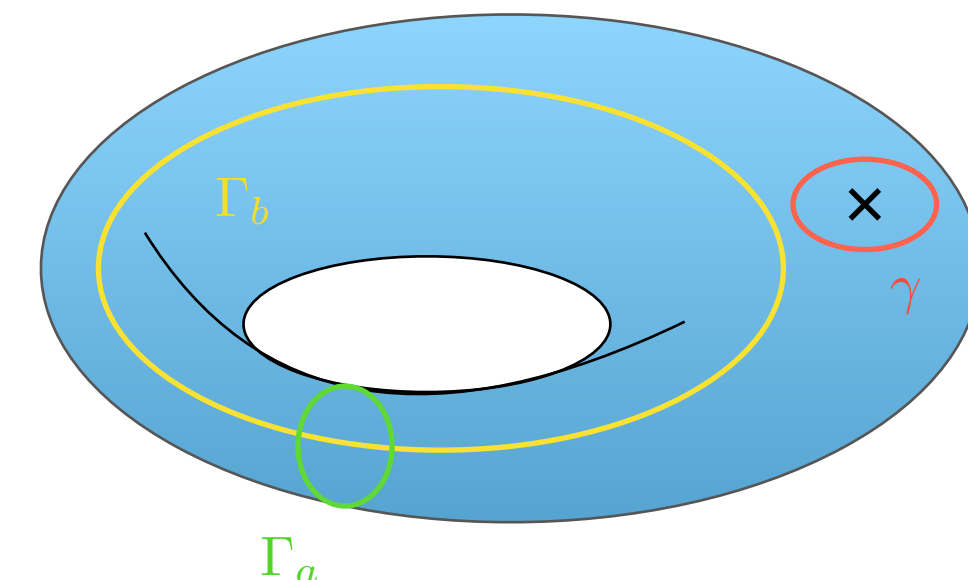
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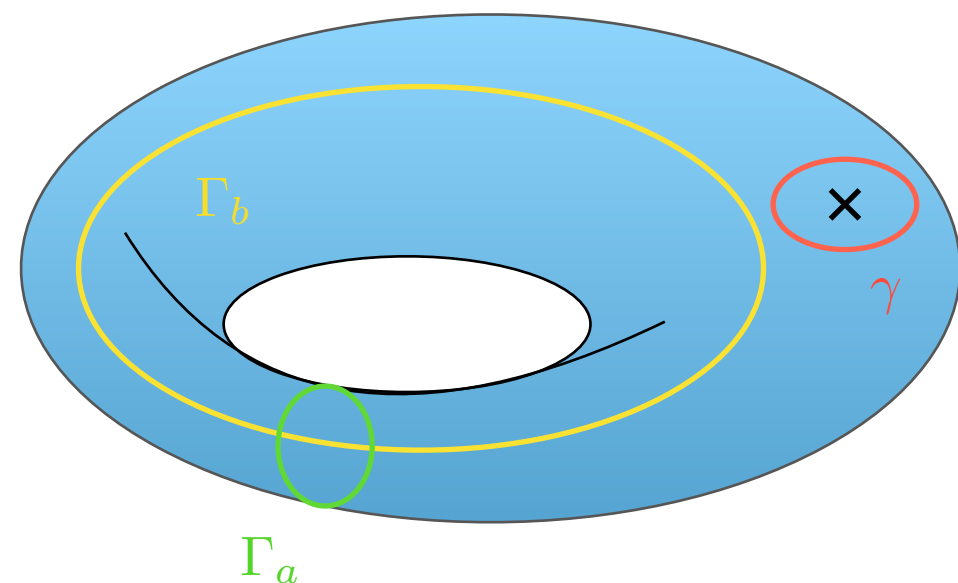
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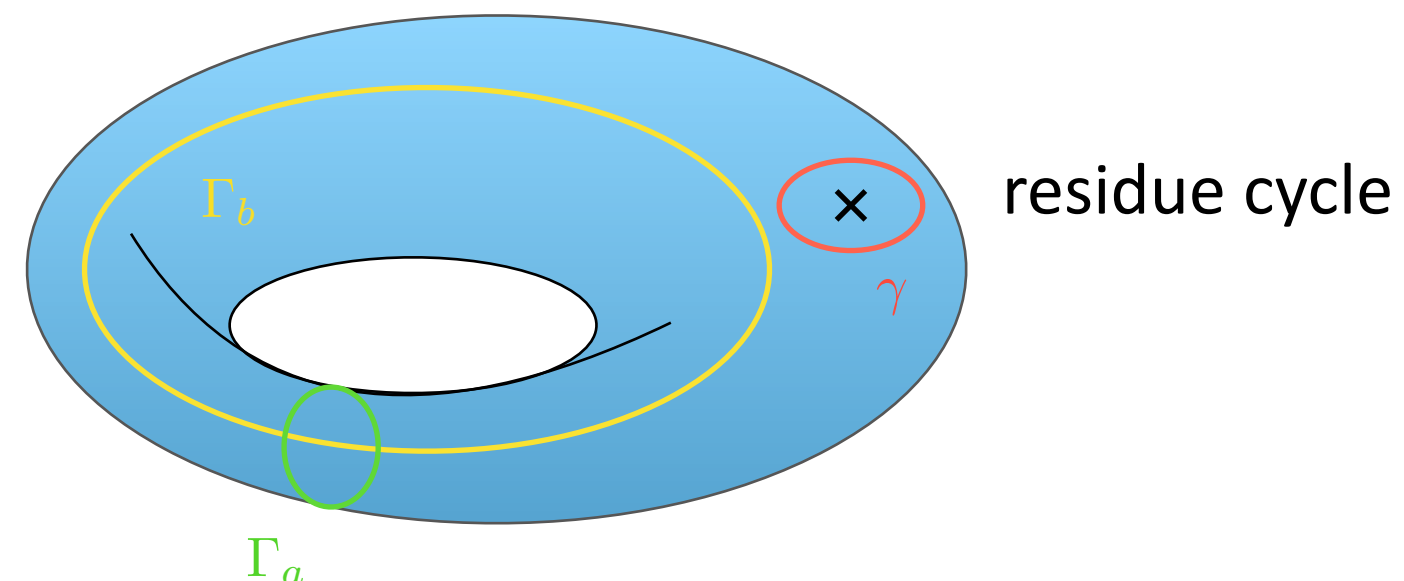
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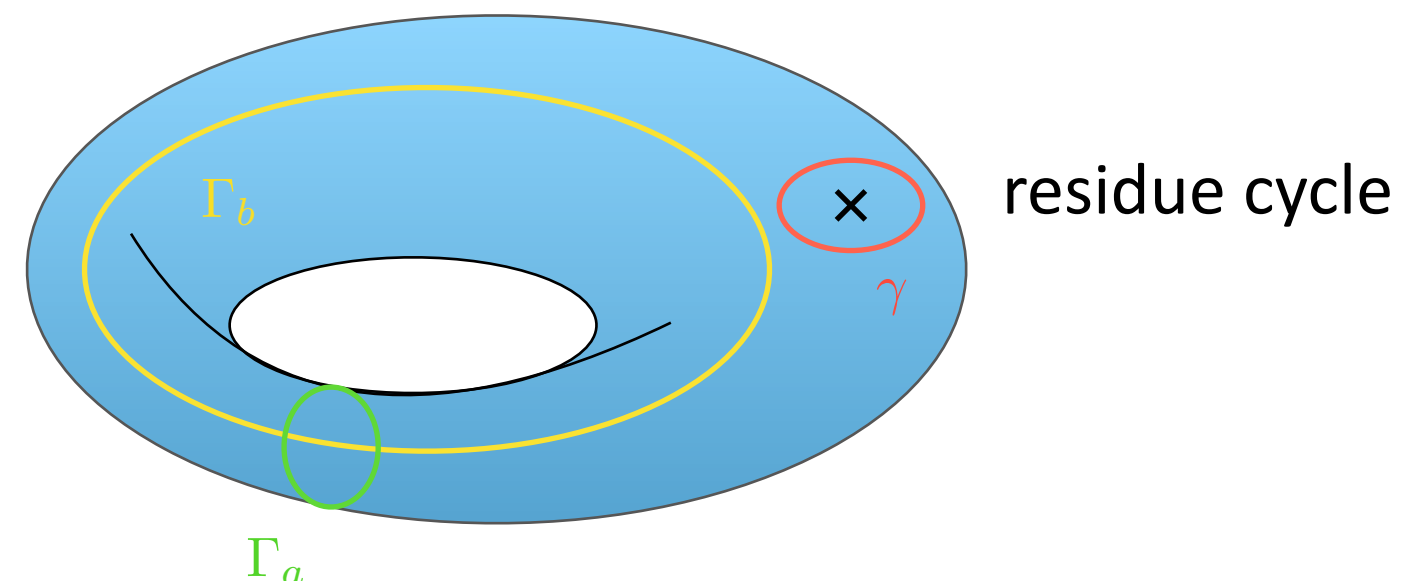
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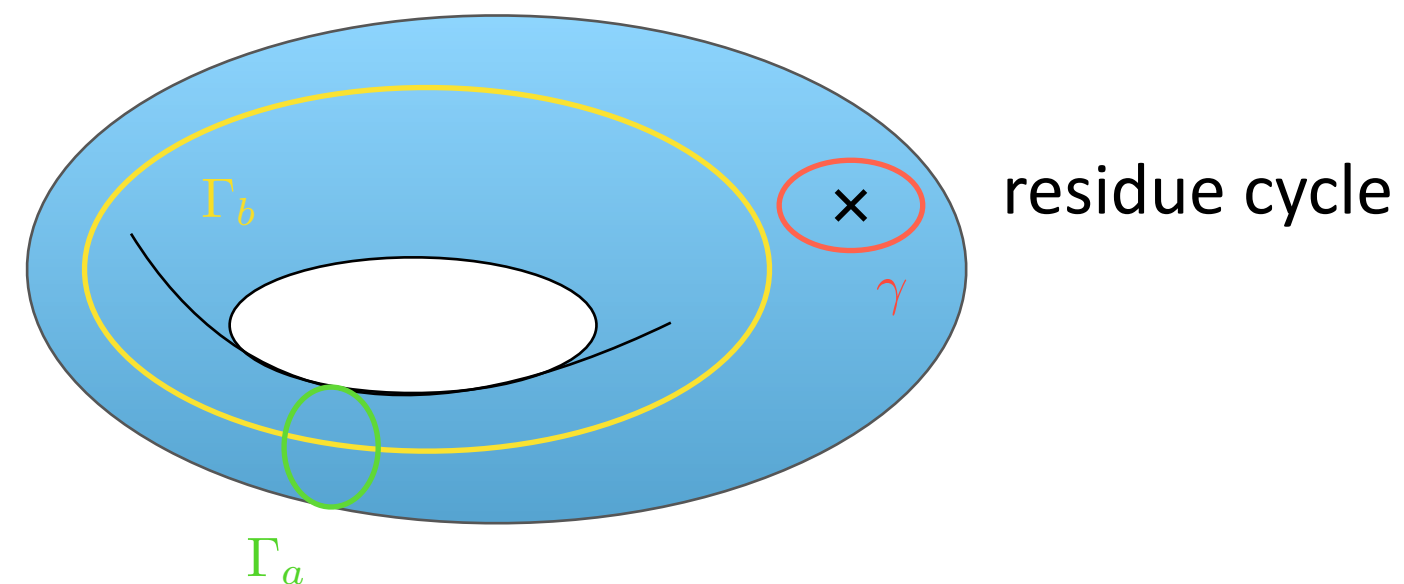
