

Local factorization of *arbitrary* two-loop QED amplitudes

Christian Biello

ITP ETH Zürich

in collaboration with

C. Anastasiou, A. Favorito, G. Gambuti, R. Sahoo, G. Sterman

[in preparation]

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The bottleneck of NNLO automation

Subtraction schemes for real corrections are mature enough to automate NNLO QCD predictions.

See Chiara Signorile-Signorile's talk

Two-loop amplitudes are typically the bottleneck requiring dedicated analytic calculations of growing:

See Vasily Sotnikov's talk

- Algebraic complexity
- Analytic complexity

Zoo of special functions stolen from
Simone Zoia's talk



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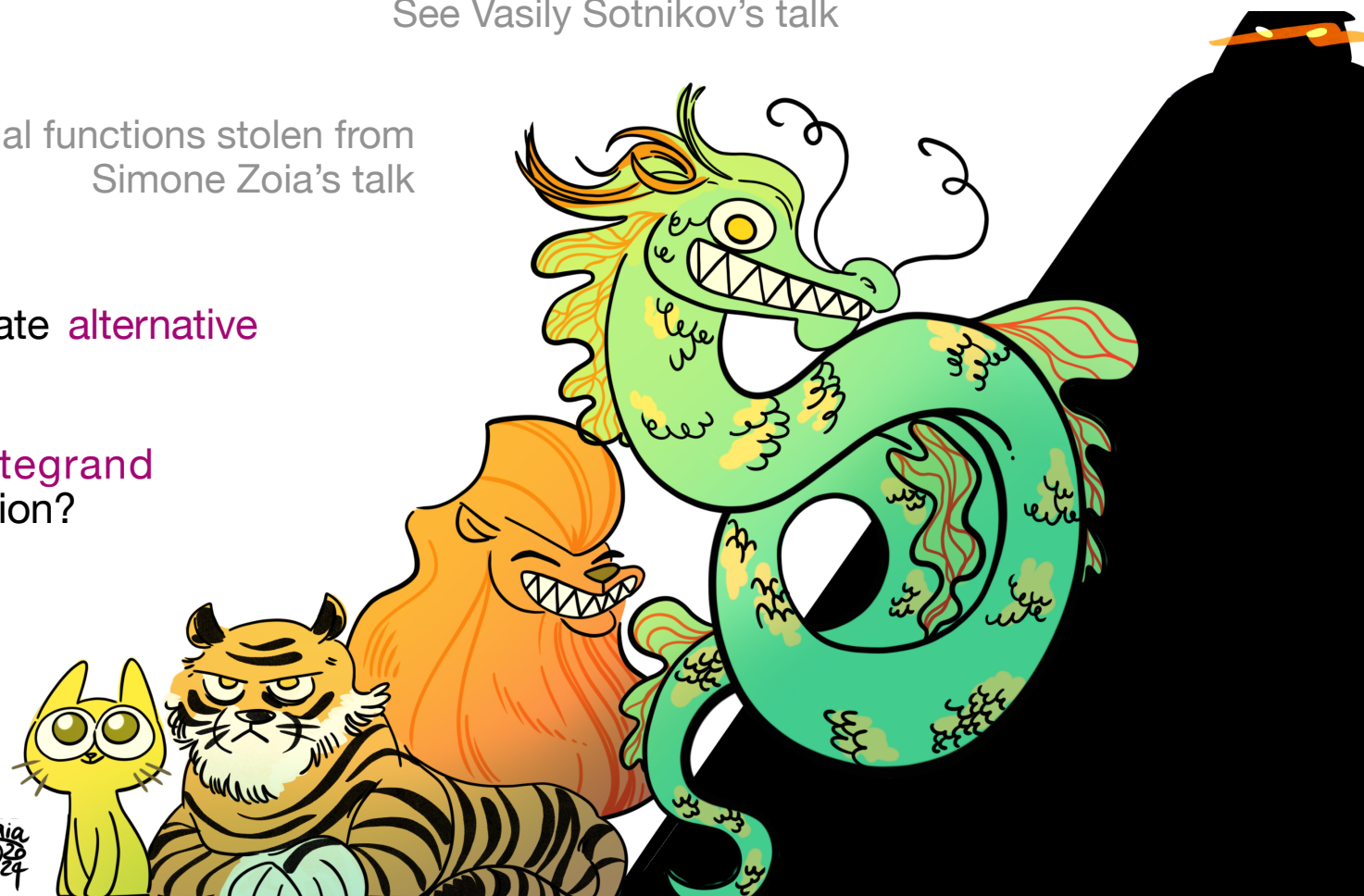
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To overcome the bottleneck, we can investigate **alternative approaches**.

