



On shell Z production at Threeloop $\mathcal{O}(\alpha_s^2\alpha)$: Form Factors and SV Cross-section

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&

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In collaboration with N. Rana, V. Ravindran and A. Vicini

High Precision for Hard Processes (HP2 2026)

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► Introduction

- Background: Why Z Production
- The Form Factors

► Calculation

- MIs and non-singlet form-factors
- Renormalization
- Singlet form-factors
- Soft-Virtual cross-section

► Conclusion and Outlook

- Conclusion and Outlook

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Background

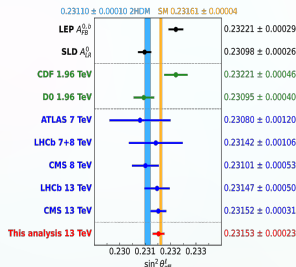
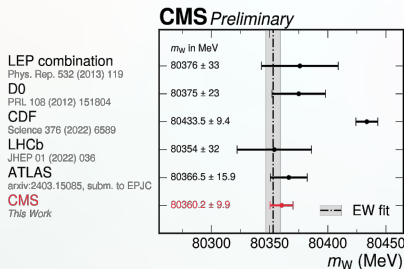
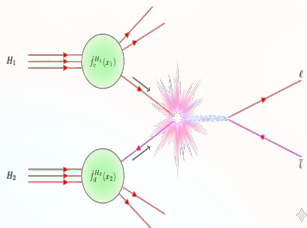
Why Z Production

Also, see talks by **Mathieu** and **Simone** on **Tue**

Why Drell-Yan

The Drell-Yan (DY) process is important:

- Quantitative test of perturbative QCD.
- Constraints the PDFs.
- Precise determination of electroweak observables: may lead to BSM signature.



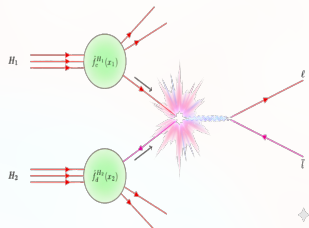
► Relative uncertainty: 0.012% [1].

► Relative uncertainty: 0.104% [2].

[1].CMS (2024) [2].CMS (2026)

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- ▶ Quantitative test of perturbative QCD.
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Projected uncertainty

- ▶ δm_W : $6 \times 10^{-3} \%$ [3].
- ▶ δs_w^2 : $4 \times 10^{-2} \%$ [3].

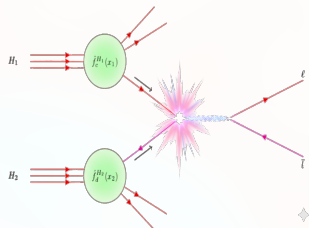


Projected uncertainty

- ▶ δm_W : $6 \times 10^{-4} \%$ [4].
- ▶ δs_w^2 : $1 \times 10^{-3} \%$ [5].

The **Drell-Yan (DY) process** is important:

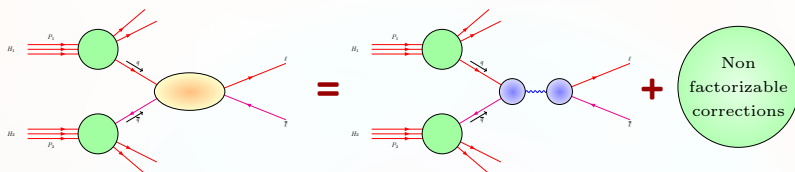
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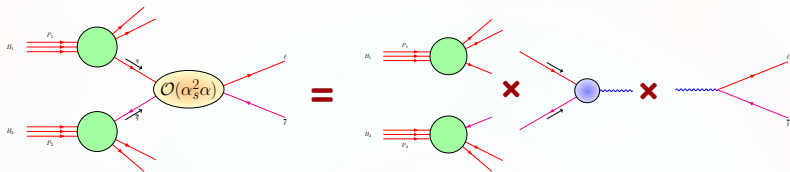
- **Standard candle:** For any process $\sigma_{pp \rightarrow Y} = \frac{N_Y}{\mathcal{L}_{DY}}$, where $\mathcal{L}_{DY} = \frac{N_{\bar{l}l}}{\sigma_{\bar{l}l}}$

Improving the Drell – Yan precision is indispensable

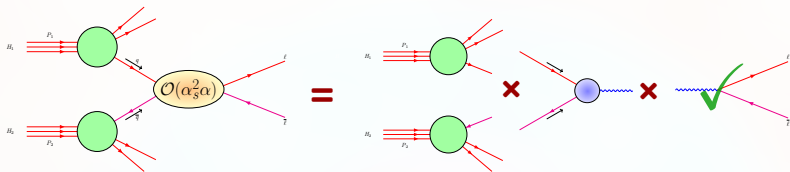
Drell-Yan at threshold



Drell-Yan at threshold



Drell-Yan at threshold



Drell-Yan at threshold

$$\sigma_H = \sum_{a,b} \int dx_1 dx_2 d\Phi_{\text{FS}} \underbrace{f_a(x_1, \mu_F)}_{\text{Parton density functions}} \underbrace{f_b(x_2, \mu_F)}_{\text{Parton density functions}} \underbrace{\hat{\sigma}_{ab \rightarrow X}(\hat{s}, \mu_F, \mu_R)}_{\text{Parton-level cross section}} + \mathcal{O}\left(\frac{\Lambda_{QCD}}{Q}\right)$$

Phase-space integral

$$\sigma_H = \sigma_0 \sum_{i,j} f_i^{h1}(\mu_F) \otimes \Delta_{ij}(\mu_R, \mu_F) \otimes f_j^{h2}(\mu_F)$$

σ_0 is the Born level cross-section.

Drell-Yan at threshold

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$$\Delta = \sum_{m,n} a_s^m a_e^n \Delta^{mn}$$

Drell-Yan at threshold

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$$\left[+ a_s^3 \Delta_g^{30} + a_s^2 a_e \left(\Delta_\gamma^{21} + \frac{1}{s_w^2 c_w^2} \Delta_Z^{21} + \frac{1}{s_w^2} \Delta_W^{21} \right) + \mathcal{O}(a_s a_e^2) + \mathcal{O}(a_e^3) \dots \right]$$

Z Production: Status

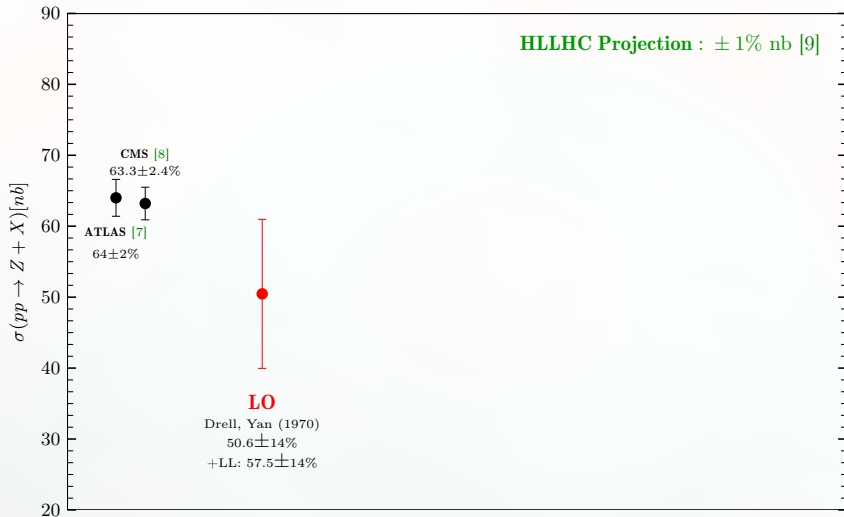
Process: $pp \rightarrow Z + X$, $\sqrt{s}=14$ TeV, PDFSet: MMHT2014, $\mu_F, \mu_R \in \left(\frac{m_Z}{2}, 2m_Z\right)$



