



Efficient Reconstruction of Scattering Amplitudes

Anton Olsson – 5th October 2026

With: Bakul Agarwal, Andreas Von Manteuffel



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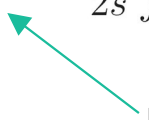
Outline

- Introduction
 - Scattering amplitudes
 - Integration-by-parts reduction
 - Partial fractioning IBP coefficients
- Reconstructing IBP coefficients in partial-fractioned form
 - Constraining ansätze with singularity tests
 - Partial-fractioning with graphs
 - Constraint generation via graph traversing
- Examples
 - Five-point one-mass function at one loop
 - Five-point massless functions at two loops
- Summary

Scattering Amplitudes

- Predictions for collider physics require knowledge of the **scattering amplitude**

$$\sigma = \sum_{a,b} \int_0^1 dx_a dx_b \underbrace{f_{a/h_1}(x_a) f_{b/h_2}(x_b)}_{\text{parton distribution functions (PDFs)}} \underbrace{d\hat{\sigma}_{ab \rightarrow n}}_{\text{partonic cross section}}, \quad \hat{\sigma}_{ab \rightarrow n} = \frac{1}{2\hat{s}} \int d\Phi_n \underbrace{|\mathcal{M}_{ab \rightarrow n}(p_1, \dots, p_n)|^2}_{\text{scattering amplitude}}$$

 **Hard process**

- Scattering amplitudes are evaluated in perturbation theory

$$\mathcal{M} \approx \sum_{k=0}^{\infty} g^{2k} \mathcal{M}_k = \mathcal{M}_0 + g^2 \mathcal{M}_1 + g^4 \mathcal{M}_2 \quad \xrightarrow{\text{Higher-order terms involve Feynman loop integrals}} \quad I \sim \int \prod_{l=1}^L \frac{d^D k_l}{i\pi^{\frac{D}{2}}} \frac{1}{P_0 \cdots P_N}, \quad P_j = (q_j^2 - m_j^2 + i\delta)$$

- Number of Feynman diagrams explode with increasing loop order

$D = 4 - 2\epsilon$ in *dimensional regularisation*

- Tensor contraction $\rightarrow$ large number of scalar Feynman integrals
- These integrals are **not independent** – we should find a basis of *master integrals*

- Ex:** In $qq \rightarrow t\bar{t}H$ at two loops (Nf part) there were **~20 000** scalar integrals [Agarwal et al.; 2402.03301]

- Amplitude could be represented in terms of **831** master integrals

IBP Reduction

- Feynman integrals satisfy Integration-By-Parts (IBP) identities

$$\int \prod_{l=1}^L \frac{d^D k_l}{i\pi^{\frac{D}{2}}} \frac{\partial}{\partial k_i^\mu} q^\mu \prod_{j=1}^N \frac{1}{P_j^{\nu_j}(\{k\}, \{p\}, m_j)} = 0, \quad i = 1, \dots, L$$

External and loop momenta

- Insert different $q \rightarrow$ generate large system of linear equations
 - Systematic construction via the Laporta algorithm [Laporta '01]
 - Reduce system w.r.t chosen basis (master integrals)

- Amplitude represented as linear combination of master integrals

$$\mathcal{M}(\mathbf{s}; D) = \sum_k r_k(\mathbf{s}; D) I_k(\mathbf{s}; D), \quad r_k(\mathbf{s}; D) = \frac{\sum_i c_i m_i}{\prod_j^n d_j^{\nu_j}}$$

"IBP coefficient"

r_k = rational functions

m_i = monomials in $\mathbf{s}, D$

c_i = numbers

d_j = polynomials in $\mathbf{s}, D$

- Expansion in dimensional regulator (only reconstruct relevant orders)

$$r_k(\mathbf{s}; D) = \sum_{i=-r}^n \varepsilon^i r_k^{(i)}(\mathbf{s}) + \mathcal{O}(\varepsilon^{n+1}) \quad I_k(\mathbf{s}; D) = \sum_{i=-r}^n \varepsilon^i I_k^{(i)}(\mathbf{s}) + \mathcal{O}(\varepsilon^{n+1})$$

- The IBP coefficients get very large at high loop order
 - Difficult to compute, slow to evaluate..