Can we construct a **finite-remainder integrand representation** for direct numerical loop integration?



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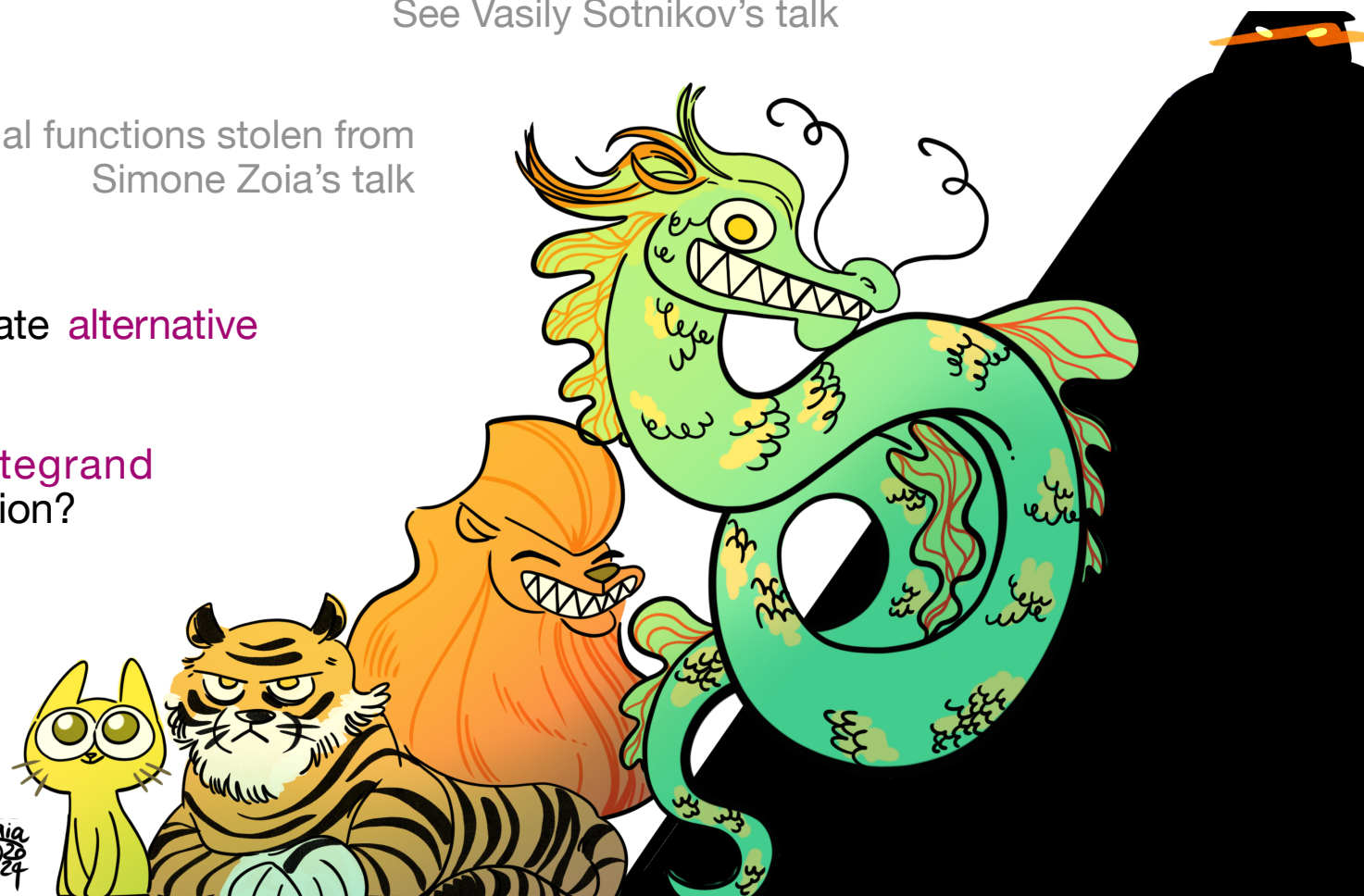
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To overcome the bottleneck, we can investigate **alternative approaches**.