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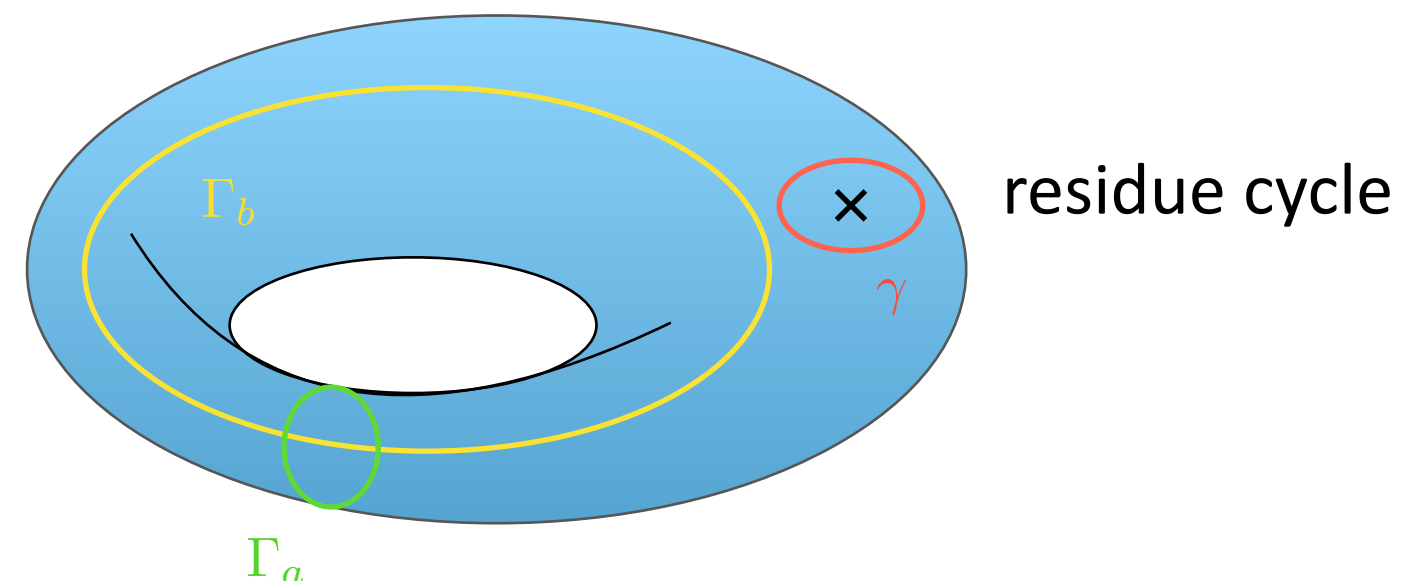
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Beyond the polylogarithmic case, leading singularities are themselves transcendental functions. Some integrals, see I_3 , can have **multiple different leading singularities**.



