

The Stieltjes Bootstrap for Feynman Integrals

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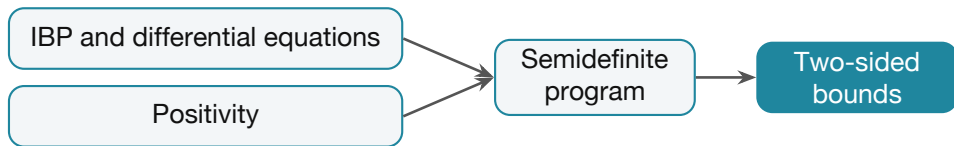
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The Stieltjes Bootstrap for Feynman Integrals, [arXiv:2609.37536](https://arxiv.org/abs/2609.37536)

Linear relations and positivity give bounds



- 2023 Integrand positivity + IBP: semidefinite program (Zeng)
- 2024 Positivity properties of Feynman integrals (JMH, Raman)
- 2025 Complete monotonicity, Stieltjes and Padé (Ditsch, JMH, Raman)
- 2026 General theorem, Hankel constraints, three loops (this work)

Outline

- 1 Feynman integrals are Stieltjes functions
- 2 Two methods: semidefinite bootstrap and Padé
- 3 Application: the three-loop electroweak ladder

Stieltjes functions have a positive density

$$f(x) = \int_R^\infty \frac{d\mu(t)}{x+t}, \quad d\mu(t) \geq 0$$

- Density on $t \geq R$: analytic off the cut $x \leq -R$
- Moments $a_k = \int d\mu(t)/(x_0 + t)^{k+1}$, Taylor coefficients up to sign
- Every $a_k > 0$: complete monotonicity
- Stronger: $H = (a_{i+j}) \succeq 0$: $c^\top H c \geq 0$ for all c



Positivity of the density constrains every moment.

Feynman integrals are Stieltjes functions

Theorem. *Let I be a convergent scalar Feynman integral with $0 < \lambda \leq 1$, where $\lambda = \sum_e a_e - LD/2$. Then I is a Stieltjes function separately in each $-p_i \cdot p_j$ and in each internal mass m_e^2 , with the other variables in the non-negative orthant (a subset of the Euclidean region).*

- any loop order, any number of legs, any masses
- key step: spanning two-forests give $\mathcal{F}$ non-negative coefficients
- on-shell legs: adapted variables do the same job

Euclidean region: the kinematics where $\mathcal{F} \geq 0$ for all Feynman parameters.

The Euclidean region is where $\mathcal{F}$ is non-negative

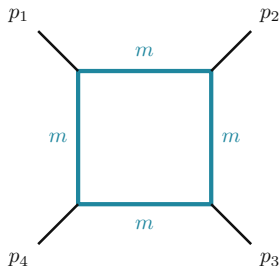
$$\mathcal{E}_G = \{z : \mathcal{F}(\alpha; z) \geq 0 \text{ for all } \alpha_e \geq 0\}$$

$\mathcal{E}_G$ is the copositivity region of $\mathcal{F}$ (Sturmfels, Telek 2025)

- Generic off-shell kinematics: all $-p_i \cdot p_j > 0$ and $m_e^2 > 0$ lie inside
- Special kinematics (on-shell legs, equal masses): a dedicated analysis finds the region and adapted variables; examples in our paper
- Massless propagators: positive coefficients are necessary and sufficient

Example: the massive box in six dimensions

$$\mathcal{F} = (-s) \alpha_1 \alpha_3 + (-t) \alpha_2 \alpha_4 + m^2 \mathcal{U}^2$$



- $I = \int d^4\alpha / \text{GL}(1) \mathcal{U}^{-2} \mathcal{F}^{-1}$, with $\mathcal{U} = \sum_e \alpha_e$
- $D = 6$: finite, and $\lambda = 4 - 6/2 = 1$
- For $-s, -t, m^2 > 0$ every coefficient of $\mathcal{F}$ is positive
- $\mathcal{F}$ is linear in each: I is Stieltjes in $-s, -t$ and m^2

The master integrals can all be chosen Stieltjes

Each family has infinitely many Stieltjes integrals: raise propagator powers and dimension together, keeping $0 < \lambda \leq 1$. For the box:

$$\vec{g} = (I_{1111}^{(6)}, I_{1110}^{(4)}, I_{1101}^{(4)}, I_{1010}^{(2)}, I_{0101}^{(2)}, I_{0003}^{(4)})$$

- Basis elements live in different dimensions, each with $\lambda = 1$
- Found by a small integer linear program: $\lambda = 1$ and UV finiteness
- Dimensional recurrence maps them back to $D = 4 - 2\varepsilon$

With higher dimensions allowed, Stieltjes integrals span every family studied.