Amplitude Reconstruction

- The $r_k^{(i)}(\mathbf{s})$ are obtained by linearly reducing the IBP system
- Performing a symbolic reduction is prohibited by intermediate expression swell
 - Instead: reduce modulo a **finite field**, $\mathbb{Z}/p\mathbb{Z}$, and **reconstruct** the result [von Manteuffel, Schabinger; 1406.4513. Peraro; 1608.01902]
 - The IBP problem is mapped to numerically reducing a linear system of integers, *many times*

$$\begin{bmatrix} s_{12}+s_{23} & -s_{34} \\ s_{45} & s_{23}-s_{51} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = 0 \rightarrow \begin{bmatrix} \frac{18}{5} & \frac{-13}{7} \\ \frac{7}{3} & \frac{-2}{11} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = 0 \xrightarrow{\mathbb{F}_{101}} \begin{bmatrix} 44 & 27 \\ 36 & 9 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = 0$$

- Performance metrics:
 - How **many IBP samples** are required?
 - How **many primes** are needed to lift the solution?
- The form of the reconstructed coefficient is important
 - In common-denominator form there are many *overlapping singularities* (less stable evaluation)
 - **In partial-fractioned form** coefficients are known to simplify

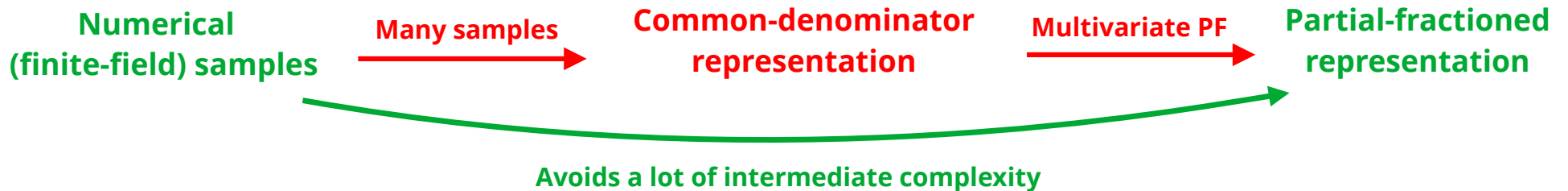
$$r(\mathbf{s}) = \frac{\sum_i c_i m_i}{\prod_j^n d_j^{\nu_j}}$$

Common-denominator form

- The most complicated function in five-point massless at 2L is **130x** larger on common-denominator form [Chawdry; 2312.03672]

Partial Fractioning IBP coefficients

- Standard technique to reduce complexity of **reconstructed** IBP coefficients
 - Applied to five-point massless 2L coefficients with MultivariateApart [Heller, Von Manteuffel, 2101.08283]
 - Univariate partial fractioning applied in many standard multi-loop workflows
- **Problem I:** Relies on having the IBP coefficient in common-denominator form
- **Problem II:** multivariate partial fractioning is very complicated (or even unfeasible)
- **Solution:** reconstruct the coefficient *directly* in partial-fractioned form



- This is not a completely new concept
 - Five-point massless [de Laurentis, Page; 2203.04269. Chawdry; 2312.03672]
 - Five-point one-mass (Vjj) [de Laurentis et al.; 2503.10595]

Generating the Ansatz

- We want to start from a common-denominator ansatz

$$r(\mathbf{s}) = \left(\sum_i c_i m_i \right) \prod_j^n q_j^{\nu_j} \quad q_j = \frac{1}{d_j}$$


 Inverse denominators

m_i = monomials in $\mathbf{s}$

c_i = numbers

d_j = polynomials in $\mathbf{s}$

1. Mass dimension of the amplitude + mass dim. of integral

→ mass dimension of IBP coefficient r_k

2. Mass dimension of IBP coefficient + *denominator guessing* [Abreu et al., 1812.04586; Heller, Von Manteuffel, 2101.08283]

