

Exponentially Converging Numerical Evaluation of Iterated Integrals

Thomas Stone

HP²: High Precision for Hard Processes

Karlsruhe, 6th October 2026

TS, Lorenzo Tancredi [[26XX.XXXX](#)]

Outline

- 1 Motivation
- 2 Exponential Suppression
- 3 Feynman Integrals
- 4 Outlook

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This talk

- Estimate a “*hard*” integral by “*simpler*” ones using *exponentially suppressed* combinations
- Error $\sim \rho^{-n}$ ($\rho > 1$) in the suppression power $n \Rightarrow$ digits grow **linearly** with n
- Works for *periods* (logs, elliptics, CY) and for *dimensionally-regulated Feynman integrals*

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Suppressing numerator $N(\mathbf{x})$, $0 \leq N < 1$ on Δ

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Choice of N : **(a)** suppress **(b)** cancel factors of $f \Rightarrow$ generates the **simpler** families

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n	correct digits
10	9.8
20	19.0
40	37.2
$\approx 0.93 n$ digits	

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- Error $\sim [4a(1+a)]^{-n}$: 4^{-n} from $\max N$, the rest from $|N|$ at the *singularity* $t = -a$
- Converges for $a > \frac{\sqrt{2}-1}{2} \approx 0.21$; e.g. $\sim 0.48, 0.91, 1.69$ digits per unit n at $a = \frac{1}{2}, 1, 3$

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Expanding the numerator gives two families of moments, $M_j = \int_0^1 t^j / \sqrt{P} dt$, related by:

$$\int_0^1 \frac{d}{dt} [t^j \sqrt{P}] dt = -\delta_{j,0} \Rightarrow j M_{j-1} - (j+1)(1+\textcolor{blue}{\lambda}) M_{j+1} + (j+2) \textcolor{blue}{\lambda} M_{j+3} = -\delta_{j,0}$$

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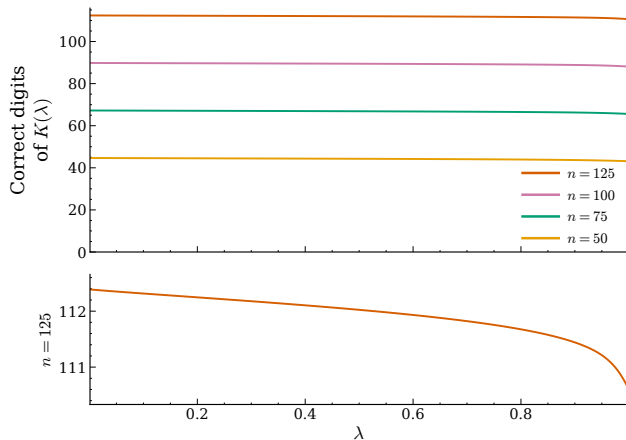
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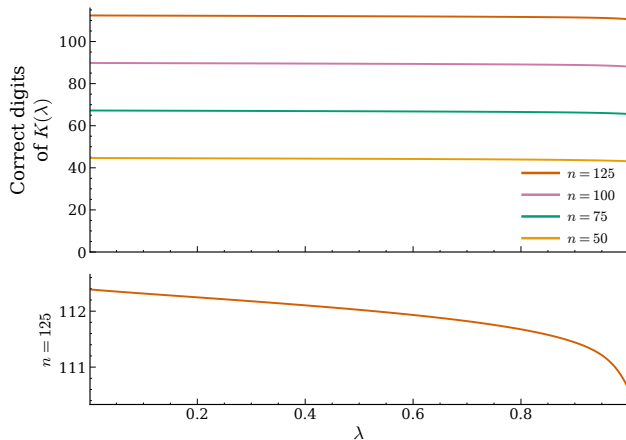
Elliptic $\approx$ logarithm $\times$ rational + rational: correct to **100+ digits** at high n