Can we construct a **finite-remainder integrand representation** for direct numerical loop integration?

Here a **general two-loop recipe for Quantum Electrodynamics** in Feynman gauge, formulated in loop-momentum space.

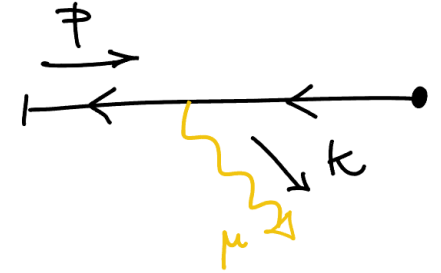


A simple Ward identity in action

Anastasiou, Sterman [2212.12162]

In the collinear limit,
the QED vertex
becomes polarized

$$\bar{u}(p) (-ie\gamma^\mu) \frac{-i(\not{p}-\not{k})}{(p-k)^2} \underset{k^\mu \sim \alpha p^\mu}{\simeq} e \left(\frac{2(1/\alpha - 1) K^\mu}{(p-k)^2} \right) \bar{u}(p)$$

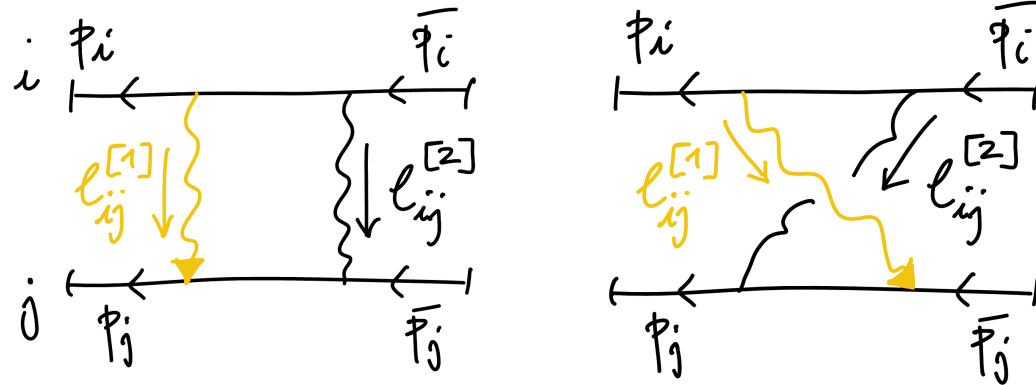


All you need is
partial fractioning

Telescoping cancellation for many legs: the flavour of a process independent construction