→ mass dimension of numerator monomials m_i

3. Write down the most general numerator ansatz based on the allowed monomials

- Construct a Groebner basis (using an ordering $\{q_{j_1}, \dots, q_{j_n}; s_{i_1}, \dots, s_{1_N}\}$)

→ polynomial reduction rules

- The reduction rules are iteratively applied to the common-denominator ansatz to generate a *partial-fractioned form*

$$r(\mathbf{s}) = \sum_i b_i Q_i(\mathbf{s}), \quad Q_i(\mathbf{s}) = a_i(\mathbf{s}) \prod_j q_j^{\mu_j}$$


 Irreducible monomials

Excluding Denominators

- Reducing the common-denominator ansatz generates **many terms**

– Example reduction rule: $q_1 q_2 q_3 q_5 q_6 q_8 \rightarrow -q_1^2 q_2 q_3 q_5 q_6 + q_1 q_2 q_3 q_5 q_8 - q_1 q_2 q_3 q_6 q_8 -$
 $q_1 q_2 q_5 q_6 q_8 - q_1 q_3 q_5 q_6 q_8 + q_1^2 q_3 q_5 q_6 q_8 + q_2 q_3 q_5 q_6 q_8$

- But we know the result should simplify massively..
 → many terms must vanish!

$$r(s) = \sum_i b_i Q_i(s), \quad Q_i(s) = \prod_j a_i(s) q_j^{\mu_j}$$

Many b_i are 0

- Extract this information by probing the coefficient's *singularity structure*

1. Select a subset of inverse denominators $q_{j_1} q_{j_2}$ (assume they appear with exponent 1 for now)
2. Impose $d_{j_1} \sim d_{j_2} \sim t$ so that $t \rightarrow 0 \implies q_{j_1}, q_{j_2} \rightarrow \infty$ (singularity)
3. Perform a *univariate reconstruction* in t
4. Check the *degree of divergence* (scaling in t)

* this works straightforwardly for linear denominators

- For a generic function we expect the result to scale as $1/t^2$
 - If the degree of divergence $< 1/t^2$, $q_{j_1} q_{j_2}$ **does not appear in the result**
 - It is the specific values of the c_i that cause the function to simplify on partial fractioning

Excluding Denominators

- 8-denominator coefficient from massless five-point 2L amplitude

** this works straightforwardly for linear denominators*

$$\prod_{j=1}^8 q_j^{\mu_j} = 1 / (s_{23} s_{34}^4 s_{45}^3 \boxed{-1 + s_{34} + s_{45}} (s_{23} - s_{51}) \times \\ \boxed{-1 + s_{34} - s_{51}}^3 (s_{23} + s_{34} - s_{51})^4 s_{51})$$

- Probe the singularity of fourth and sixth denominator vanishing simultaneously

$$d_4 = d_6 = 0 \quad \rightarrow \quad -1 + s_{34} + s_{45} = 0 \quad \text{and} \quad -1 + s_{34} - s_{51} = 0$$

- Compute a Groebner basis to find the intersection between d_4 and d_6

- While no other denominators vanish
 - In this case s_{23} is a free parameter
- $$\left\{ s_{23} \rightarrow -\frac{92}{5}, s_{34} \rightarrow -\frac{12}{5}, s_{45} \rightarrow \frac{17}{5}, s_{51} \rightarrow -\frac{17}{5} \right\}$$

- Extend each instance by a term linear in t (essentially a regulator)

$$\left\{ s_{23} \rightarrow -\frac{92}{5} + 3t, s_{34} \rightarrow -\frac{12}{5} + 5t, s_{45} \rightarrow \frac{17}{5} + 7t, s_{51} \rightarrow -\frac{17}{5} + 11t \right\} \quad (\text{when } t \rightarrow 0 \text{ the function diverges})$$

- Perform a univariate reconstruction

- We find that the function diverges as $1/t^3 \rightarrow$ the subset $q_4 q_6^3$ is excluded

- All singularity tests exclude subsets

$$q_1^3 q_3^2 q_3^3, q_2^4 q_4^2 q_6^3, q_3^3 q_4^3, q_3^3 q_6^3, q_4^3 q_6^3, q_4^4 q_7^3, q_5^3 q_6^3, q_2^4 q_5^3 q_7^4$$