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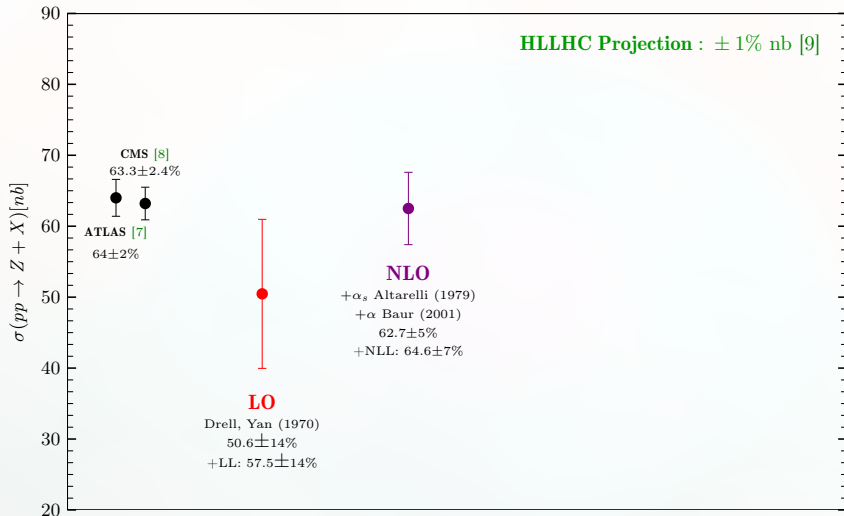
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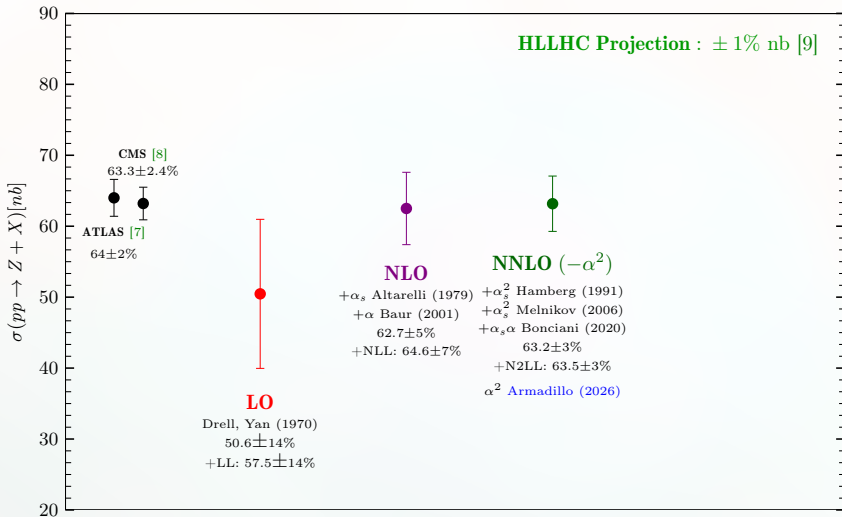
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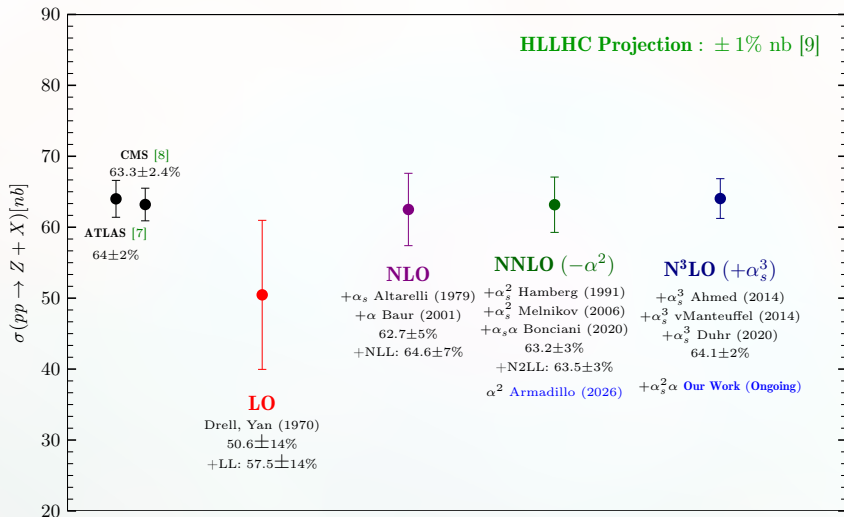
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$$\mathcal{O}(\alpha_s^2 \alpha_e) \ g \times \gamma^*$$

$$\mathcal{O}(\alpha_s^2 \alpha_e) \ g \times Z$$

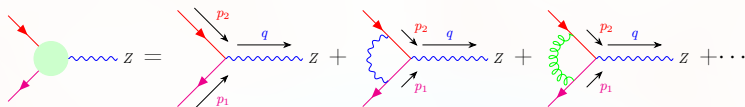
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2

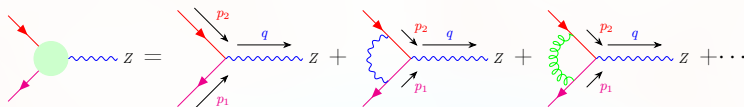
Calculations

The Form Factors

The Form Factors: Definition



The Form Factors: Definition



$\mathcal{M}^\mu$

CP even & Vector

$$v_Q \gamma^\mu$$

$$v_Q \frac{i}{2m} \sigma^{\mu\nu} q_\nu$$

CP even & Axial

$$a_Q \gamma^\mu \gamma^5$$

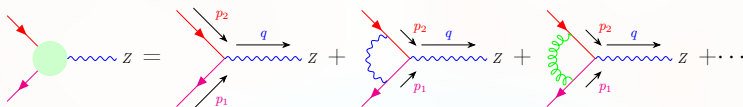
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CP odd & V + A

$$v_Q q^\mu$$

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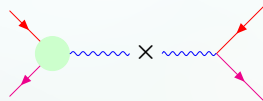
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Three-loop contribution to σ_{parton}

$$\sigma \propto \int d\Pi_1 \underbrace{\mathcal{M}_3^\mu \mathcal{M}_\mu^{\text{Born}}}_{\text{combination of FFs}} + \int d\Pi_1 \mathcal{M}_2^\mu \mathcal{M}_{1\mu}^\dagger + \dots$$



The Form Factors: Workflow

Generation of diagrams

QGRAF [Nogueira]

The Form Factors: Workflow

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Mapping to integral families

Reduze [von Manteuffel]
Mapping the diagrams to a few sets of
integral families using momenta shifts.