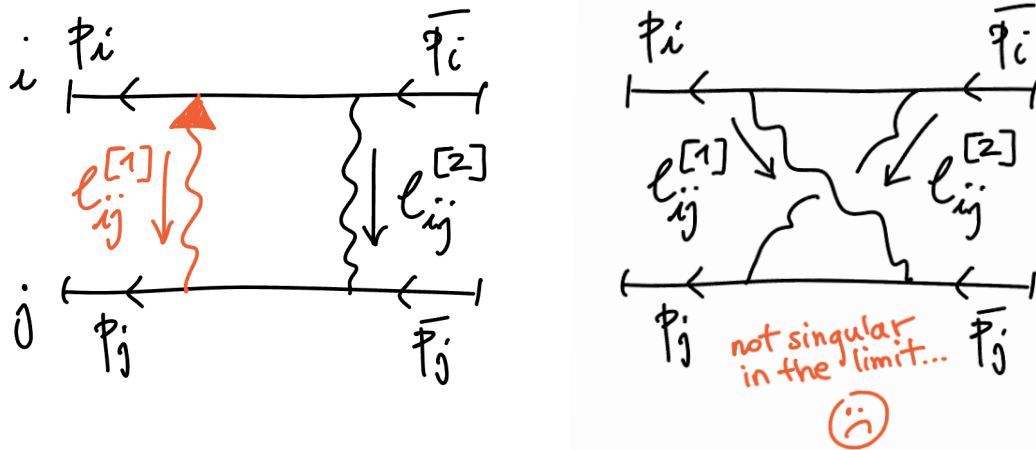
A simple Ward identity in action

Let's consider the ladder diagrams of two different fermion currents:



Local cancellation of collinear singularities
calls for a **specific loop-momenta labeling**

$$\ell_{ij}^{[1]} \parallel p_i$$

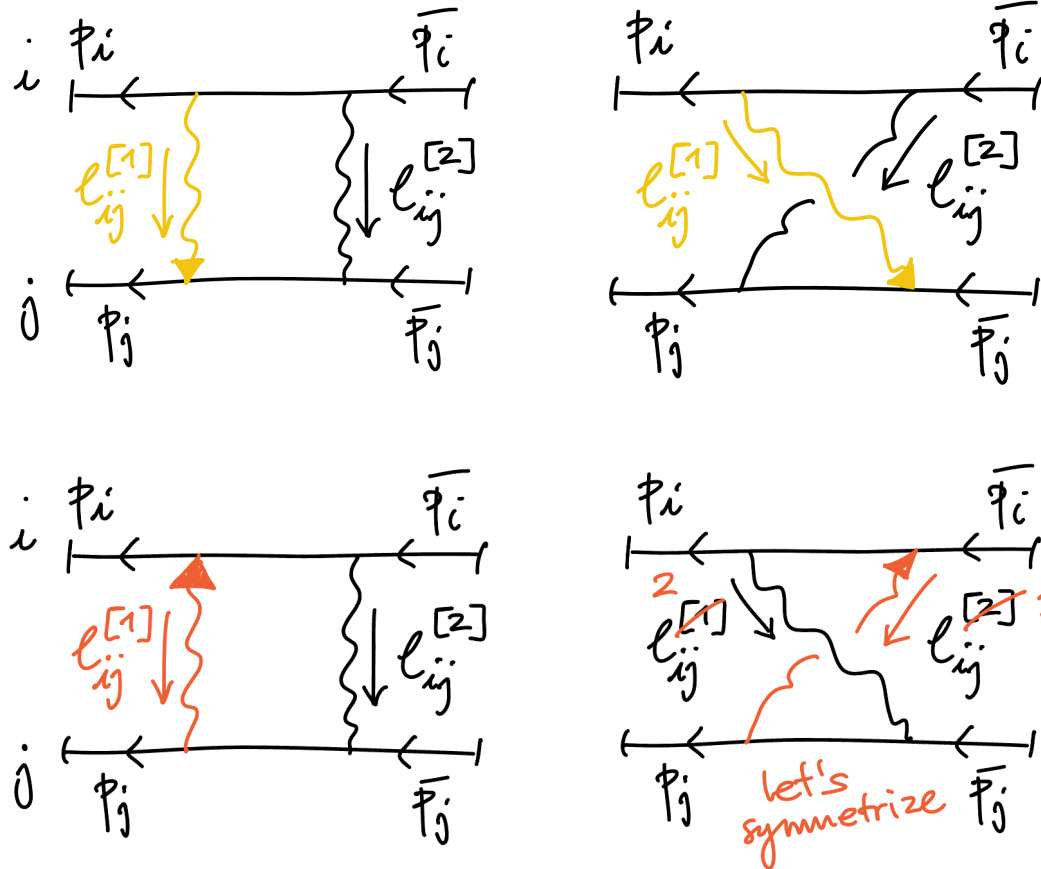


The naive routing does not work for the
collinear limits of the second line

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$$\ell_{ij}^{[1]} \parallel p_j$$