Quadratic Denominators

- 8-denominator coefficient from one-mass five-point 1L amplitude

$$\prod_{j=1}^8 q_j^{\mu_j} = 1 / (s_{12}(-1 + s_{12} + s_{15} - s_{34})(s_{12}s_{15} - s_{34})s_{34}(-s_{15} + s_{23} + s_{34})^2 \\ \times (-1 + s_{12} + s_{23} - s_{45})(s_{12} - s_{34} - s_{45})s_{45})$$

- Probe the singularity of second and third denominator vanishing simultaneously (d_3 is quadratic)

$$d_2 = d_3 = 0 \quad \rightarrow \quad -1 + s_{12} + s_{15} - s_{34} = 0 \quad \text{and} \quad s_{12}s_{15} - s_{34} = 0$$

- This time we find **2 solutions**: $\{s_{12} \rightarrow 1, s_{34} \rightarrow s_{15}\}$ and $\{s_{34} \rightarrow s_{12}, s_{15} \rightarrow 1\}$

- We obtain **2** different univariate parametrisations & reconstructions

$$\left\{ s_{12} \rightarrow 1 + 3t, s_{23} \rightarrow -\frac{1}{2} + 5t, s_{34} \rightarrow -\frac{12}{5} + 7t, s_{45} \rightarrow -\frac{9}{5} + 11t, s_{15} \rightarrow -\frac{12}{5} + 13t \right\} \longrightarrow \text{Diverges as } 1/t$$

$$\left\{ s_{12} \rightarrow -\frac{12}{5} + 3t, s_{23} \rightarrow -\frac{1}{2} + 5t, s_{34} \rightarrow -\frac{12}{5} + 7t, s_{45} \rightarrow -\frac{9}{5} + 11t, s_{15} \rightarrow 1 + 13t \right\} \longrightarrow \text{Diverges as } 1/t^2$$

In this case we don't exclude this subset

- For **higher degree denominators**, we can have:
 - Multiple solution branches (pick the most divergent one)
 - No intersections between q_1 and q_2 on rational points
 - $\rightarrow$ higher than linear parametrisation

Graph Generation

- For each excluded subset $q_{j_1} q_{j_2}$: Compute a Groebner basis and derive polynomial reduction rules
- Partial fraction each common-denominator term w.r.t $q_{j_1} q_{j_2}$
 - Keep track of all intermediate expressions in a graph structure
- Ex.:** five-point one-mass function at one loop
 - 5 denominators, 376 numerator coefficients in ansatz

$$q_1 q_2 q_3 q_4 q_5^3 = 1 / (s_{34} (-s_{12} + s_{34} + s_{45}) s_{45} (-s_{15} + s_{23} + s_{34}) (1 - s_{12} - s_{15} + s_{34})^3)$$
 - Partial fraction a few numerator nodes w.r.t to $q_1 * q_3$