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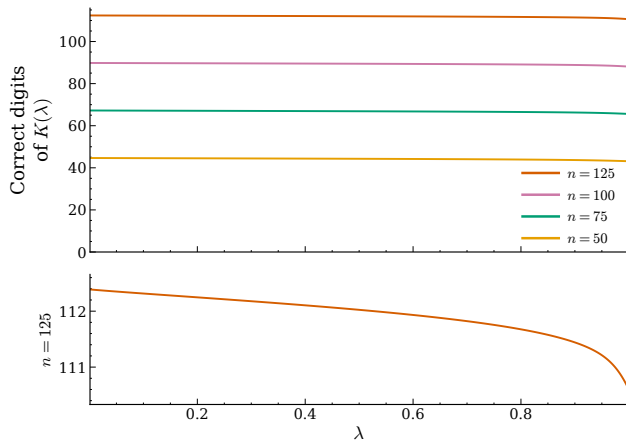


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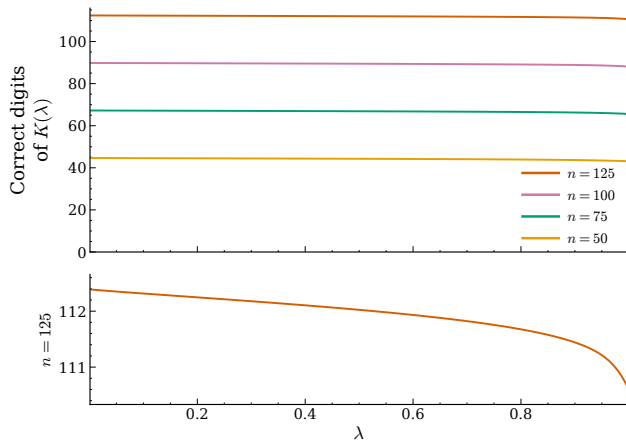
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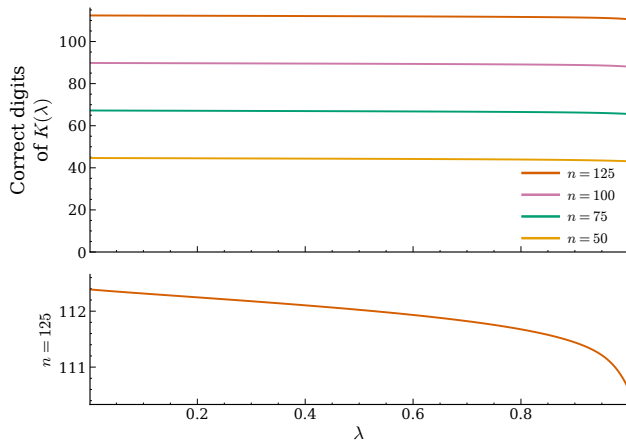
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- $n = 125$: **110+ digits** for all
 $0 < \lambda < 1$

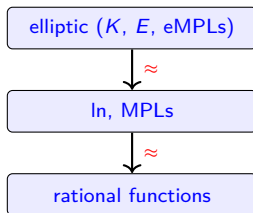
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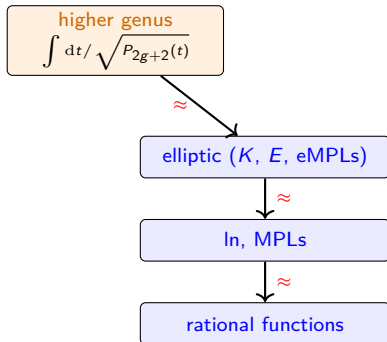
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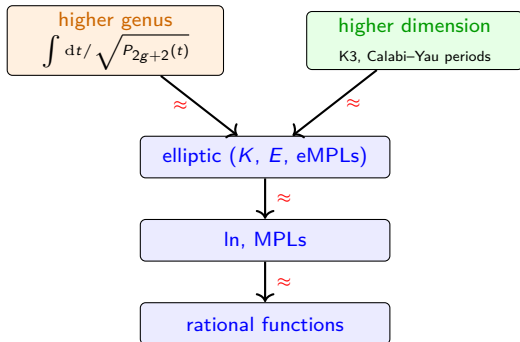