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Mapping the diagrams to a few sets of integral families using momenta shifts.

Feynman Rules, Dirac, Color Algebra

Apply projectors, perform Dirac Traces, color algebra using in-house routines written in FORM [Vermaseren].

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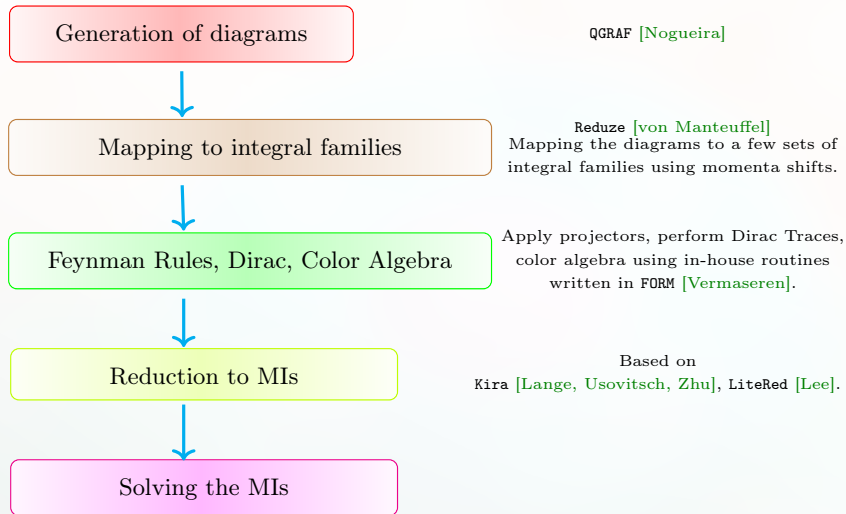
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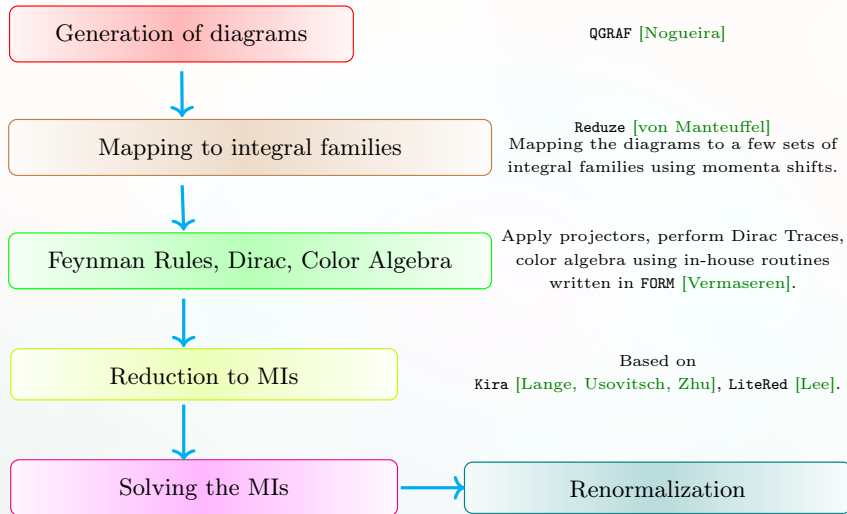
Reduction to MIs

Based on
Kira [Lange, Usovitsch, Zhu], LiteRed [Lee].

The Form Factors: Workflow



The Form Factors: Workflow



3

Calculations

MI and nonsinglet FFs

TP, Rana N (JHEP 10 (2025) 233) [arXiv: 2506.15363](#)

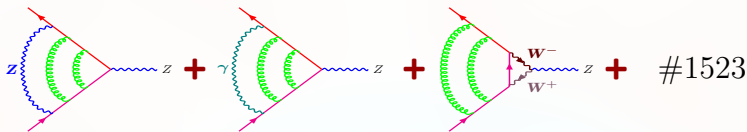
TP, Rana N, Ravindran V (JHEP 05 (2026) 220) [arXiv: 2512.22992](#)

The Form Factors: Evaluation

The image shows three Feynman diagrams representing different contributions to the production of a Z boson in a quark-antiquark annihilation process. Each diagram consists of a triangular loop of quarks (green lines) with a Z boson (blue wavy line) attached to the top vertex. The incoming quark and antiquark lines are red. The diagrams are separated by red plus signs. The third diagram includes a vertex correction with a W boson loop (green wavy lines) and a Z boson (blue wavy line) attached to the top vertex. The outgoing Z boson is labeled 'Z'. The diagrams are followed by a red plus sign and the text '#1523'.

$$+ \quad + \quad + \quad \#1523$$

The Form Factors: Evaluation



Evaluation of the Master Integrals

- ✓ **Analytic evaluation** of all (308+4) **Master Integrals** in terms of GPLs.
 - Required three independent rational transformation, massive symbolic parallelization. Computational cost in final run: $\sim 10^4$ core-h.

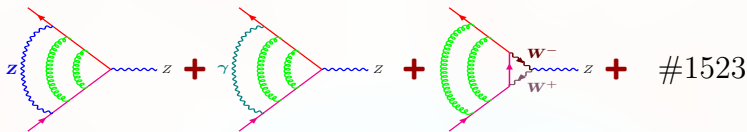
The Form Factors: Evaluation

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \dots$$

The non-singlet Form Factors

► Defn of **Non-Singlet**: $\{\text{Diagrams}\} \cap \left\{ \text{Diagram} \right\} = \emptyset$

The Form Factors: Evaluation



The non-singlet Form Factors

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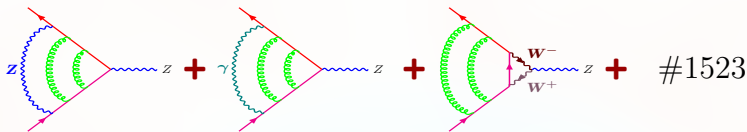
Internal Z

- Initial expression size: 50 GiB
- Final expression size: 2.1 MiB

Internal W

- Initial expression size: 40 GiB
- Final expression size: 3.7 MiB

The Form Factors: Evaluation



The non-singlet Form Factors

► Defn of **Non-Singlet**: $\{\text{Diagrams}\} \cap \left\{ \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\} = \emptyset$

► The final expression is **complete analytic** in terms of Generalized Polylogs:
But, contains **divergences**:

$$|\mathcal{M}|^2 = \frac{A(q^2, m_V)}{\epsilon^5} + \frac{B(q^2, m_V)}{\epsilon^4} + \frac{C(q^2, m_V)}{\epsilon^3} + \frac{D(q^2, m_V)}{\epsilon^2} + \frac{E(q^2, m_V)}{\epsilon} + G$$

4

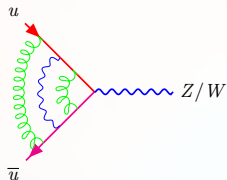
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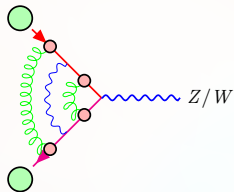
Renormalization