Unit Leading Singularities

- Second, we have to generalize the notion of **unit leading singularities**:

A basis of integrals has unit leading singularities if, locally, all integrals have maximal transcendental weight, all corresponding integrands have at most single poles, and all leading singularities associated with the holomorphic cycle(s) and with the cycles encircling the single poles are constant to order ϵ^0 .



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$$J_2 = \frac{1}{\partial_a \tau} \frac{\partial}{\partial a} \left(\frac{1}{\varpi_0} I_1 \right) = \frac{\varpi_0^2}{\Delta(a)} \frac{\partial}{\partial a} \left(\frac{1}{\varpi_0} I_1 \right)$$

$$\Delta(a) = \varpi_0 \varpi_1' - \varpi_0' \varpi_1$$



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$$I_2 = \frac{\partial}{\partial a} \left(\frac{1}{\varpi_0} I_1 \right) \longrightarrow \begin{cases} A\text{-cycle: } \text{LS}[I_2]_A = 0, \\ B\text{-cycle: } \text{LS}[I_2]_B = \partial_a \tau(a) \sim \frac{1}{a}, \end{cases}$$

different transcendental weight $\rightarrow \epsilon^{-1}$

$$J_2 = \frac{1}{\partial_a \tau} \frac{\partial}{\partial a} \left(\frac{1}{\varpi_0} I_1 \right) = \frac{\varpi_0^2}{\Delta(a)} \frac{\partial}{\partial a} \left(\frac{1}{\varpi_0} I_1 \right)$$

$$\Delta(a) = \varpi_0 \varpi_1' - \varpi_0' \varpi_1$$

$$W_{\text{ss}}^{-1} = \begin{pmatrix} \frac{1}{\varpi_0} & 0 \\ -\frac{\varpi_0'}{\Delta(a)} & \frac{\varpi_0}{\Delta(a)} \end{pmatrix}$$

Producing integrals with **unit leading singularities** is similar to multiplying by the **inverse semi-simple** part of the **period matrix/Wronskian**.



Unit Leading Singularities

A basis of integrals has unit leading singularities if, locally, all integrals have maximal transcendental weight, all corresponding integrands have at most single poles, and all leading singularities associated with the holomorphic cycle(s) and with the cycles encircling the single poles are constant to order ϵ^0 .

Example:

Elliptic case:

- If the integrals are extended to ϵ -dimensions, we have to analyze the **second integral** because of the **weight drop to one order higher in ϵ** .

$$K_2 = \frac{1}{\epsilon} \frac{\varpi_0^2}{\Delta(a)} \partial_a J_1 - \varpi_0^2(a) J_1$$

→ This normalizes the leading singularities up to ϵ^0 .



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- For the **third integral**, I_3 , we have to **remove additional leading singularities**.

$$J_3 = y(b)I_3 - GJ_1 = y(b)I_3 - G \frac{I_1}{\varpi_0} \longrightarrow \begin{cases} \text{residue: } \text{LS}[J_3]_{x=b} = 1, \\ \text{A-cycle: } \text{LS}[J_3]_A = 0, \\ \text{B-cycle: } \text{LS}[J_3]_B \sim c \log(a), \end{cases}$$



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Additional LS must be removed, requiring new transcendental functions, which we collectively call **G-functions.**



Integrand Analysis Beyond Polylogarithms: Summary

Integrand Analysis, Leading Singularities and Canonical Bases beyond Polylogarithms [1]

● In summary, we have established the following connections:

- We have identified the **periods** as **leading singularities** for $\epsilon = 0$ that have to be normalized through $(W^{ss})^{-1}$.
→ Works for **arbitrary geometries** (mathematically: **Mixed Hodge Structure**)
- Choosing the **second-kind forms as derivatives** guarantees that our **procedure terminates** after finitely many steps.
- **Higher poles** of differential forms produce a **weight drop**, which requires studying the integrals to higher orders in ϵ .
→ For **polylogarithmic integrals**, forms with **higher poles are not independent**, and a basis containing **only single poles** exists. Therefore, additional steps are not needed.
- The **clean-up step normalizes** leading singularities **at higher orders** in ϵ . In other words, **G-functions** are just **LS** of our integrals, either for $\epsilon = 0$ or at higher orders in ϵ .
- In [1], we have also shown the **equivalence** between **normalizing LS** at the **integrand level** and via the **differential equations**.
- **Canonical bases** constructed via **our procedure** should **define/be related to** the correct **pure multiple polylogarithms** on the **considered geometries**.



Two-Loop Corrections to Top-Pair Production

- ◎ The current **state-of-the-art application** of our **canonicalization method** is our upcoming article [\[4\]](#) about **analytic two-loop corrections to top-pair production** at hadron colliders.

Preliminary Results!

$$q(p_1) + \bar{q}(p_2) \rightarrow t(p_3) + \bar{t}(p_4)$$

quark annihilation channel

$$g(p_1) + g(p_2) \rightarrow t(p_3) + \bar{t}(p_4)$$

gluon channel

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad m_t^2$$

two-parameter problem



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- Number of **diagrams**:

Process	tree level	one loop	two loops
$gg \rightarrow t\bar{t}$	3	34	789
$q\bar{q} \rightarrow t\bar{t}$	1	11	218

- **14 integral families** (5 planar, 9 non-planar) requiring **417 master integrals**.
- These master integrals contain **17 elliptic curves** and **1 genus-two curve**.