$s_{45}^3 s_{51}^3 q_1 q_2 q_3 q_4 q_5^3$

$s_{34} s_{45}^2 s_{51}^3 q_1 q_2 q_3 q_4 q_5^3$

$s_{34} s_{45}^3 s_{51}^2 q_1 q_2 q_3 q_4 q_5^3$

- Numerator
- Intermediate
- Irreducible

$s_{45}^2 s_{51}^3 q_1 q_2 q_4 q_5^3$

$s_{45}^2 s_{51}^3 q_2 q_3 q_4 q_5^3$

$s_{45} s_{51}^3 q_2 q_4 q_5^3$

$s_{45}^3 s_{51}^2 q_2 q_3 q_4 q_5^3$

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Generating Constraints

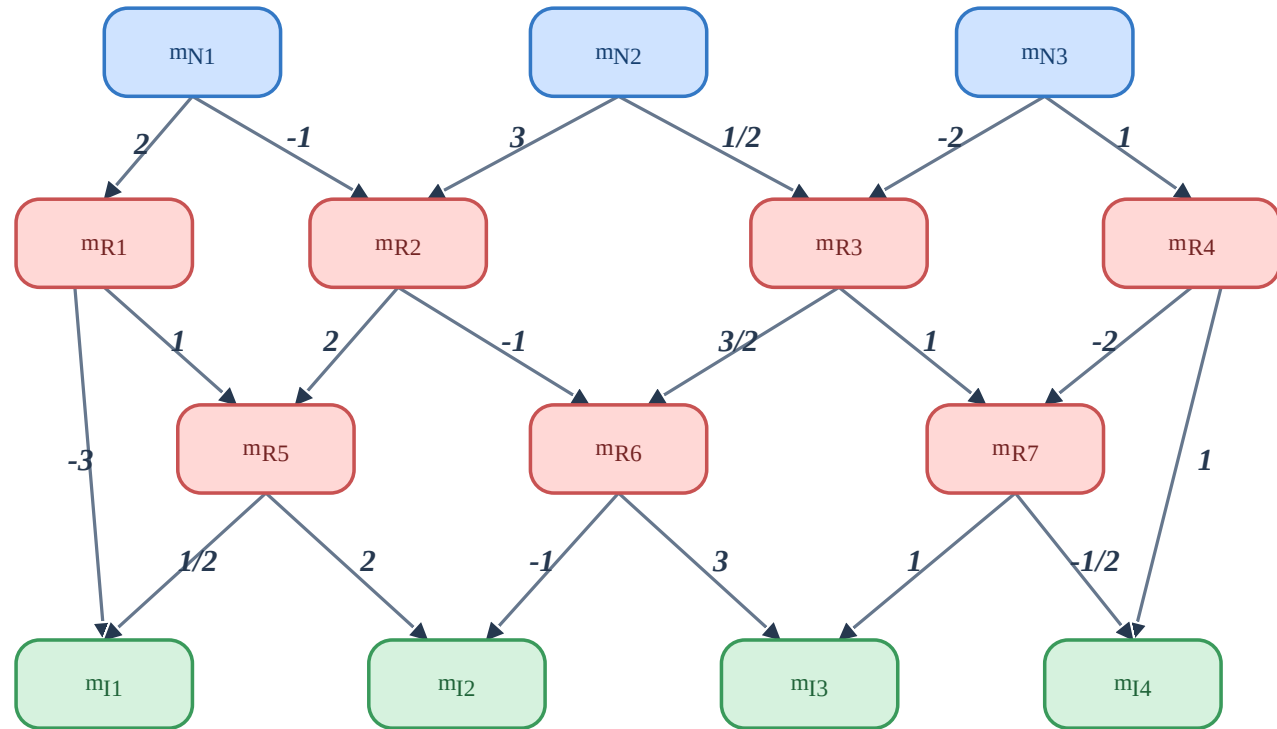
- Each successful singularity test excludes a subset of $q_{j_1} q_{j_2}$
- Irreducible nodes resulting from PF w.r.t $q_{j_1} q_{j_2}$, that contain $q_{j_1} q_{j_2}$, have vanishing coefficient

Constraints via traversal:

- c_i coefficients are related to b_i coefficients
- Relations are encoded by the graph edges
- Connect irreducible nodes to top layer via a full graph traversal
- Each *excluded* irreducible node generates one constraint

