

Relations between electroweak parameters to order α_s^3

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Introduction

At tree-level

$$\left(1 - \frac{M_W^2}{M_Z^2}\right) = \sin^2 \theta_w$$

Loop-induced deviation from this relation parameterized by $\Delta_\rho, \Delta_r, \Delta_\kappa$.

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Loop-induced deviation from this relation parameterized by $\Delta\rho$, Δr , $\Delta\kappa$.

$$(1 + \Delta r)\pi\alpha = \sqrt{2}G_F M_W^2 \left(1 - \frac{M_W^2}{M_Z^2}\right)$$

$\Delta\kappa$ relates the weak mixing angle θ_w to the ratio of the weak boson masses, viz.

$$\sin^2 \theta_w = (1 + \Delta\kappa) \left(1 - \frac{M_W^2}{M_Z^2}\right).$$

Leading top mass correction $\Delta\rho$ universal to both Δr , $\Delta\kappa$

$$\Delta\rho = \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}$$

- Δ_ρ : QCD top-mass corrections known up to four loops
Veltman '77; Djouadi, Verzegnassi '87; Djouadi '88; Kniehl, Kühn, Stuart '88; Avdeev, Fleischer, Mikhailov, Tarasov '95; Chetyrkin, Kühn, Steinhauser '95, Schröder, Steinhauser '05; Chetyrkin, Faisst, Kühn, Maierhöfer, Sturm '06; Boughezal, Czakon '06

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- Δ_r, Δ_κ QCD top-mass subleading terms up to three loops
Halzen, Kniehl, Stong '93; Chetyrkin, Kühn, Steinhauser '95; Pati, Rana, Vicini '26

Calculation

Δ_r and Δ_κ are given by a combination of gauge boson self-energies

$$\Delta_r = \frac{c^2}{s^2} \Re \left(\frac{\Pi_{WW}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2} \right) + \lim_{q^2 \rightarrow 0} \frac{\Pi_{\gamma\gamma}(q^2)}{q^2} + \frac{\Pi_{WW}(0) - \Re \Pi_{WW}(M_W^2)}{M_W^2}$$
$$\Delta_\kappa = -\frac{c}{s} \frac{\Pi_{\gamma Z}(M_Z^2)}{M_Z^2} - \frac{c^2}{s^2} \Re \left(\frac{\Pi_{WW}(M_W^2)}{M_W^2} - \frac{\Pi_{ZZ}(M_Z^2)}{M_Z^2} \right),$$

To avoid non-perturbative contribution use instead $\Delta_{\bar{r}}$

$$\Delta_{\bar{r}} = \Delta_r - \lim_{q^2 \rightarrow 0} \frac{\Pi_{\gamma\gamma}(q^2)}{q^2} + \frac{\Re \Pi_{\gamma\gamma}(M_Z^2)}{M_Z^2}$$

Calculation cont'd

The leading m_t^2 contribution is universal for both Δ_r and Δ_κ and contained in Δ_ρ

$$\Delta_\rho = \frac{\Pi_{ZZ}(0)}{M_Z^2} - \frac{\Pi_{WW}(0)}{M_W^2}$$

Δ_r and Δ_κ receive further power (M_M^2/m_t^2) suppressed contributions

$$\Delta\bar{r} = -\frac{c^2}{s^2}\Delta\rho + \mathcal{O}\left(\frac{M_Z^2}{m_t^2}\right),$$
$$\Delta\kappa = \frac{c^2}{s^2}\Delta\rho + \mathcal{O}\left(\frac{M_Z^2}{m_t^2}\right).$$

Calculation cont'd

The gauge-boson self-energies can be expressed by vector and axial-vector contributions to quark current correlators

- $\Pi_{WW} \rightarrow$ non-diagonal correlators

$$\Pi_{WW} = \frac{g^2}{8} \left[\Pi_{tb,tb}^V(q^2) + \Pi_{tb,tb}^A(q^2) \right]$$

- $\Pi_{ZZ}, \Pi_{\gamma\gamma}, \Pi_{Z\gamma} \rightarrow$ diagonal correlators

$$\Pi_{ZZ} = \frac{g^2}{16c^2} \sum_{q,q'=t,b} \left[v_q v_{q'} \Pi_{qq,q'q'}^V(q^2) + \Pi_{qq,q'q'}^A(q^2) \right],$$

$$\Pi_{\gamma\gamma} = g^2 s^2 \sum_{q,q'=t,b} Q_q Q_{q'} \Pi_{qq,q'q'}^V(q^2),$$

$$\Pi_{\gamma Z} = -\frac{g^2 s}{4c} \sum_{q,q'=t,b} Q_q v_{q'} \Pi_{qq,q'q'}^V(q^2),$$