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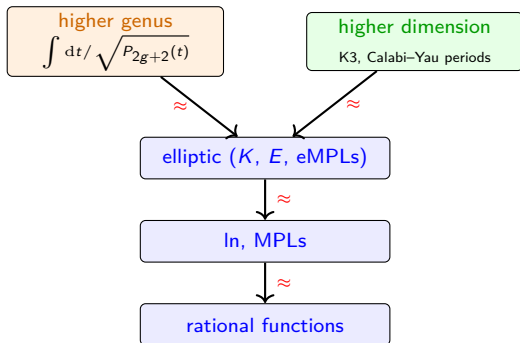
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→ e.g. $\iint \frac{a^n b^m da db}{\sqrt{a b (1-a)(1-b)(1-x a b)}}$

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Feynman integrals live in **(2)**: *multivariate* parametric integrals with non-trivial polynomials

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Feynman Parameter Families

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Parametric family on the simplex $\Delta = \{x_i \geq 0, \sum_i x_i = 1\}$

$$J[\nu] = \int_{\Delta} d^{N-1} \mathbf{x} \, \mathbf{x}^{\nu} \underbrace{\prod_i x_i^{c_i} \prod_j P_j(\mathbf{x}; \mathbf{s})^{\gamma_j}}_{\Phi(\mathbf{x}): \text{twist}}, \quad \gamma_j \in \{a + b\epsilon\}$$

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- N^k peaks ever more sharply at $\mathbf{x}^*$: the node is $\Phi(\mathbf{x}^*)$ times the integral of N^k , plus corrections in $t_k \sim \frac{1}{Nk}$
- $D_k = \int_{\Delta} N^k$: **exponentially** small
- $\beta_0 = \Phi(\mathbf{x}^*)$: known **exactly**
- $b_{m \geq 1}$: *unknown*, **s**- and ϵ -dependent

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All done **inside** the reduction in e.g. **ratracer** [Magerya]

Add the fit relations as extra equations $\rightarrow$ returns $J[\text{target}] \approx R(\mathbf{s}, \epsilon) \beta_0 + \sum R'(\text{known})$

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Integrate over Δ : by (i), only the faces $x_i = 0$ contribute

$$\sum_{\nu'} c_{\nu'}(\mathbf{s}, \epsilon) J[\nu'] = - \sum_i \int_{\Delta \cap \{x_i=0\}} \left(\mathbf{x}^{\nu} W_i \Phi \right) \Big|_{x_i=0}$$

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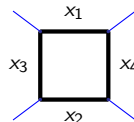
Finding all W

- (i) + (ii) are *linear* in the coefficients of W_i and $G_j \Rightarrow$ solve *degree by degree* over $\mathbb{Q}(s)$
- Same **syzygy problem** as for the K3/CY periods: $F P_a + G P_b = Q P$
- Plus the trivial relation $J[\nu] = \sum_i J[\nu + \mathbf{e}_i]$ from $\sum_i x_i = 1$

Example 1: One-Loop Box in $D = 8 - 2\epsilon$

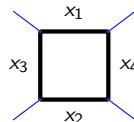
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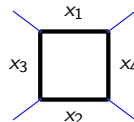
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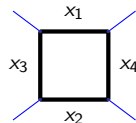
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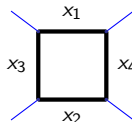
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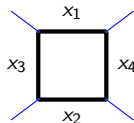