TP, Rana N, Ravindran V (JHEP 05 (2026) 220) [arXiv: 2512.22992](#)

TP, Rana N, Vicini A (PLB.2026.140967) [arXiv: 2604.18516](#)

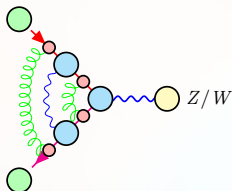
Renormalization Checklist





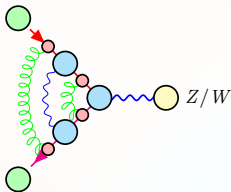
Renormalization Checklist

- External NS quark wave-function
- Strong coupling



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- Electroweak charge / Vertex
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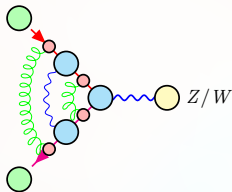
Quark wave-function + Strong coupling

► Define: $\psi_0 \rightarrow \sum_{i,j} \alpha_s^i \alpha_e^j Z_{i,j} \psi_R$ and $\alpha_s^0 \rightarrow \sum_{i,j} \alpha_s^i \alpha_e^j Z_{i,j}^\alpha \alpha_s$. Then define:

$$F_{\text{ren}} = |Z_\psi|^2 F_{\text{Bare}}$$

$$F_{21}^{\text{ren}} = F_{21} + Z_{21} + F_{10} Z_{11} + F_{20} Z_{01} - \frac{\beta_{00} Z_{11}}{\epsilon} - \frac{\beta_{00} F_{10} Z_{01}}{\epsilon} - \frac{\beta_{00} F_{11}}{\epsilon} - \frac{\beta_{01} F_{10}}{2\epsilon}$$

Renormalization Checklist



- ✓ External NS quark wave-function
- ✓ Strong coupling
- Electroweak charge / Vertex
- Vector Boson wave-function

EW charge + Vector Boson wave-function

Weak charge + VB wavefunction

Vertex Oblique correction [10]

$$\frac{g_0}{c_w^0} Z_{ZZ}^{1/2} \rightarrow \sqrt{4\sqrt{2}G_F m_Z^2} \left(1 - \frac{1}{2}\Delta r + \frac{1}{2}\delta c_Z \right)$$

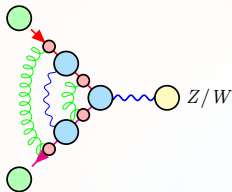
where: $\delta c_Z = \delta Z_{ZZ} + 2\delta Z_e + \frac{(s_w^2 - c_w^2)}{c_w^2} \frac{\delta s_w^2}{s_w^2}$

$$v_q \rightarrow v_q - 2Q_q s_w^2 \Delta\kappa$$

where: $\Delta\kappa = \frac{\delta s_w^2}{s_w^2} + \frac{2c_w}{s_w} \delta Z_{AZ}$

[10].Bonciani (2021)

Renormalization Checklist



✓ External NS quark wave-function

✓ Strong coupling

Electroweak charge / Vertex

Vector Boson wave-function

EW charge + Vector Boson wave-function

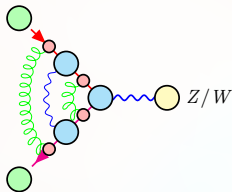
$$\Delta r = \frac{s_w^2}{c_w^2} \left(\frac{\delta c_w^2}{c_w^2} \right) + 2\delta Z_e + \frac{1}{m_W^2} \left(\Pi_{WW}^T(0) - \text{Re}[\Pi_{WW}^T(m_W^2)] \right) + V_W + B_W$$

$$\Delta r = 0 + \alpha \Delta r^{01} [11] + \alpha_s \alpha \Delta r^{12} [9] + \alpha^2 \Delta r^{02} [13] + \alpha_s^2 \alpha \Delta r^{21} [14]$$

► Similar perturbative expansions hold for $\delta c_{Z/W}$ or $\Delta\kappa$. We performed full off-shell calculation to evaluate δc_V^{21} , $\Delta\kappa^{21}$, Δr^{21} with analytic series expansion.

[11].Sirlin (1980,83) [12].Freitas(2000) [13].Awramik (2002,03,04) [14].Chetyrkin, (1995)

Renormalization Checklist



- ✓ External quark wave-function
- ✓ Strong coupling
- ✓ Electroweak charge / Vertex
- ✓ Vector Boson wave-function

After complete UV renormalization

$$F_{21}^{\text{ren}} = \frac{B_1(q^2, m_V)}{\epsilon^4} + \frac{C_1(q^2, m_V)}{\epsilon^3} + \frac{D_1(q^2, m_V)}{\epsilon^2} + \frac{E_1(q^2, m_V)}{\epsilon} + G_1$$

► These poles are pure IR: $F_{21}^{\text{IR-fin}} = Z_{\text{IR}} F_{21}^{\text{fin}}$ gives IR finite FFs: $F_{21}^{\text{IR-fin}}$.

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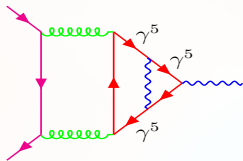
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The singlet FFs

TP, Rana N [arXiv: 2606.20342](#)

TP, Rana N (Under progress) [arXiv: 261x.xxxxx](#)

The Singlet FFs



Nonvanishing Vector FFs

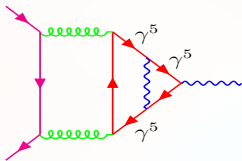
✓ $\text{Tr} (\dots \gamma^5 \dots \gamma^5 \dots)$ using NDR

Nonvanishing Axial FFs

⚠ $\text{Tr} (\dots \gamma^5 \dots) \text{Tr} (\dots \gamma^5 \dots)$

⚠ $\text{Tr} (\dots \gamma^5 \dots) \text{Tr} (\gamma^5 \dots \gamma^5 \dots \gamma^5 \cdot)$

The Singlet FFs



Nonvanishing Vector FFs

✓ $\text{Tr} (\cdots \gamma^5 \cdots \gamma^5 \cdots)$ using NDR

Nonvanishing Axial FFs

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⚠ $\text{Tr} (\cdots \gamma^5 \cdots) \text{Tr} (\gamma^5 \cdots \gamma^5 \cdots \gamma^5 \cdots)$

Problem with γ^5 in d -dim

► It is **not possible** to satisfy both anti-commutativity and trace-condition:

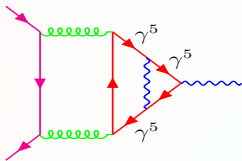
$$\{\gamma^\mu, \gamma^5\} = 0$$

$$\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$$

► **Solution:** Abandon any one of the condition and **renormalize** it at the end.