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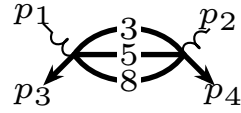
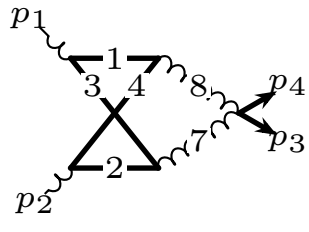
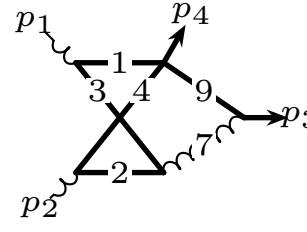
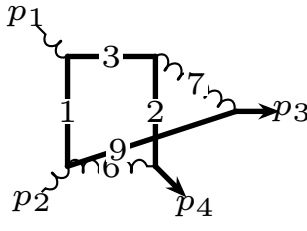
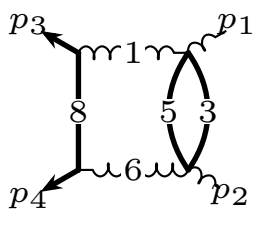
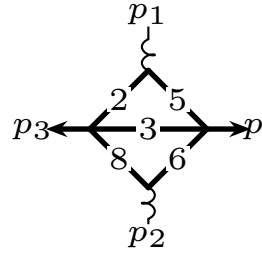
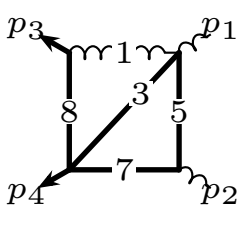
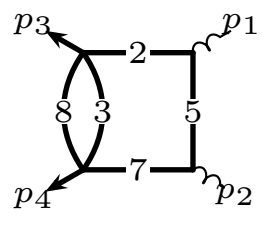
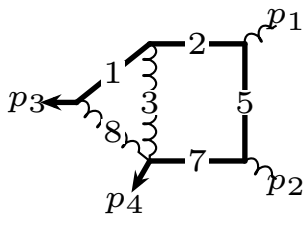
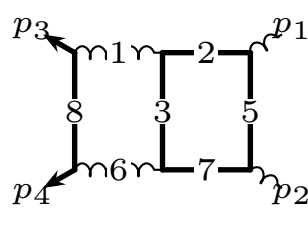
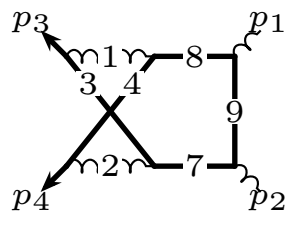
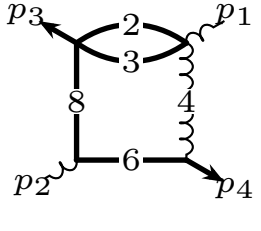
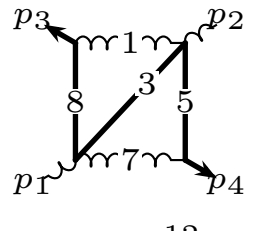
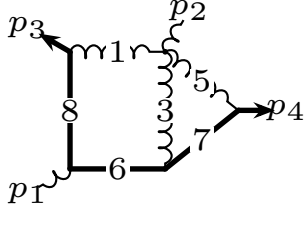
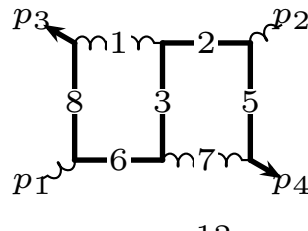
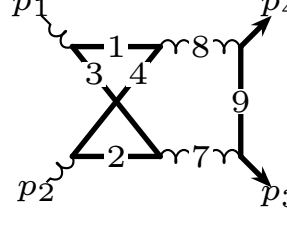
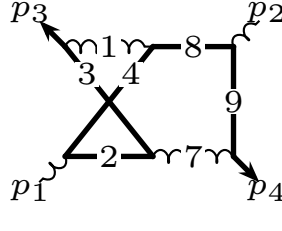
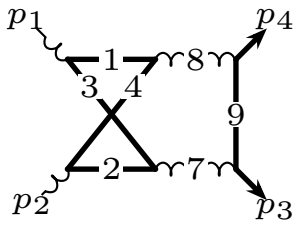
**Multiple different geometries
&
Two-parameter problem**

HARD



Two-Loop Corrections to Top-Pair Production

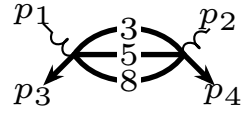
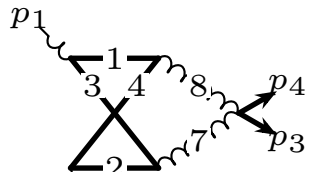
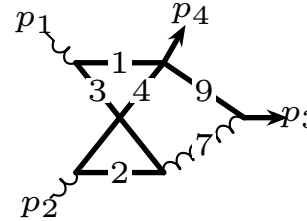
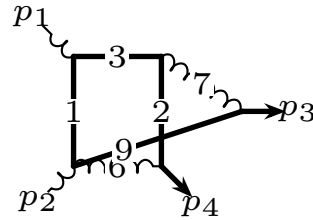
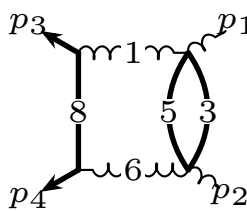
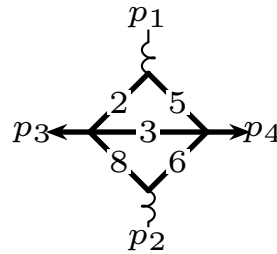
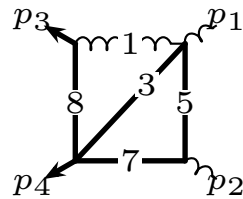
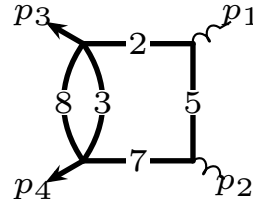
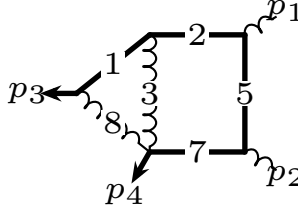
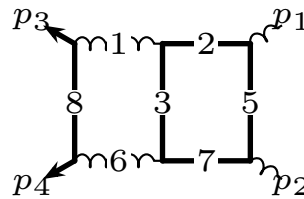
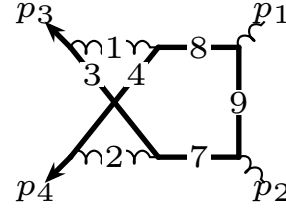
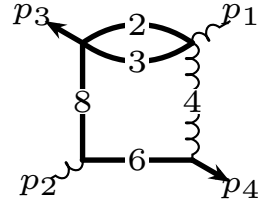
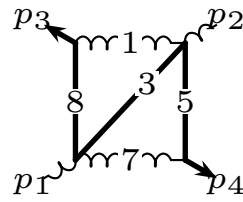
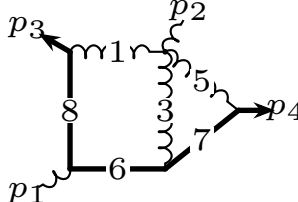
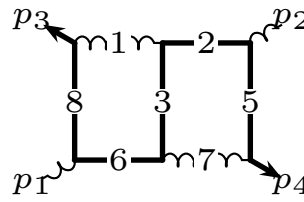
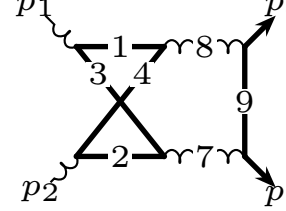
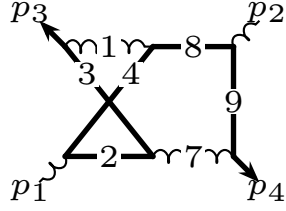
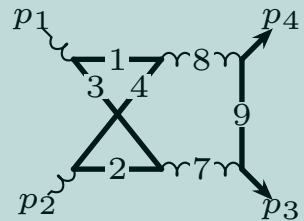
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Two-Loop Corrections to Top-Pair Production