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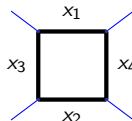
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$$J[0, 0, 0, 0] \approx R_1 (-\mathbf{s})^{-\epsilon} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} + R_2 (-\mathbf{t})^{-\epsilon} \frac{\Gamma(1-\epsilon)^2}{\Gamma(2-2\epsilon)} + R_3 \beta_0, \quad R_i \in \mathbb{Q}(\mathbf{s}, \mathbf{t}, \epsilon)$$

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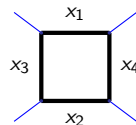
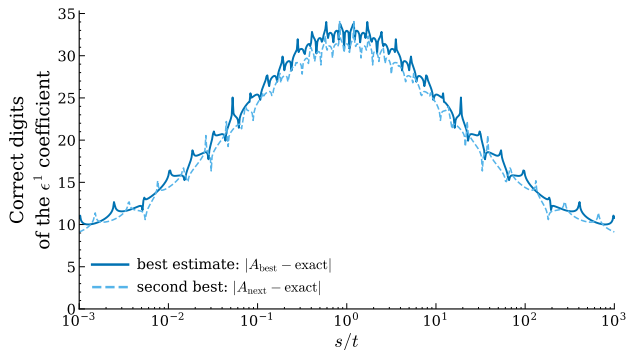
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ϵ^0 : **exact** ($= \frac{1}{6}$) $\rightarrow$ test the ϵ^1 coefficient against the analytic result

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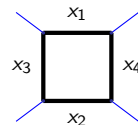
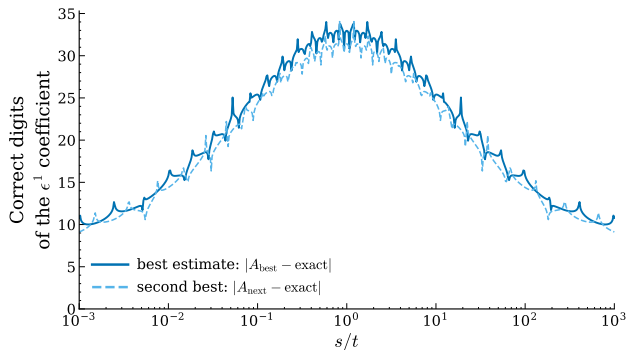
[PRELIMINARY]



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[PRELIMINARY]



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- True error
(estimate vs. **analytic** result)
- Second best (fewer nodes) a few digits **below**
 $\Rightarrow |A_{\text{best}} - A_{\text{next}}|$: *error estimate*

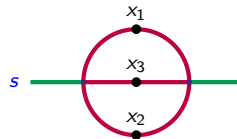
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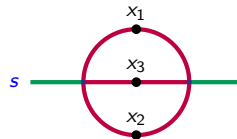
one dot on each propagator

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one dot on each propagator

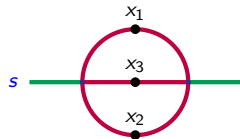
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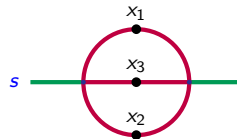
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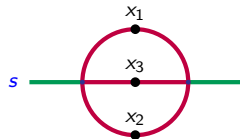
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- Top sector: **3** masters $\Rightarrow$ **3** fit relations

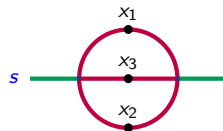
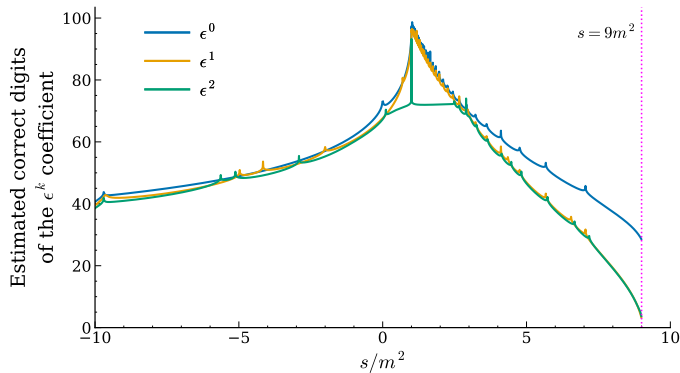
$$\beta_0 = \mathcal{U}^{-3+3\epsilon} \mathcal{F}^{-2\epsilon} \Big|_{\mathbf{x}^*} = 27^{1+\epsilon} (9m^2 - s)^{-2\epsilon}, \quad D_k = \frac{(k!)^3}{(3k+2)!}$$