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Method A: positivity as a semidefinite program

$$H_0 \succeq 0, \quad H_1 \succeq 0, \quad H_0 - (x_0+R) H_1 \succeq 0, \quad H_1 - (x_0+R) H_2 \succeq 0$$

- $H_M = (a_{i+j+M})$: Hankel matrices of the moments, for every integral
- Differential equation: all a_k linear in the unknown values $\vec{g}(x_0)$
- Minimise and maximise each value, with one normalisation fixed
- High-precision solvers, as in the conformal bootstrap

Input: differential equation, one normalisation. Output: an interval per integral.

Toy example: Approximation of $\log 2$

$$f(u, v) = \int_0^\infty \frac{dt}{(u+t)(v+t)} = \frac{\log(u/v)}{u-v}$$

Stieltjes in u and in v , with $R = 0$. Its differential equation at $u = 1$:

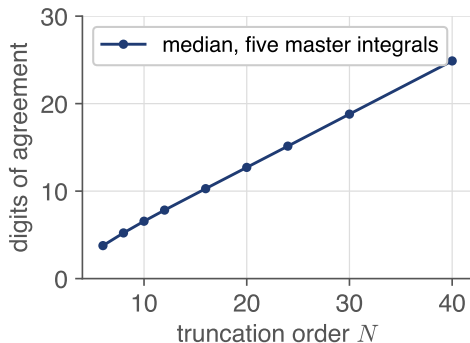
$$\partial_v \begin{pmatrix} f \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1-v} & \frac{1}{v(v-1)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ 1 \end{pmatrix}$$

At $v = 2$ every moment is linear in $y = f(1, 2)$: $a_1 = y - \frac{1}{2}$, $a_2 = y - \frac{5}{8}$, ...

$$N = 8 : \quad 0.6931464 \dots = \frac{445}{642} \leq \log 2 \leq \frac{3743}{5400} = 0.6931481 \dots$$

Without ever evaluating f , positivity pins down $\log 2$.

Massive box: the bounds close exponentially

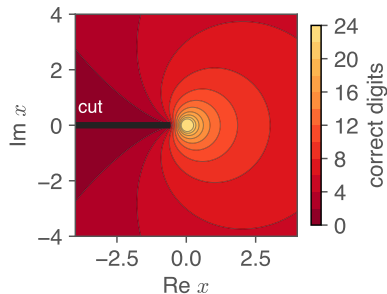


- point $(s, t, m^2) = (-3, -7, 1)$
- five unknowns, $g_6 = 1$ fixed
- $N = 40$: 24.9 digits in 38 s
- Exponential convergence
- Solver: SDPB (Simmons-Duffin 2015)

About 0.6 digits per unit of N , from the differential equation alone.

Method B: Padé gives two-sided bounds

- Expand in x : $(s, t) \rightarrow (xs, xt)$; tadpole at $x = 0$
- Stieltjes: Padé converges in the cut plane (classical)
- Two-sided: $[N-1/N] \leq f \leq [N/N]$
- g_1 at $x = 1$: $[10/10]$ gives 12.5 digits



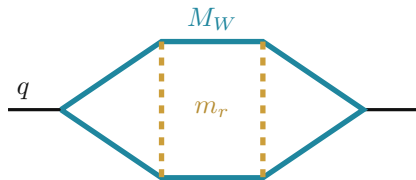
massive box: correct digits of $[10/10]$

Method B needs values at one point; method A only the differential equation.

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Target: the three-loop electroweak ladder



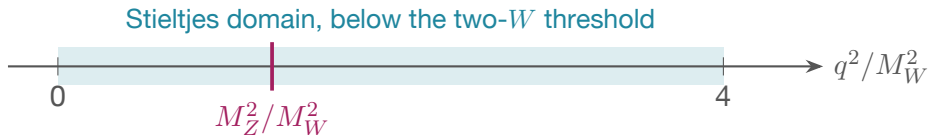
six W propagators; rungs $m_r = M_Z$
or M_H

- FCC-ee needs three-loop electroweak corrections
- neutral-boson self-energies at $q^2 = M_Z^2$
- 60 master integrals in 35 sectors, two mass scales M_W, m_r

Can the differential equation plus positivity alone determine them?