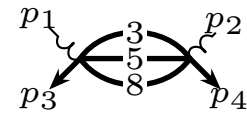
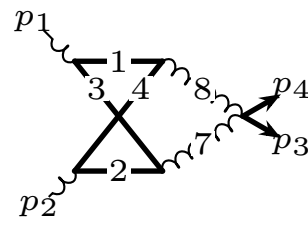
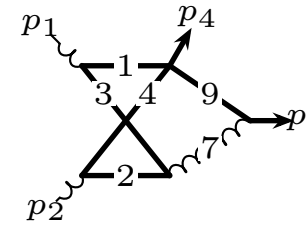
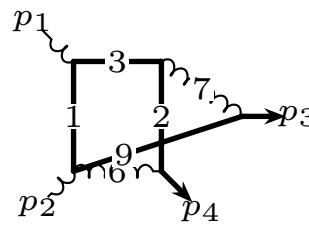
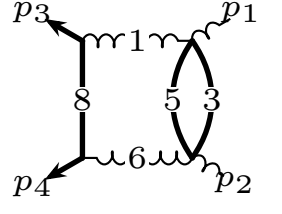
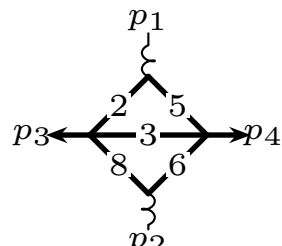
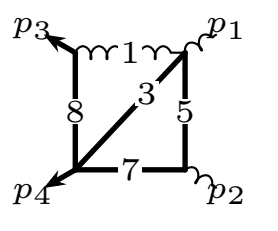
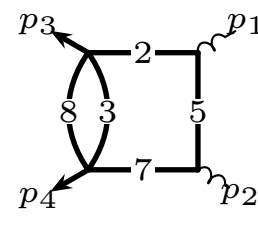
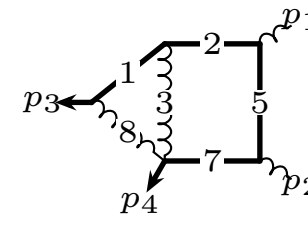
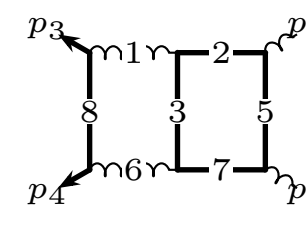
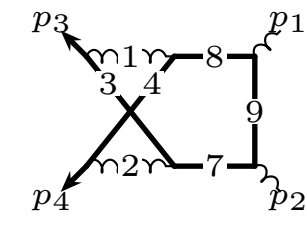
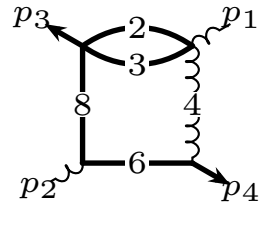
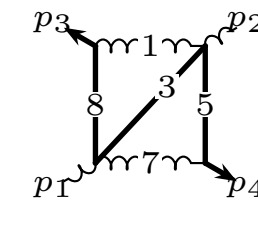
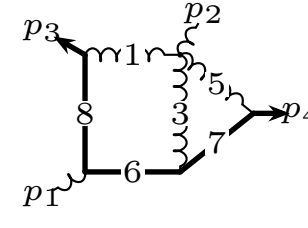
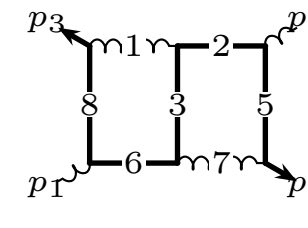
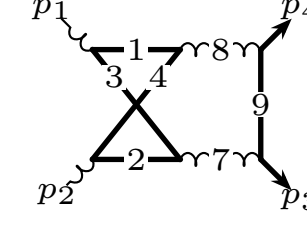
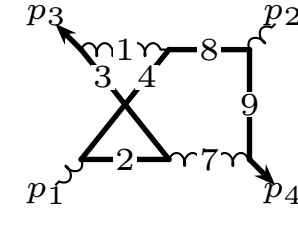
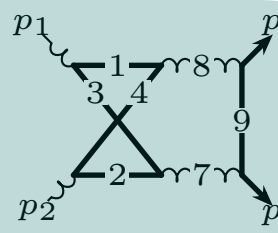
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Two-Loop Corrections to Top-Pair Production

- These master integrals contain **17 elliptic curves** and **1 genus-two curve**. More concretely:

Curve	Sectors	Curve	Sectors	Curve	Sectors	
pB148 elliptic	 pB148 2 (0)	nB207 elliptic	 nB207 2 (0)	nB335 elliptic	 nB335 4 (2)	
nB359 elliptic	 nB359 5 (0)	pB181 elliptic	 pB181 3 (0)	pE182 elliptic	 pE182 3 (0)	
pB214 elliptic	 pB213 4 (0)	 pB214 3 (0)	 pC215 3 (0)	 pB247 5 (0)	 nC463 ^{x12} 4 (0)	
pE174 elliptic	 pE174 3 (0)	 pE213 ^{x12} 5 (0)	 pD245 ^{x12} 3 (0)	 pE247 ^{x12} 4 (1)	 nB463 5 (0)	 nI463 6 (1)
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Using our method, we found a fully canonical basis that also includes the genus-two curve.



Analytic Results

- What are the **benefits** of our **analytic results**?