► Proposed schemes: 't-H-V (1972), BM (1977), Larin (1991) KKS (1992), $\cdots$

The Singlet FFs



Nonvanishing Vector FFs

✓ $\text{Tr} (\cdots \gamma^5 \cdots \gamma^5 \cdots)$ using NDR

Nonvanishing Axial FFs

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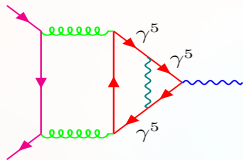
Larin's prescription

- ▶ Abandon the anticommutativity and for axial-vector, replace:

$$\gamma^\mu \gamma^5 \rightarrow -\frac{i}{3!} \epsilon^{\mu abc} \gamma_a \gamma_b \gamma_c$$

- ▶ Contract ϵ^{abcd} using the **4 - dim** identity resulting **d - dim** Lorentz indices.
- ▶ Compensate the violation by restoring the form of **axial Ward identity**.

Axial FFs for γ

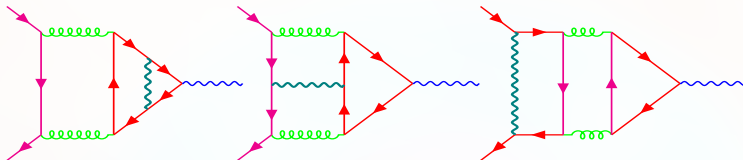


Nonvanishing Axial FFs

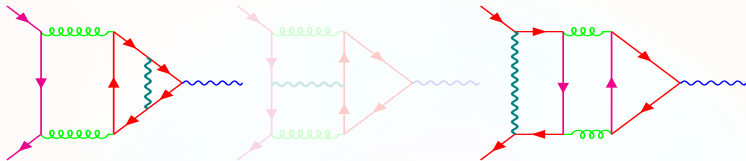
- ✗ $\text{Tr} (\dots \gamma^5 \dots) \text{Tr} (\cdot \gamma^5 \dots \gamma^5 \dots \gamma^5 \cdot)$
- 💡 $\text{Tr} (\dots \gamma^5 \dots) \text{Tr} (\dots \gamma^5 \dots)$

The Singlet FFs: γ

Axial FFs for γ



Axial FFs for γ



- UV renormalized IR subtracted Born-normalized **pure axial** FFs at $\mu^2 \rightarrow -q^2$:

$$F_{\text{sing},\gamma}^{21} = \sum_{\substack{u \in \{u,c\} \\ d \in \{d,s,b\}}} \frac{a_u(Q_d^2 a_d + Q_u^2 a_u)}{2c_w^4 s_w^4} C_F \left\{ -\frac{16}{\epsilon} - 196 + 32\zeta(2) + 48\zeta(3) \right\} \\ + \sum_q \frac{Q_u^2 a_q a_u}{2c_w^4 s_w^4} C_F \left\{ \frac{28}{\epsilon} + 582 - 96\zeta(2) - \frac{192}{5}\zeta(2)^2 \right\}$$

The Singlet FFs: γ

► We start with the $SU(N) \times U(1)$ axial anomaly eqn:

$$(\partial_\mu J_\mu^5)_R = a_s n_F T_F (G\tilde{G})_R + a_e (F\tilde{F})_R$$

$$\text{with, } G\tilde{G} = \epsilon_{\mu\nu\rho\sigma} G_{a,\mu\nu} G_{\rho\sigma}^a \quad F\tilde{F} = \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

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- Since anticommutativity is violated, current renormalization: $J_\mu^R = Z_{\overline{MS}} Z_5 J_\mu^B$

$$\left[\text{Triangle Diagram with Gluon} \right] \times p^\mu = a_s n_F T_F \left[\text{Triangle Diagram with Photon} \right] \times \epsilon^{\mu\nu\rho\sigma} p_1 p_2$$

The Singlet FFs: γ

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- Since anticommutativity is violated, current renormalization: $J_\mu^R = Z_{\overline{MS}} Z_5 J_\mu^B$

$$\left[\text{Diagram 1} \right] \times p^\mu = a_s n_F T_F \left[\text{Diagram 2} \right] \times \epsilon^{\mu\nu\rho\sigma} p_1 p_2$$

- The renormalization of gauge-field density operators is **non-trivial**:

$$\begin{pmatrix} (\partial_\mu J_\mu^5)_R \\ (G\tilde{G})_R \\ (F\tilde{F})_R \end{pmatrix} = \begin{pmatrix} Z_{JJ} & 0 & 0 \\ Z_{GJ} & Z_{GG} & 0 \\ Z_{FJ} & 0 & Z_{FF} \end{pmatrix} \begin{pmatrix} (\partial_\mu J_\mu^5)_B \\ (G\tilde{G})_B \\ (F\tilde{F})_B \end{pmatrix}$$

The Singlet FFs: γ

- We start with the $SU(N) \times U(1)$ axial anomaly eqn:

$$(\partial_\mu J_\mu^5)_R = a_s n_F T_F (G\tilde{G})_R + a_e (F\tilde{F})_R$$

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- Since anticommutativity is violated, current renormalization: $J_\mu^R = Z_{\overline{MS}} Z_5 J_\mu^B$
- After UV renormalization and IR subtraction, the constants read:

$$Z_{\overline{MS},\gamma}^{21} = \sum_{\substack{u \in \{u,c\} \\ d \in \{d,s,b\}}} \frac{a_u (Q_d^2 a_d + Q_u^2 a_u)}{2c_w^4 s_w^4} C_F \left\{ \frac{16}{\epsilon} \right\} + \sum_q \frac{Q_u^2 a_q a_u}{2c_w^4 s_w^4} C_F \left\{ -\frac{28}{\epsilon} \right\}$$

$$Z_{5,\gamma}^{21} = \sum_{\substack{u \in \{u,c\} \\ d \in \{d,s,b\}}} \frac{a_u (Q_d^2 a_d + Q_u^2 a_u)}{2c_w^4 s_w^4} C_F \{196 - 48\zeta(3)\} + \sum_q \frac{Q_u^2 a_q a_u}{2c_w^4 s_w^4} C_F \{-173\}$$

The Singlet FFs: γ

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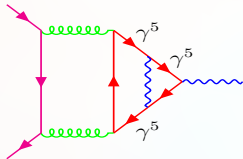
$$(\partial_\mu J_\mu^5)_R = a_s n_F T_F (G\tilde{G})_R + a_e (F\tilde{F})_R$$

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- Since anticommutativity is violated, current renormalization: $J_\mu^R = Z_{\overline{MS}} Z_5 J_\mu^B$
- Combine everything, the finite **pure axial** FFs at $\mu^2 \rightarrow q^2$:

$$F_{\text{sing},\gamma}^{21} = \sum_{\substack{u \in \{u,c\} \\ d \in \{d,s,b\}}} \frac{a_u (Q_d^2 a_d + Q_u^2 a_u)}{2c_w^4 s_w^4} C_F \{32\zeta(2)\} \\ + \sum_q \frac{Q_u^2 a_q a_u}{2c_w^4 s_w^4} C_F \left\{ 409 - 96\zeta(2) - \frac{192}{5}\zeta(2)^2 \right\}$$