$$c_{j_1} + c_{j_2} + \dots + c_{j_n} = 0$$

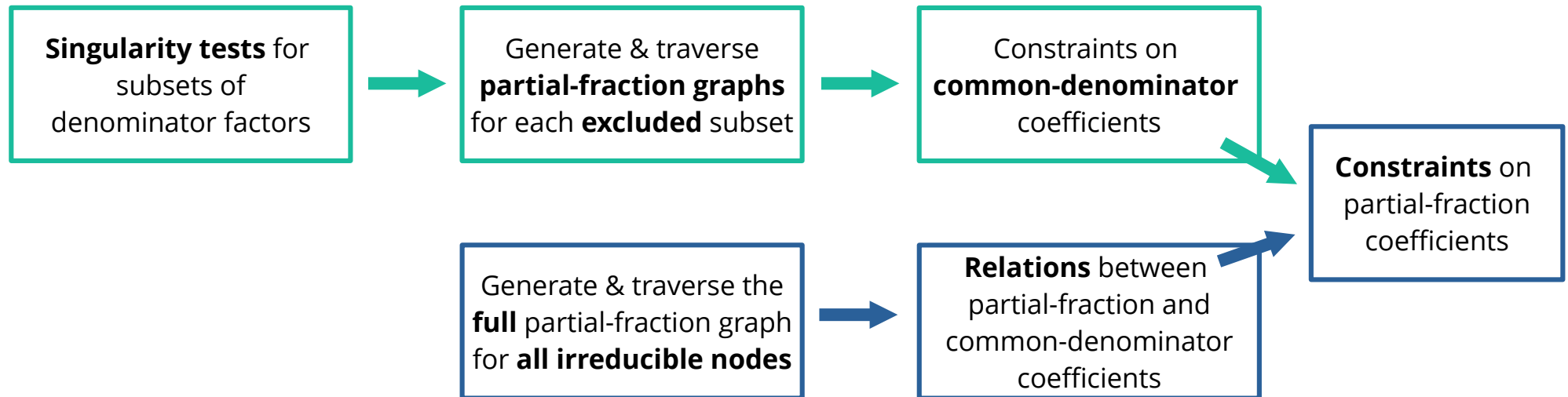
- Row reduce the system of equations to find independent constraints on the top-level coefficients



Constraining the Partial-Fraction Form

- Repeat the graph generation and traversal for **each excluded denominator subset**
 - Reduce the combined set of constraints on the numerator nodes
 - We use *Finred* for all linear reductions [private code; von Manteuffel]
- Perform a final graph generation with a GB based on the set of all denominator factors
 - Traverse *top-down* to generate relations between the c_i and b_i
 - Combine with constraints from the singularity tests and row reduce
 - Constraints on the partial-fraction coefficients

Workflow summary



One-Loop Example

- One-mass five-point function at one loop

$$r^{(0)}(\mathbf{s}) = \sum_{i=1}^{2331} c_i m_i \prod_{j=1}^8 q_j^{\mu_j} \quad \prod_{j=1}^8 q_j^{\mu_j} = 1 / (s_{12}(-1 + s_{12} + s_{15} - s_{34})(s_{12}s_{15} - s_{34})s_{34} \times \\ (-s_{15} + s_{23} + s_{34})^2(-1 + s_{12} + s_{23} - s_{45})(s_{12} - s_{34} - s_{45})s_{45})$$

- 2331 numerator monomials, 8 denominator factors
- Singularity test excludes 46 subsets
 - Traversing each graph generates 3340 equations → 1295 **independent constraints**
 - There are 1036 free coefficients
 - Append constraint matrix with 1036 IBP samples
- Partial fraction w.r.t to all 8 denominator factors
 - 4786 irreducible nodes
 - Traverse & combine with constraint matrix → **all $\mathbf{b_i}$ are determined**
 - *4609 of these coefficients are 0*
 - Only 177 terms remain in the final result (the together form has 634 terms after constraining)
- **Result**
 - Less than **half** as many function samples required
 - The result is more than **3x** more compact than the common-denominator representation (counting terms)

Two-Loop Example

- Massless five-point function at two loops

$$r^{(0)}(\mathbf{s}) = \sum_{i=1}^{9245} c_i m_i \prod_{j=1}^{10} q_j^{\mu_j} \left| \prod_{j=1}^{10} q_j^{\mu_j} = 1 / (s_{23}^3 s_{34}^4 s_{45}^3 (-1 + s_{34} + s_{45})(s_{23} - s_{51})(-1 + s_{34} - s_{51})^4 \right. \\ \left. \times (-1 + s_{23} + s_{34} - s_{51})^3 (s_{23} + s_{34} - s_{51})^4 s_{51}) \right|$$