Preliminary Results!



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● **Simplified expressions** for the (helicity) **amplitudes** (up to finite parts).

● **Analytic control** over **function space** (iterated integrals).

● **Analytic pole cancellation** after UV renormalization and IR subtraction. No additional relations needed.

● There is an **interesting dropping pattern** of **integration kernels** when comparing the **differential-equation level** with the **amplitude level**.

Differential equations

dlog-kernels: 21 (rational) + 17 (algebraic)

elliptic-kernels: 283 (simple) + 43 (quadratic)

genus-two-kernels: 85

vs.

Amplitudes

dlog-kernels: 16 (rational) + 17 (algebraic)

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● Also, the **numerical evaluation** of the **integrals/amplitudes** is **much simpler** in the **canonical basis**.



Numerical Results

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1) Around the **threshold**, we use a **series expansion** obtained from the canonical differential equations.

$$s = 4m_t^2(1 + x_1), \quad t = -m_t^2(1 + x_1x_2), \quad x_1, x_2: \text{blow-up variables (disentangle singularities)}$$

$(x_1, x_2) = (0,0)$: **threshold, boundary values** computed with **AMFlow**



Very fast numerical evaluation of expanded amplitudes.

We use **Bernoulli-type variables** to **enlarge the convergence region**.



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Code is **fast** for **solving ϵ -factorized differential equations**.

New transcendental functions themselves satisfy **differential equations** with **rational coefficients**.



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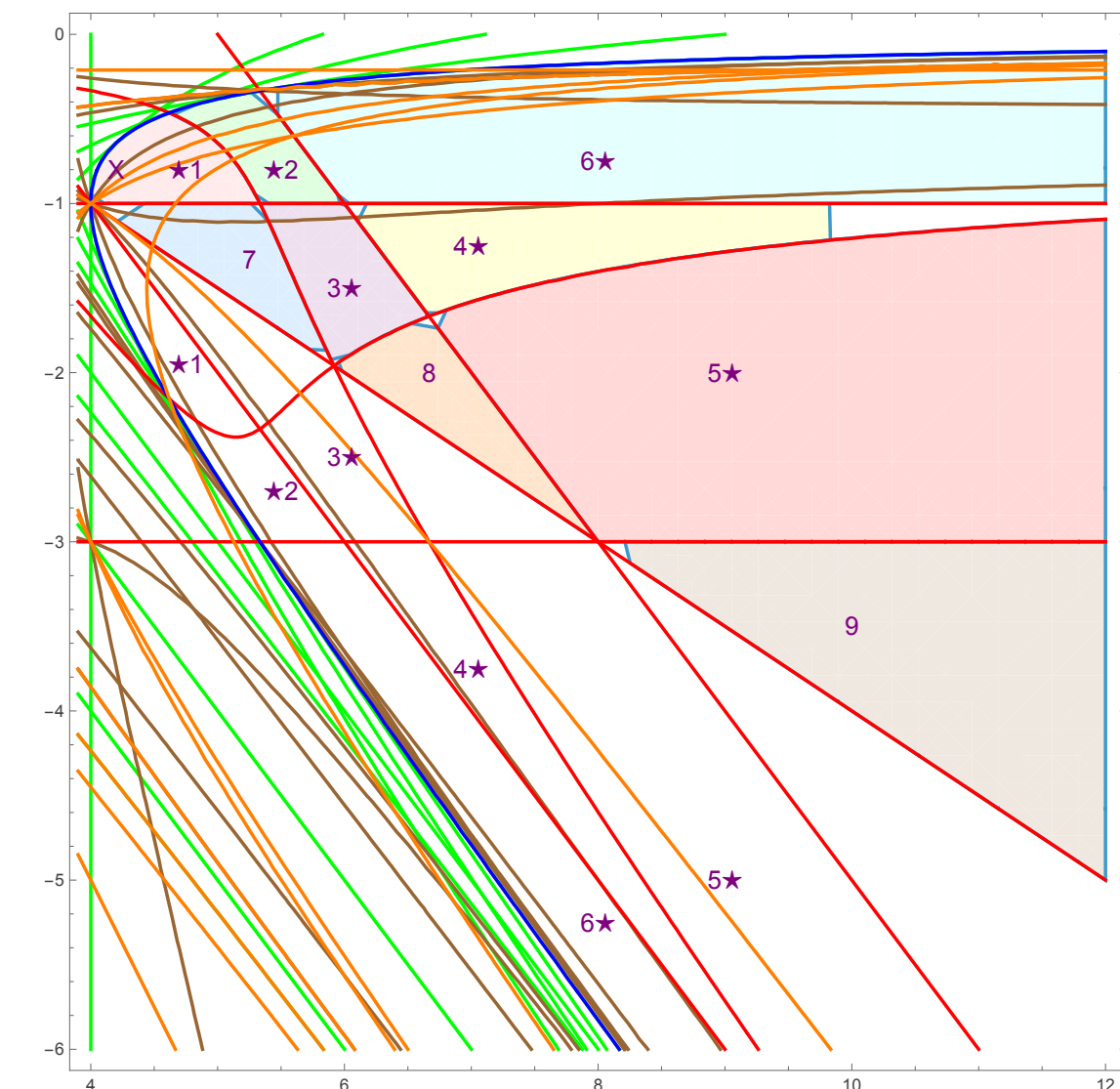
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all nodes	3402	0.61	0.15	0.73	1.07	20.7	46.8	2083
transported (no series)	2696	0.77	0.18	0.82	1.22	21.9	46.8	2080
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~1/2-hour



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series (threshold series)	706	0.005	0.004	0.005	0.028	3	0.2%
single (one mixed evaluation)	2056	0.17	0.16	0.27	0.64	356	17.1%
checked (two mixed evaluations)	590	0.88	0.72	1.29	6.51	522	25.0%
dd_check (ends in dd_real)	48	25.0	22.2	43.1	46.8	1200	57.6%
on_line (exactly on a line)	2	1.20	1.20	1.24	1.24	2	0.1%