A Stieltjes basis for the electroweak ladder

- in $D = 4 - 2\varepsilon$, Stieltjes integrals span only 17 of the 60 master integrals
- in $D = 6 - 2\varepsilon$: 149 Stieltjes integrals, giving a basis of 60
- Stieltjes in $-q^2/M_W^2$ (threshold 4) and in m_r^2/M_W^2
- $\lambda = 1 + 3\varepsilon$ requires $-1/3 < \varepsilon \leq 0$



The physical point $q^2 = M_Z^2$ needs no analytic continuation.

All 60 three-loop master integrals to 20 digits

- $\varepsilon = -1/10$, $N = 22$: all 59 unknowns to at least 20.3 digits
- Values agree with AMFlow
- ε -expansion below: method B at 28 values of ε

top sector	ε^0	ε^1	ε^2
g_{55}	0.120105581249...	-0.539284773...	2.27447255...
g_{58}	0.037079419645...	-0.168758700...	0.68197213...
g_{59}	0.065088876705...	-0.293958933...	1.21653195...
g_{60}	0.120049044401...	-0.539227087...	2.27426314...

method B, at $q^2 = M_Z^2$, $m_r = M_H$, $M_W = 1$; digits truncated

Comparing methods on the electroweak ladder

method	extra input	digits	solve	setup
A: Stieltjes bootstrap	one normalisation	20	20 min	12 min
complete-monotonicity bootstrap	one normalisation	10	28 min	13 min
B: soft expansion + Padé	values at $q^2 = 0$	24	44 s	27 min
LINE	values at $q^2 = 0$	56	26 s	27 min
AMFlow	family definition	80	24 min	–

60 integrals at $q^2 = M_Z^2$, $\varepsilon = -1/10$; times refer to wall clock

Bootstraps: 128 EPYC cores; AMFlow: 24 threads; LINE, Padé: one core

Setup: building the Hankel constraints from the differential equation, or AMFlow values at $q^2 = 0$

AMFlow is faster today; this first bootstrap is comparable to the other methods.

Discussion

- Feynman integrals are Stieltjes in each invariant
 - Method A: Differential equation and positivity as a semidefinite program
 - Method B: Padé approximants from an expansion point
 - In both, Stieltjes positivity gives two-sided bounds that converge
- *Fits the usual workflow: one extra reduction of the Stieltjes integrals*

Outlook

- Faster: leaner semidefinite programs, analytic boundary values for Padé
- Analytic continuation to the physical region above thresholds



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Backup

At each order, two of the four inequalities suffice

$$H_1^{(N)} = \left(H_1^{(N-1)} - (x_0+R) H_2^{(N)} \right) + (x_0+R) \tilde{H}_0^{(N)}$$

- $\tilde{H}_0^{(N)}$: $H_0^{(N)}$ without its first row and column
- Even N : $H_0 \succeq 0$ and $H_1 - (x_0+R)H_2 \succeq 0$ imply the other two
- Odd N : $H_1 \succeq 0$ and $H_0 - (x_0+R)H_1 \succeq 0$ suffice

Half the constraints, the same feasible set: smaller semidefinite programs.

Special kinematics: adapted variables

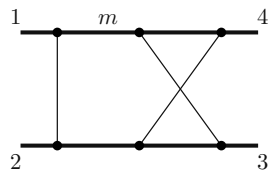
On-shell legs and momentum conservation can make the positive orthant unreachable. Instead, find variables x_a with

$$\mathcal{F} = A_0(\alpha) + \sum_a x_a A_a(\alpha), \quad A_0, A_a \geq 0.$$

- Hexa-box (five points, non-planar): Stieltjes in five adapted variables
- Massless double pentagon: provably no Euclidean region with interior
- Equal-mass non-planar Bhabha: positive subregion, three adapted variables
- Three-loop electroweak ladder: Stieltjes up to the two- W threshold

The adapted variables are exactly the Stieltjes variables of the integral.

Bhabha: quadratic forms give a positive cone



equal-mass non-planar
Bhabha

$$m^2 = u + v + w, \quad y_{12} = -u + v - w, \quad y_{13} = -u - v + w$$

- Coefficient test: only a degenerate region
- Quadratic forms: $\mathcal{F} = u A_u + v A_v + w A_w$, all $A \geq 0$
- Stieltjes in u, v, w on $\{u, v, w \geq 0\}$

With masses the coefficient test is only sufficient; copositivity is the full answer.