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[PRELIMINARY]

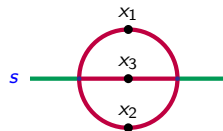
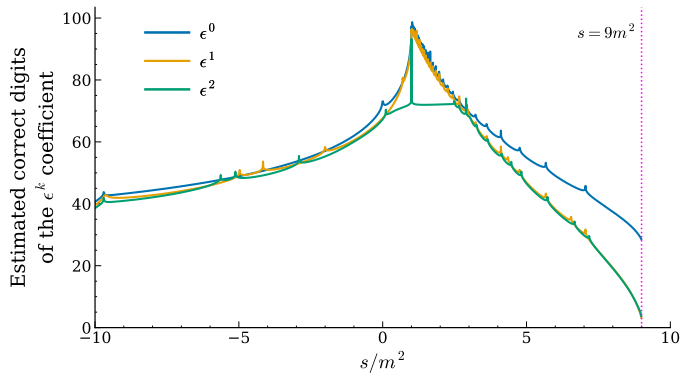
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[PRELIMINARY]



- Estimate error from three node windows
- Up to ϵ^2 , below threshold
- Digits **fall** as $s \rightarrow 9m^2$:
 β_0 branches

Example 3: One-Mass Sunrise in $D = 6 - 2\epsilon$, Minkowski Regime

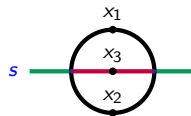
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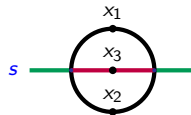
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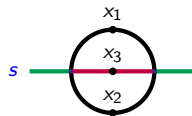
Same-sign decomposition

$$J(s) = m^{4-6\epsilon} J^+(s) + (-1 - i\delta^+)^{-2\epsilon} m^{2-2\epsilon} (s - m^2)^{2-4\epsilon} J^-(s)$$

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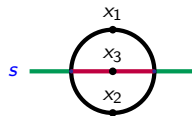
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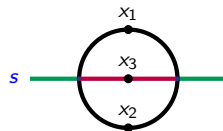
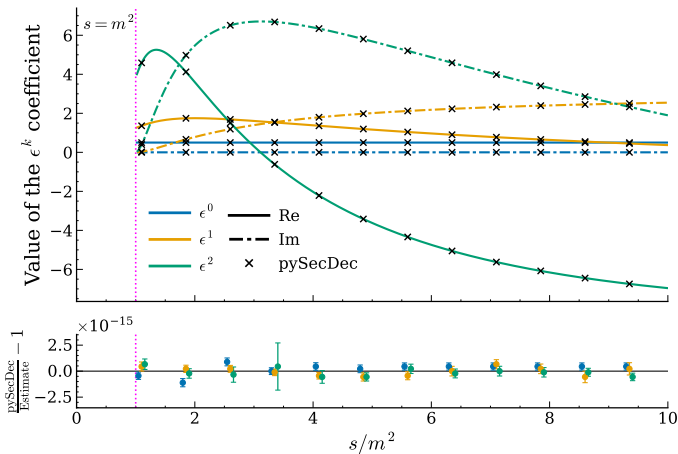
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Both *real* and *positive* $\Rightarrow$ *each* is a same-sign family: *fit each piece separately*