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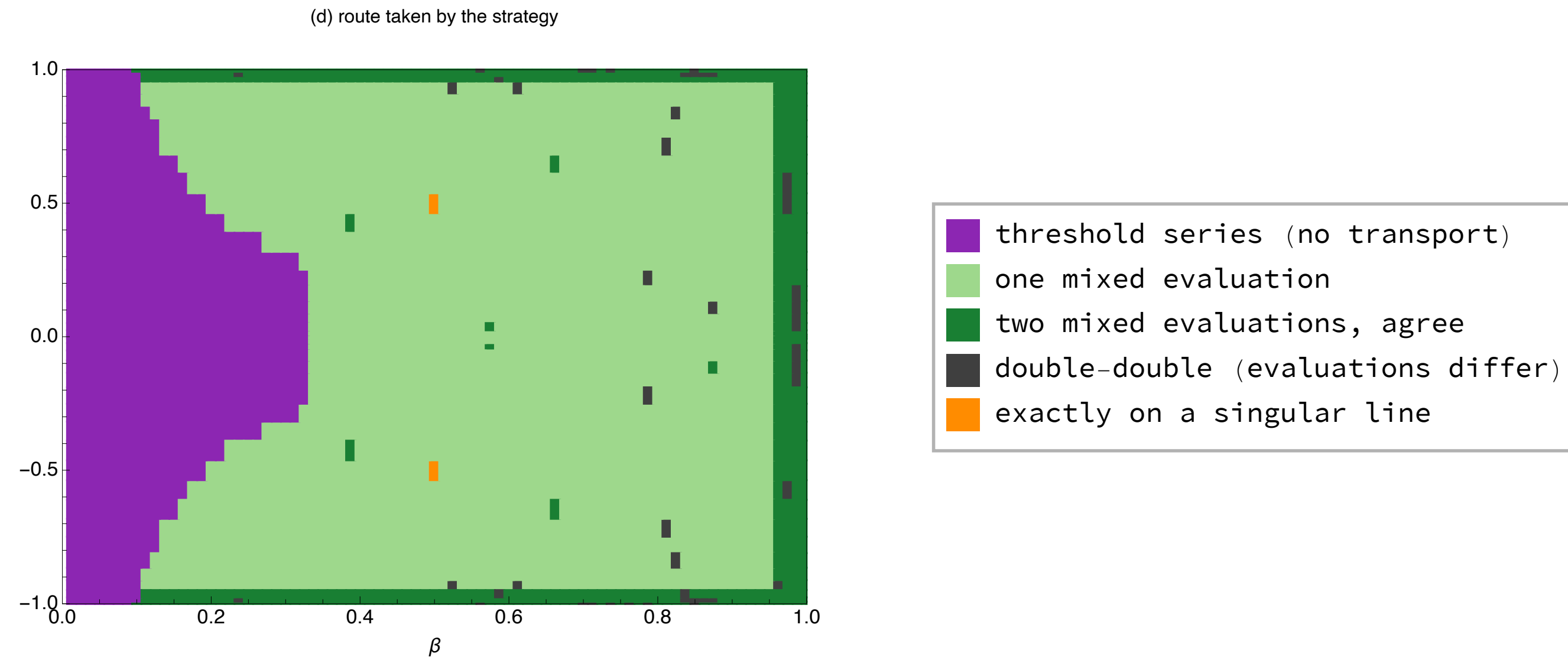
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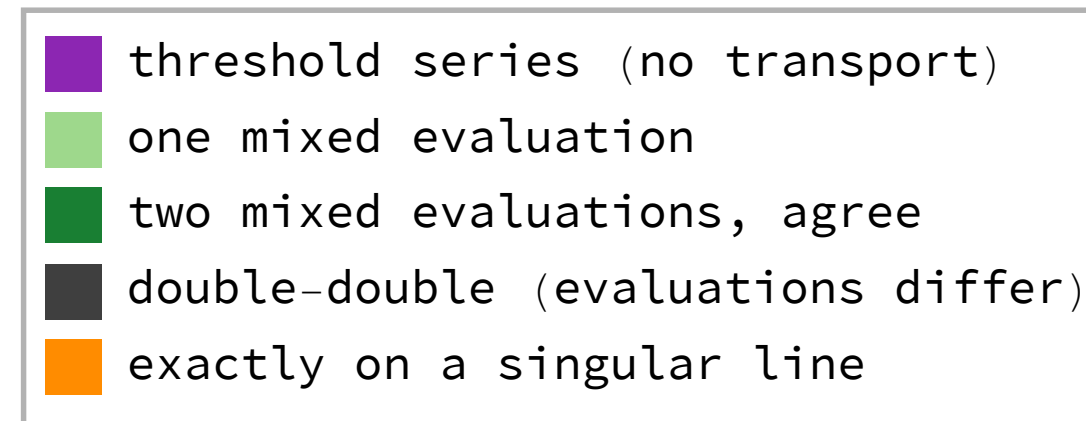
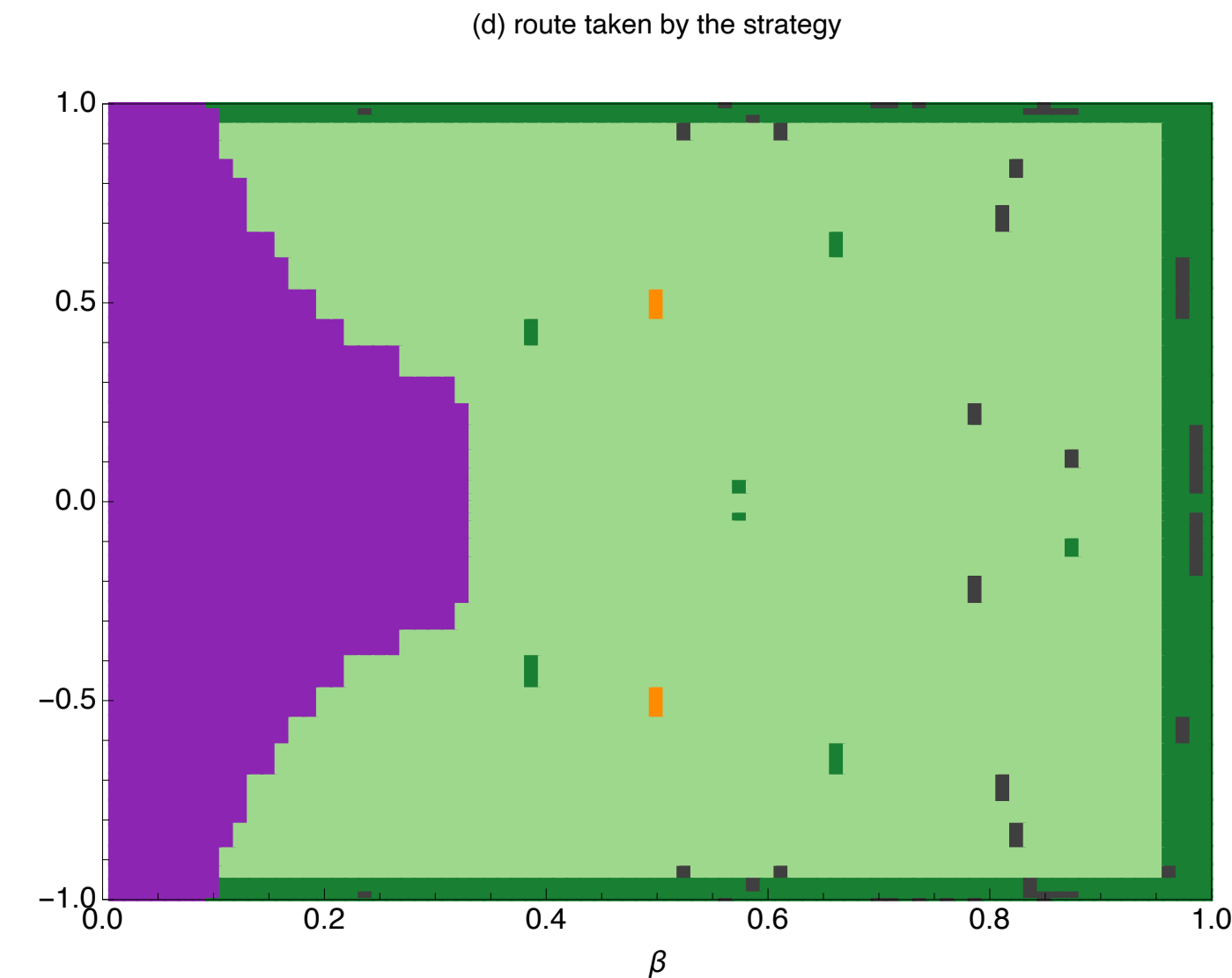