Calculation cont'd

Need expansions of the quark correlators aka moments

↪ most pieces already known for a long time

- three loops: up to 30 moments

Chetyrkin, Harlander, Kühn, Steinhauser '97; Chetyrkin, Kühn, Steinhauser '97; Chetyrkin, Steinhauser '01; Chetyrkin, Steinhauser '01; Boughezal, Czakon, Schutzmeier '06; Maier, Maierhöfer, Marquard '08; Maier, Marquard '12

- four loops: up to 4 moments

Kiyo, Maier, Maierhöfer, Marquard '09; Maier, Marquard '18; Chetyrkin, Kühn, Sturm '06; Boughezal, Czakon, Schutzmeier '06; Maier, Maierhöfer, Marquard '08; Sturm '08; Maier, Maierhöfer, Marquard, Smirnov '10; Maier, Marquard '15; Chetyrkin, Maier '10

Calculation cont'd

Need expansions of the quark correlators aka moments

↪ **most pieces already known for a long time**

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- four loops: up to 4 moments

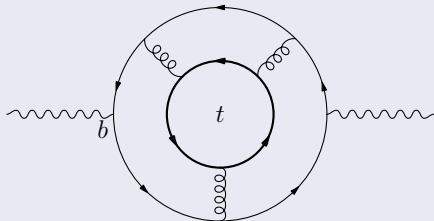
Kiyo, Maier, Maierhöfer, Marquard '09; Maier, Marquard '18; Chetyrkin, Kühn, Sturm '06; Boughezal, Czakon, Schutzmeier '06; Maier, Maierhöfer, Marquard '08; Sturm '08; Maier, Maierhöfer, Marquard, Smirnov '10; Maier, Marquard '15; Chetyrkin, Maier '10

↪ **missing pieces**

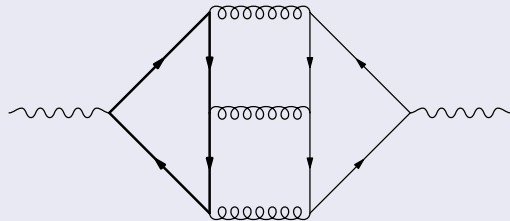
- (higher) moments of (3)4-loop top-quark corrections to non-singlet bottom correlators
- singlet contribution to vector and axial vector correlators at four loops

Calculation cont'd: missing pieces

(higher) moments of (3)4-loop top-quark corrections to non-singlet bottom correlators



singlet contribution to vector and axial vector correlators at four loops



Calculation cont'd

Calculated 6 terms in the expansion of the missing pieces up to four loops by performing an asymptotic expansion:

Standard setup:

- Diagram generation: `qgraf` [Nogueira '93]
- Family mapping: `dynast` [Maier] using `nauty`, `traces`
- Simplification: `FORM 5` [Davies, Kaneko, Marinissen, Ueda, Vermaseren '26]
- Tensor reduction: `OPITeR` [Goode, Herzog, Teale '25]
- IBP reduction: `Crusher`, `tinbox` [Maier, Marquard]
- Master integrals known
- Larin scheme to treat γ_5 where necessary

Results – Δ_r

$$\begin{aligned}
 & -\frac{8\sqrt{2}\pi^2 s^2}{3G_F c^2} \Delta \bar{r}_{\overline{\text{MS}}} = 1.000 + \frac{\alpha_s}{\pi} \left[-0.1932 \right] + \frac{\alpha_s^2}{\pi^2} \left[-3.970 \right] + \frac{\alpha_s^3}{\pi^3} \left[-1.680 \right] \\
 & + \frac{M_Z^2}{\bar{m}_t^2} \left\{ 0.3333 - 0.5926s^2 + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.8889s^2 - 0.6667 \right] \right. \\
 & + \frac{\alpha_s}{\pi} \left[0.2137 - 2.268s^2 + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.8889s^2 - 0.6667 \right] \right] \\
 & + \frac{\alpha_s^2}{\pi^2} \left[7.204s^2 - 5.649 + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[1.253s^2 + 0.08829 \right] + \log^2 \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.4722 - 0.8519s^2 \right] \right] \\
 & + \frac{\alpha_s^3}{\pi^3} \left[38.09s^2 - 14.14 + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[30.95 - 38.57s^2 \right] \right. \\
 & \left. \left. + \log^2 \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.01596 - 3.475s^2 \right] + \log^3 \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[1.088s^2 - 0.6034 \right] \right] \right\}
 \end{aligned}$$