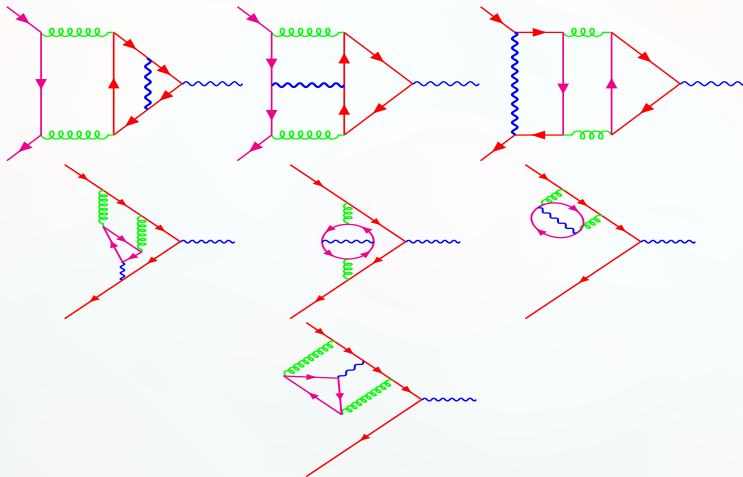
Axial FFs for $Z/W^\pm$



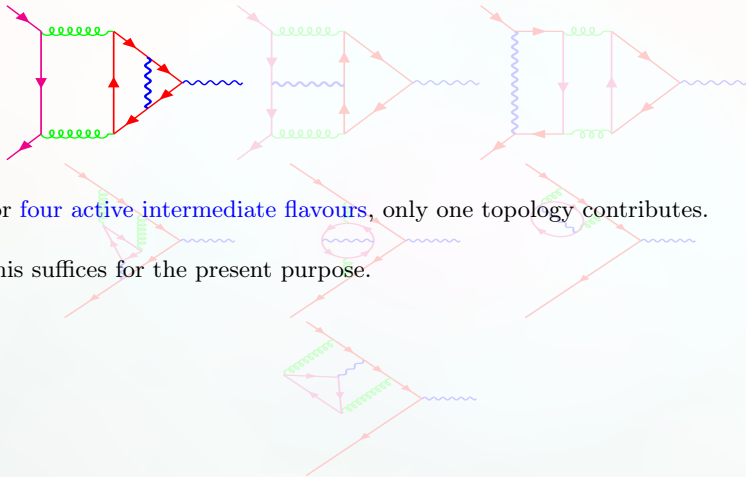
Nonvanishing Axial FFs

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Axial FFs for $Z/W^\pm$



Axial FFs for $Z/W^\pm$



- For four active intermediate flavours, only one topology contributes.
- This suffices for the present purpose.

Axial FFs for $Z/W^\pm$

- Massive Boson does not introduce new UV divergence, so:

$$Z_{\overline{MS},Z}^{2,1} = \sum_{\substack{u \in \{u,c\} \\ d \in \{d,s\}}} \frac{a_u(v_d^2 a_d + v_u^2 a_u)}{c_w^4 s_w^4} \left\{ \frac{16}{\epsilon} \right\}$$
$$Z_{5,Z}^{2,1} = \sum_{\substack{u \in \{u,c\} \\ d \in \{d,s\}}} \frac{a_u(v_d^2 a_d + v_u^2 a_u)}{c_w^4 s_w^4} \{196 - 48\zeta(3)\}$$

Axial FFs for $Z/W^\pm$

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- In the **on-shell limit** $q^2 \rightarrow m_Z^2$ for intermediate Z the finite axial FF:

$$F_{\text{sing},Z}^{21} = \sum_{\substack{u \in \{u,c\} \\ d \in \{d,s\}}} \frac{a_u(v_d^2 a_d + v_u^2 a_u)}{c_w^4 s_w^4} \left\{ -31.1341672998045 + 9.20241201912141 i \right\}$$

Axial FFs for $Z/W^\pm$

► Similarly, for intermediate W with $x = -\frac{m_Z^2}{m_W^2}$ the finite axial FF:

$$\begin{aligned}
 F_{\text{sing}, W}^{21} = & \sum_{\substack{u \in \{u, c\} \\ d \in \{d, s\}}} \frac{a_u(v_d + v_u)}{8c_w^2 s_w^4} \left\{ \frac{136(-1 + 2x + 3x^2)H_{-1}(x)}{x^2} - \frac{144(-1 + x^2)H_{-1, -1}(x)}{x^2} \right. \\
 & + \frac{8(51 - 153x + 4\pi^2(-2 + 3x))H_0(x)}{3x} + \frac{32(-1 + x^2)H_{-1, 0}(x)}{x^2} - \frac{64(-3 + x)H_{0, -1}(x)}{x} \\
 & + \frac{8(-8 + 11x)H_{0, 0}(x)}{x} + \frac{8(15 + 8\pi^2(-1 + x) - 33x)(-1 + x)H_{1, 0}(x)}{3x^2} \\
 & - \frac{192(-1 + x^2)H_{-1, -1, 0}(x)}{x^2} + \frac{96(-1 + x^2)H_{-1, 0, -1}(x)}{x^2} - 96H_{0, -1, -1}(x) \\
 & - \frac{96(-4 + 3x)H_{0, 0, -1}(x)}{x} + \frac{16(-2 + 3x)H_{0, 0, 0}(x)}{x} - \frac{192(-1 + x)^2 H_{1, 0, -1}(x)}{x^2} \\
 & \left. - \frac{64(1 - 4x + x^2)H_{0, -1, -1, -1}(x)}{x^2} + \frac{64(1 - 4x + x^2)H_{0, -1, -1, 0}(x)}{x^2} + \dots \right\}
 \end{aligned}$$

6

Calculations

The SV Cross-section

TP, Rana N (Under progress) [arXiv: 2611.xxxxx](#)

Z Production Threshold

Squared matrix element **factorizes**
[19]. Schematically:

$$|\mathcal{M}_{n+1 \text{ soft}}|^2 \simeq |\mathcal{M}_n|^2 \otimes V_{\text{soft}}$$

The differential PS **factorizes**.
Schematically:

$$d\Phi_{n+1 \text{ soft}} \simeq d\Phi_n \otimes d\Phi_{\text{soft}}$$

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► Collect the soft-collinear divergences to define finite **soft-virtual** X-section
[20]:

$$\Delta_q^{SV}(z, q^2, \mu_R^2, \mu_F^2) = Z_q^2(\mu_R^2) |\hat{F}_q(q^2)|^2 \delta(1-z) \otimes \mathcal{C} e^{2\Phi_q(q^2)} \otimes \Gamma_{qq}(\mu_F^2) \otimes \Gamma_{\bar{q}q}(\mu_F^2)$$

[19]. Catani (1996, 00) [20]. Ravindran (2006)

Z Production Threshold

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► Collect the soft-collinear divergences to define finite **soft-virtual** X-section [20]:

$$\begin{aligned} \Delta_q^{SV}(z, q^2, \mu_R^2, \mu_F^2) &= \underbrace{Z_q^2(\mu_R^2) |\hat{F}_q(q^2)|^2 \delta(1-z)}_{|F_q^{\text{fin}}|^2 Z_{IR} \delta(1-z)} \otimes \underbrace{\mathcal{C} e^{2\Phi_q(q^2)}}_{\mathcal{C} e^{2\Phi_q^{\text{fin}}} \otimes \mathcal{C} e^{2\Phi_q^{\text{div}}}} \otimes \Gamma_{qq}(\mu_F^2) \otimes \Gamma_{\bar{q}q}(\mu_F^2) \\ &= |F_q^{\text{fin}}|^2 \delta(1-z) \otimes \mathcal{C} e^{2\Phi_q^{\text{fin}}} \end{aligned}$$