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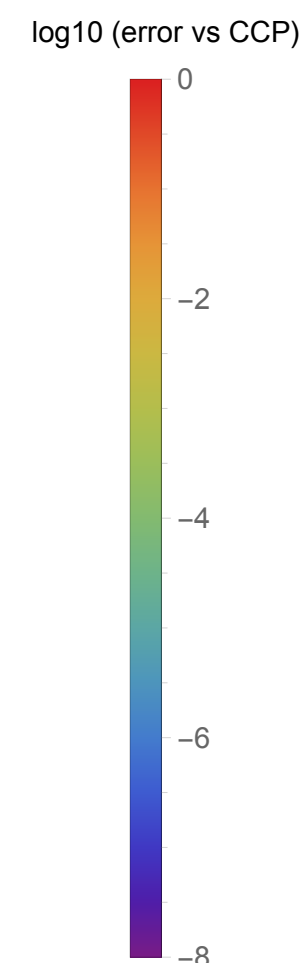
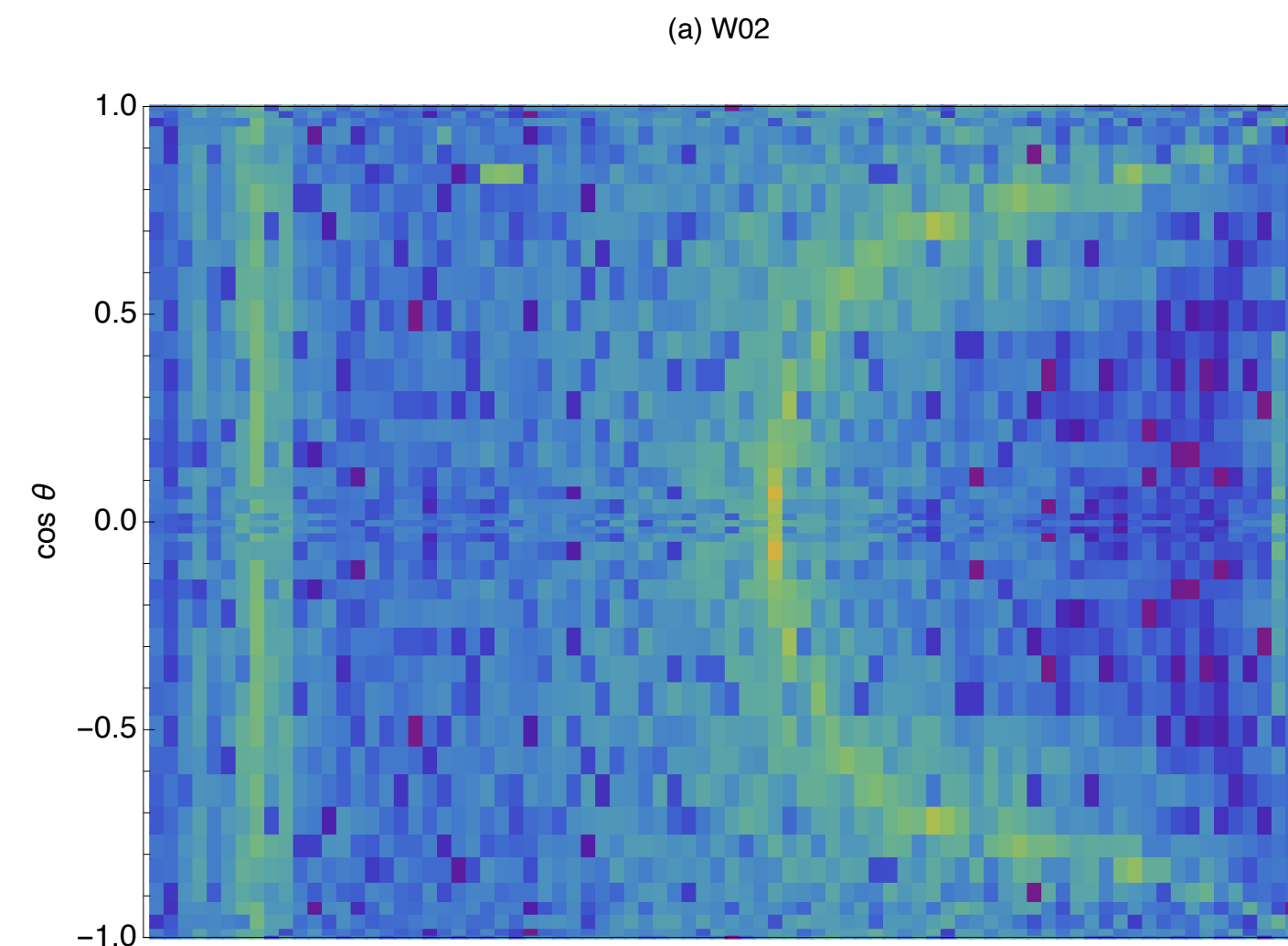


Numerical Results

Comparison with the **grid** produced in [Chen-Czakov-Poncelet '18].



Preliminary Results!



Difference between the **grid** from [Chen-Czakov-Poncelet '18] and **our numerical implementation**.

On-the-fly detection of instabilities allows us to identify regions where more precision is needed.



Summary

Choosing a canonical basis is beneficial because it provides significant analytic control and simplifies numerical evaluation.

Do not be afraid of new transcendental functions. They satisfy rational differential equations and can be easily evaluated numerically.



Thank you for your attention