Results – Δ_κ

$$\begin{aligned}
 \frac{8\sqrt{2}\pi^2 s^2}{3G_F c^2} \Delta_{\kappa_{\overline{\text{MS}}}} = & 1.000 + \frac{\alpha_s}{\pi} \left[-0.1932 \right] + \frac{\alpha_s^2}{\pi^2} \left[-3.970 \right] + \frac{\alpha_s^3}{\pi^3} \left[-1.680 \right] \\
 & + \frac{M_Z^2}{\bar{m}_t^2} \left\{ 0.3333 - 0.2963s^2 + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.4444s^2 - 0.6667 \right] + \frac{\alpha_s}{\pi} \left[0.2137 - 1.134s^2 \right. \right. \\
 & + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.4444s^2 - 0.6667 \right] \left. \right] + \frac{\alpha_s^2}{\pi^2} \left[3.602s^2 - 5.649 \right. \\
 & + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.6263s^2 + 0.08829 \right] + \log^2 \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.4722 - 0.4259s^2 \right] \left. \right] \\
 & + \frac{\alpha_s^3}{\pi^3} \left[19.04s^2 - 14.14 + \log \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[30.95 - 19.29s^2 \right] + \log^2 \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.01596 - 1.737s^2 \right] \right. \\
 & \left. \left. + \log^3 \left(\frac{M_Z^2}{\bar{m}_t^2} \right) \left[0.5442s^2 - 0.6034 \right] \right] \right\}
 \end{aligned}$$

Numerical impact: Δ_r on-shell scheme

$\Delta M_W [\text{MeV}]$	α_s^0	α_s^1	α_s^2	α_s^3
M_t^2	539.6253	-55.0977	-9.5853	-2.0953
M_t^0	134.0347	-5.0638	-2.3777	-0.7162
M_t^{-2}	-9.5827	-1.1332	-0.2217	-0.0724
$M_t^{\leq -4}$	-0.6911	-0.1172	-0.0347	-0.0124

Estimated uncertainty of FCC-ee

$$\Delta M_W^{\text{sys}} = 0.16 \text{ MeV}$$

$$\Delta M_W^{\text{stat}} = 0.18 \text{ MeV}$$

Numerical impact: Δ_r $\overline{\text{MS}}$ scheme

$\Delta M_W [\text{MeV}]$	α_s^0	α_s^1	α_s^2	α_s^3
$\bar{m}_t^2$	475.8288	-3.3032	-2.3425	-0.0342
$\bar{m}_t^0$	122.7257	1.7872	-0.8006	-0.2218
$\bar{m}_t^{-2}$	-10.6056	-0.3080	0.0047	0.0041
$\bar{m}_t^{\leq -4}$	-0.8721	0.0182	-0.0003	0.0004

Parametric uncertainty

$$\alpha_s \pm 0.001 \rightarrow M_W \pm 0.7 \text{ MeV}$$

$$M_t \pm 0.27 \text{ GeV} \rightarrow M_W \pm 1.9 \text{ MeV}$$

Numerical impact: Δ_κ

$\Re(\Delta_\kappa) [\text{‰}]$	α_s^0	α_s^1	α_s^2	α_s^3
M_t^2	32.46451	-3.18297	-0.55684	-0.12184
M_t^0	8.98207	-0.25739	-0.15307	-0.04734
M_t^{-2}	-0.61033	-0.06955	-0.01378	-0.00457
$M_t^{\leq -4}$	-0.04466	-0.00748	-0.00223	-0.00080

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$\Re(\Delta_\kappa) [\text{‰}]$	α_s^0	α_s^1	α_s^2	α_s^3
$\bar{m}_t^2$	28.77623	-0.19207	-0.13628	-0.00199
$\bar{m}_t^0$	8.36201	0.19283	-0.04978	-0.01509
$\bar{m}_t^{-2}$	-0.68856	-0.01563	0.00062	0.00028
$\bar{m}_t^{\leq -4}$	-0.05763	0.00150	0.00001	0.00003

Conclusions

- Calculated the relevant 4-loop QCD corrections to Δ_r and Δ_{κ}
- In general: corrections in the on-shell scheme larger than in the $\overline{\text{MS}}$ scheme
- Contribution small, but esp. the $(\frac{M_Z^2}{m_t^2})^0$ corrections not negligible
- Would be useful to have the 5-loop corrections to Δ_ρ