Example 3: One-Mass Sunrise vs. pySecDec

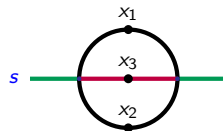
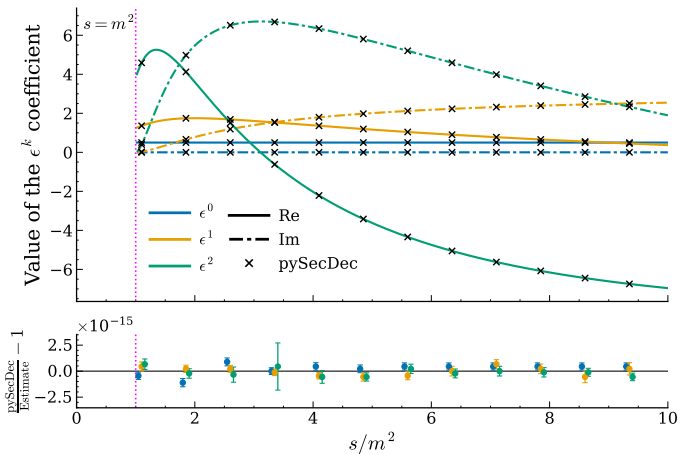
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[PRELIMINARY]



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[PRELIMINARY]



- Re and Im up to ϵ^2 above threshold
- Im entirely from $(-1 - i\delta^+)^{-2\epsilon}$
- Lower panel: agreement within pySecDec errors

Outline

- 1 Motivation
- 2 Exponential Suppression
- 3 Feynman Integrals
- 4 Outlook

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Thank you for listening!

Outline

5 Backup

Backup: A First Higher-Dimensional Period

$$I_{n,m}(x) = \int_0^1 \int_0^1 \frac{a^n b^m da db}{\sqrt{P(a,b)}}, \quad P = ab(1-a)(1-b)(1-xab), \quad x < 1$$

- IBP + symmetry $\rightarrow$ masters $\{I_{0,0}, I_{1,0}, I_{2,0}\}$; $I_{0,0} = 4K(\chi)^2$, $\chi = \frac{1}{2}(1 - \sqrt{1-x})$
- Suppressing numerator (cancels factors of $\sqrt{P}$):

$$\bar{I}_k = \iint \frac{\left([\sqrt{a(1-a)} - a(1-a)] [\sqrt{b(1-b)} - b(1-b)] \right)^k}{\sqrt{P}}, \quad 0 < \bar{I}_k < 2^{4-2k}(1-\ln 2)$$

- Expansion generates simpler families I^L (logs) and $I^E, I^{E'}$ (elliptic): 9 masters in total
- Function basis: $\{1, \pi, \sqrt{1-x}, \ln \frac{1+\sqrt{1-x}}{2}, E(x), K(x)\}$
- Three suppressed integrals $\bar{I}_{k_1,2,3} \approx 0 \Rightarrow I_{0,0}, I_{1,0}, I_{2,0} \approx \sum_i R_i^{(k_1,k_2,k_3)}(x) f_i$

Backup: Syzygies for the K3 Banana Periods

$$P = a b (a + 1)(b + 1)(ab - x)((a + 1)(b + 1) - x), \quad \Delta = [-1, 0]^2, \quad x < 0$$

- IBPs without raising $\sqrt{P}$: $F \partial_a P + G \partial_b P = Q P$, F, G vanishing on $\partial\Delta$
- Two *separable* solutions + one **non-separable**:

$$F = -a(1 + a) b (ab - x), \quad G = b(1 + b) a (ab - x), \quad Q = (a - b)(2ab - x)$$

- Only the non-separable syzygy reduces the masters $5 \rightarrow 4$: $\{l_{0,0}, l_{0,2}, l_{1,1}, l_{1,3}\}$
- Differential equations expose one more *constant* relation:

$$l_{1,1} - \frac{1 + 2x}{6} l_{0,0} = \frac{\pi^2}{3}$$

Same structure as the Feynman integral systems: syzygies + hidden relations