[1].Catani(1996,00) [2].Ravindran(2006) [3].Goyal(2025)

The SV Cross-Section

$$\begin{aligned}
 \Delta_{SV,\gamma}^{2,1} = & \mathcal{D}_5 (1536 C_F^2 e_q^2) + \mathcal{D}_4 C_F \left(-\frac{7040 C_A e_q^2}{9} + \frac{1280 e_q^2 n_f}{9} \right) + \mathcal{D}_3 \left[C_F^2 e_q^2 (-6144 - 1536 \pi^2) \right. \\
 & + C_F \left(-\frac{2560 e_q^2 n_f}{9} + C_A e_q^2 \left(\frac{17152}{9} - \frac{256 \pi^2}{3} \right) \right) \left. \right] + \mathcal{D}_2 \left[30720 C_F^2 e_q^2 \zeta_3 \right. \\
 & + C_F \left(e_q^2 n_f \left(\frac{256}{9} - \frac{1024 \pi^2}{9} \right) + 32 \sum e_q^2 + C_A e_q^2 \left(-\frac{4480}{9} + \frac{5632 \pi^2}{9} + 1344 \zeta_3 \right) \right) \left. \right] \\
 & + \mathcal{D}_1 \left\{ C_F^2 e_q^2 \left(6132 + 1488 \pi^2 - \frac{1184 \pi^4}{5} - 2880 \zeta_3 \right) + C_F \left[C_A e_q^2 \left(-\frac{35572}{9} - \frac{5824 \pi^2}{27} \right. \right. \right. \\
 & + \left. \frac{304 \pi^4}{15} - 5184 \zeta_3 \right) + \left(\sum e_q^2 \right) \left(-\frac{440}{3} + 128 \zeta_3 \right) + e_q^2 n_f \left(\frac{5608}{9} + \frac{1024 \pi^2}{27} + 1152 \zeta_3 \right) \left. \right] \left. \right\} \\
 & + \mathcal{D}_0 \left\{ C_F \left[e_q^2 n_f \left(-\frac{3584}{27} + \frac{1408 \pi^2}{81} - \frac{64 \pi^4}{27} - \frac{5120 \zeta_3}{9} \right) + \left(\sum e_q^2 \right) \left(\frac{3422}{27} - \frac{16 \pi^2}{3} \right. \right. \right. \\
 & - \left. \frac{16 \pi^4}{45} - \frac{608 \zeta_3}{9} \right) + C_A e_q^2 \left(\frac{25856}{27} - \frac{6208 \pi^2}{81} + \frac{352 \pi^4}{27} + \frac{26240 \zeta_3}{9} - \frac{736}{3} \pi^2 \zeta_3 \right) \left. \right] \\
 & + C_F^2 e_q^2 \left(-12288 \zeta_3 - 3072 \pi^2 \zeta_3 + 36864 \zeta_5 \right) \left. \right\} + \delta(1-z) \left[C_F^2 \left(-\frac{5599}{2} - \frac{65 \pi^2}{3} + \frac{103 \pi^4}{15} \right. \right. \\
 & - \left. \frac{23092 \pi^6}{2835} - 1380 \zeta_3 + 40 \pi^2 \zeta_3 + 10336 \zeta_3^2 \right) + \dots + \text{Terms from } F_{\text{fin},\gamma}^{2,1} \left. \right]
 \end{aligned}$$

[21] .Ajmath(2019)

The SV Cross-Section: Z

SV Cross-Section: Z

$$\Delta_{SV,Z}^{2,1} = (v_q^4 + 6v_q^2 a_q^2 + a_q^4) \left\{ \mathcal{D}_3 \# + \mathcal{D}_2 \# + \mathcal{D}_1 \# + \mathcal{D}_0 \# \right. \\ \left. + \delta(1-z) [\# + \text{Terms from } F_{\text{fin},Z}^{2,1}] \right\}$$

SV Cross-Section: W

► Define the following combination of vector couplings:

$$V_q = (v_q + a_q)^2 \quad V_{du} = (v_d + a_d)(v_u + a_u)$$

$$\Delta_{SV,W}^{2,1} = \mathcal{D}_3 [\# V_q + \# V_{du}] + \mathcal{D}_2 [\# V_q + \# V_{du}] + \mathcal{D}_1 [\# V_{du} + \# V_q] \\ + \mathcal{D}_0 [\# V_{du} + \# V_q] + \delta(1-z) [\# V_q + \# V_{du} + \text{Terms from } F_{\text{fin},W}^{2,1}]$$

$\#$: Coefficients determined from the lower order.

The SV Cross-Section: Preliminary Results

Input Parameters

$$m_Z = 91.153 \text{ GeV}, \quad m_W = 80.358 \text{ GeV}, \quad n_F = 4, \quad \alpha(0) = 7.297 \times 10^{-3}$$
$$G_F = 1.166378 \times 10^{-5} \text{ GeV}^{-2}, \text{ PDF: NNPDF31_nnlo_as_0118_luxqed_nf_4}$$

Order (G_μ Scheme) (Scaled with σ^{LO})	$\sigma_{\text{Full}}^{\text{NLO}}$	$\sigma_{\text{Full}}^{\text{NLO}} + \sigma_{\text{SV}}^{\text{N2LO}}$	$\sigma_{\text{Full}}^{\text{NNLO}}$	$\sigma_{\text{Full}}^{\text{NNLO}} + \sigma_{\text{SV}}^{\text{N3LO(EW)}}$
COM Energy ($\sqrt{s} = 14 \text{ TeV}$)	$1.1355^{+3.94\%}_{-2.48\%}$	$1.1297^{+0.99\%}_{-1.24\%}$	$1.1322^{+0.00\%}_{-0.97\%}$	$1.1323^{+0.00\%}_{-0.92\%}$

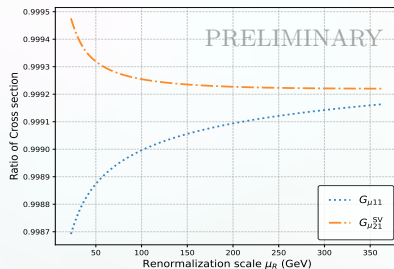
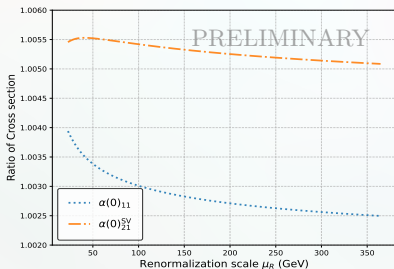
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7

Conclusion

Conclusion and Outlook

Conclusion and Outlook

- ▶ We presented the analytic calculation of non-singlet, and singlet ($n_F = 4$) Form-Factors for Drell-Yan on-shell Z production.
- ▶ We performed all the required UV renormalizations.
- ▶ We performed required IR subtractions to calculate the finite SV cross-sections.
- ▶ From order-of-magnitude estimation of the SV cross-section, N3LO QCD-EW correction of $q\bar{q} \rightarrow V$ is **indispensable** for FCCs.