- 9245 numerator monomials, 10 denominator factors
- Singularity test excludes 27 subsets
 - Traversing each graph generates 11143 equations → 6191 **independent constraints**
 - There are 3054 free coefficients
 - Append constraint matrix with 3054 IBP samples
- Partial fraction w.r.t to all 10 denominator factors
 - 52699 irreducible nodes
 - Traverse & combine with constraint matrix → **all $\mathbf{b}_i$ are determined**
 - 41517 *of these coefficients are immediately 0*
 - Only **803** terms remain in the final result (the together form has **4871** terms after constraining)
- **Result**
 - Less than **a third** as many function samples required
 - The result is **6x** more compact than the common-denominator representation (counting terms)

Two-Loop Example II

- Massless five-point function at two loops

$$r^{(0)}(\mathbf{s}) = \sum_{i=1}^{13393} c_i m_i \prod_{j=1}^8 q_j^{\mu_j} \quad \prod_{j=1}^8 q_j^{\mu_j} = 1 / (s_{23} s_{34}^4 (1 + s_{23} - s_{45})^3 s_{45}^3 (-1 + s_{34} + s_{45}) (-1 + s_{34} - s_{51})^3 \\ \times (s_{23} + s_{34} - s_{51})^4 (s_{23} - s_{45} - s_{51})^3 s_{51})$$

- 13393 numerator monomials, 8 denominator factors
- Singularity test excludes 13 subsets
 - Graph traversing $\rightarrow$ 7202 independent constraints
 - 6323 free coefficients remain
- Partial fraction w.r.t to full denominator
 - 40847 irreducible nodes (39974 are 0)
 - Only **873** terms remain in the final result (the together form has 10012 numerator terms)
- **Result**
 - Less than **half** function samples required
 - The result is more than **10x** more compact than the common-denominator representation (counting terms)
- We observe on these examples that more complicated functions *simplify more*

Summary

New method for reconstructing amplitudes directly in partial-fractioned form

- Requires fewer IBP samples
- The partial-fractioned form is much smaller
- More complicated functions benefit more

The partial-fraction ansatz is constrained via singularity tests

- This only involves cheap univariate reconstructions
- The univariate reconstructions can be used to generate additional IBP samples
- Massive particles complicate the parametrisation → consider multiple solution branches

We generate constraints on common-denominator coefficients via graph traversing

- Combining constraints with relations between top- and bottom-layer nodes yield constraints on partial-fraction coefficients
- Constraint generation is a fully automated process

Outlook

- Select kinematic point smartly between different subsets to reduce the number of numerical samples for tests
- Systematize higher degree parametrisation (linear case is automated)

Summary

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Outlook

- Select kinematic point smartly between different subsets to reduce the number of numerical samples for tests
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Thank you!

Backup: Partial Fractioning

- Partial fractioning splits a rational function into a sum of “simpler” terms

- In one variable the decomposition is unique:
$$\frac{x}{(x-1)(x+1)^2} = -\frac{1}{4(x+1)} + \frac{1}{2(x+1)^2} + \frac{1}{4(x-1)}$$

- A generalisation to multiple variables is iterated partial fractioning (variable-by-variable)

- The result is ambiguous because it depends on the **variable ordering**
 - This procedure may introduce additional denominators (spurious singularities)

$$\frac{1}{(x+y)(x-y)} = \frac{1}{2y} \frac{1}{(x-y)} - \frac{1}{2y} \frac{1}{(x+y)}$$

- A better algorithm is based on Groebner bases (GB)

- Define inverse denominators: $q_i = 1/d_i$
 - Define an ideal of polynomial relations: $I = \langle d_1 q_1 - 1, \dots, d_n q_n - 1 \rangle$
 - Select a variable ordering: $\{q_{j_1}, \dots, q_{j_n}; s_{i_1}, \dots, s_{1_N}\}$ and compute a GB

- The Groebner basis provides polynomial reduction rules that result in a unique decomposition

- Ex:** $\frac{2y-x}{y(x+y)(y-x)}$. Using the ordering: $\{\{q_3, q_1\}, \{q_2\}, \{x, y\}\}$ $\longrightarrow$ $\frac{3}{2y(x+y)} + \frac{1}{2y(y-x)}$

- Such an algorithm is implemented in *MultivariateApart* [Heller, Von Manteuffel; 2101.08283]

- All examples on this slide from [2101.08283]