Immediate Goal

- ▶ Extending the calculation for five active intermediate flavours ($n_F = 5$).
- ▶ Calculating the SV cross-section for full Drell-Yan.

Long-Term Goal

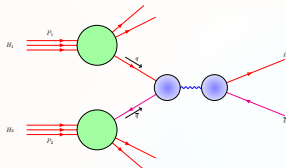
- ▶ The real-virtual and real calculations at order $\mathcal{O}(\alpha_s^2\alpha)$.

Thank you for attention

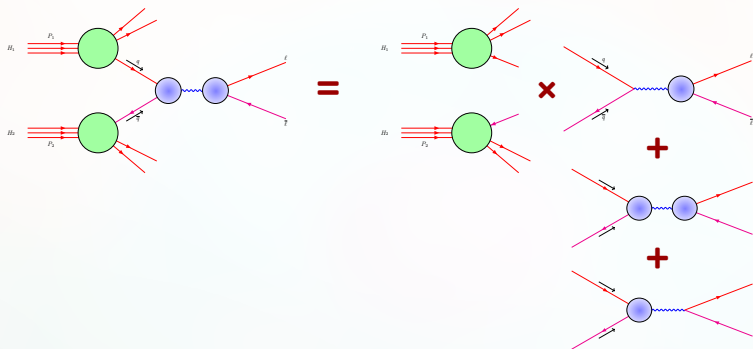
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Appendix Additional Details

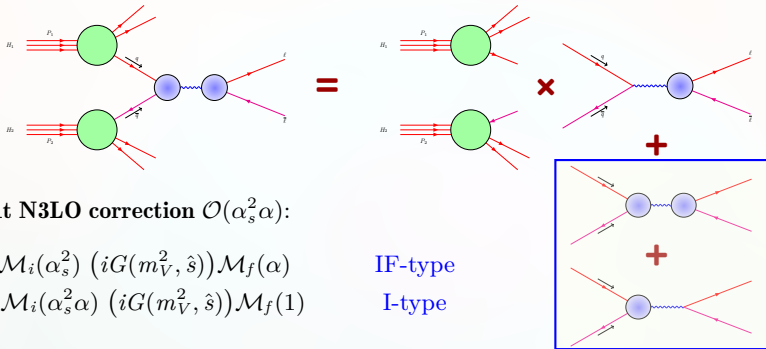
Drell-Yan at threshold



Drell-Yan at threshold



Drell-Yan at threshold

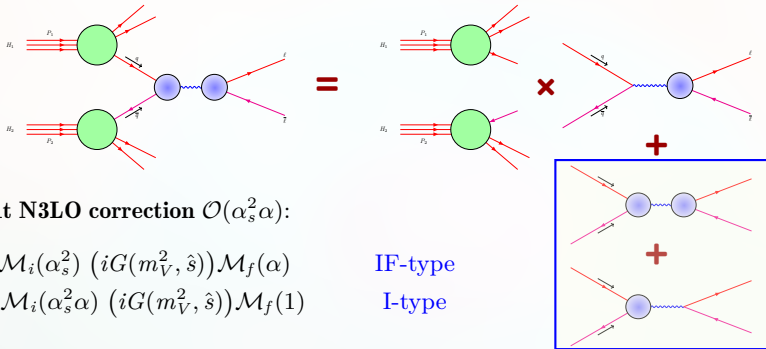


► At N3LO correction $\mathcal{O}(\alpha_s^2\alpha)$:

✓ $\mathcal{M}_i(\alpha_s^2) (iG(m_V^2, \hat{s})) \mathcal{M}_f(\alpha)$
 ✗ $\mathcal{M}_i(\alpha_s^2\alpha) (iG(m_V^2, \hat{s})) \mathcal{M}_f(1)$

IF-type
I-type

Drell-Yan at threshold



► At N3LO correction $\mathcal{O}(\alpha_s^2\alpha)$:

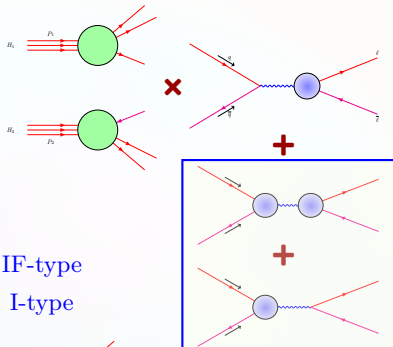
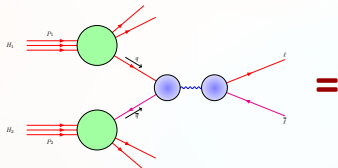
✓ $\mathcal{M}_i(\alpha_s^2) (iG(m_V^2, \hat{s})) \mathcal{M}_f(\alpha)$
 ✗ $\mathcal{M}_i(\alpha_s^2\alpha) (iG(m_V^2, \hat{s})) \mathcal{M}_f(1)$

IF-type
I-type

► Neutral current DY process with intermediate Z using Breit-Wigner approximation:

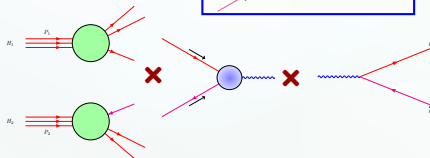
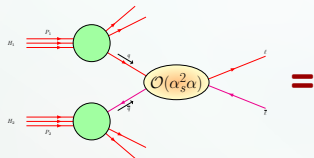
$$\hat{\sigma}(q\bar{q} \rightarrow l\bar{l}) = \frac{\pi}{3} \sqrt{2} G_F m_Z^2 F_Z^2 \frac{\Gamma(Z \rightarrow l\bar{l})}{\Gamma_Z} \delta(\hat{s} - m_Z^2)$$

Drell-Yan at threshold

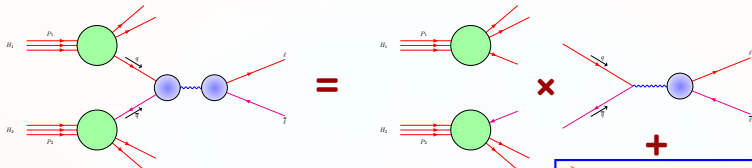


► At N3LO correction $\mathcal{O}(\alpha_s^2\alpha)$:

- ✓ $\mathcal{M}_i(\alpha_s^2) (iG(m_V^2, \hat{s})) \mathcal{M}_f(\alpha)$
- ✗ $\mathcal{M}_i(\alpha_s^2\alpha) (iG(m_V^2, \hat{s})) \mathcal{M}_f(1)$



Drell-Yan at threshold



► At N3LO correction $\mathcal{O}(\alpha_s^2\alpha)$:

✓ $\mathcal{M}_i(\alpha_s^2) (iG(m_V^2, \hat{s})) \mathcal{M}_f(\alpha)$
 ✗ $\mathcal{M}_i(\alpha_s^2\alpha) (iG(m_V^2, \hat{s})) \mathcal{M}_f(1)$

IF-type
I-type