This requires **symmetrising** the ladder
amplitude

A simple Ward identity in action

Anastasiou, CB, Favorito, Gambuti,
Sahoo, Sterman [in preparation]

$$\hat{S} \left[\begin{array}{c} i \quad p_i \quad \bar{p}_i \\ \downarrow \ell_{ij}^{[1]} \quad \downarrow \ell_{ij}^{[2]} \\ j \quad p_j \quad \bar{p}_j \end{array} + \begin{array}{c} p_i \quad \bar{p}_i \\ \downarrow \ell_{ij}^{[1]} \quad \downarrow \ell_{ij}^{[2]} \\ p_j \quad \bar{p}_j \end{array} \right]$$

A simple Ward identity in action

$$\hat{S} \left[\begin{array}{c} i \quad p_i \quad \bar{p}_i \\ \leftarrow \quad \leftarrow \\ \downarrow \ell_{ij}^{[1]} \quad \downarrow \ell_{ij}^{[2]} \\ j \quad p_j \quad \bar{p}_j \\ \leftarrow \quad \leftarrow \end{array} + \begin{array}{c} p_i \quad \bar{p}_i \\ \leftarrow \quad \leftarrow \\ \downarrow \ell_{ij}^{[1]} \quad \downarrow \ell_{ij}^{[2]} \\ p_j \quad \bar{p}_j \\ \leftarrow \quad \leftarrow \end{array} \right]$$

$$\hat{S} \left[\begin{array}{c} i \quad \ell_{ik}^{[2]} \quad k \\ \swarrow \quad \swarrow \\ \downarrow \ell_{ij}^{[1]} \\ j \end{array} + \begin{array}{c} i \quad \ell_{ik}^{[1]} \quad k \\ \swarrow \quad \swarrow \\ \downarrow \ell_{ij}^{[1]} \\ j \end{array} + \dots \right]$$

→ Symmetrization operator

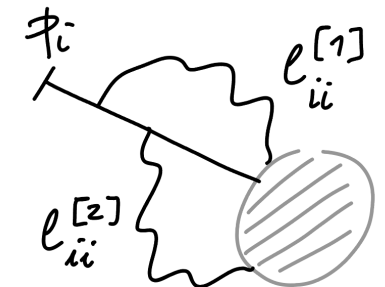
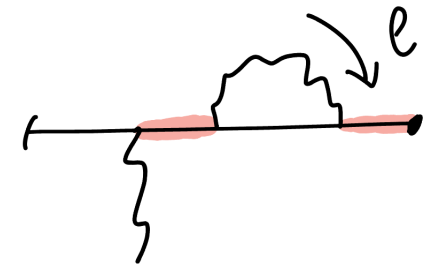
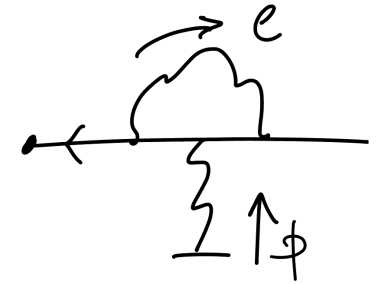
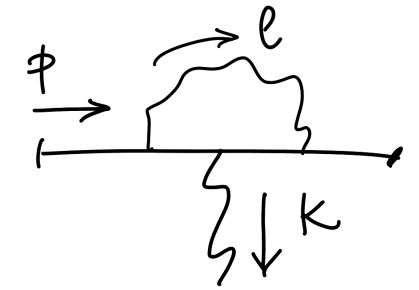
r is free of collinear singularities
 $\checkmark$ soft exponentiates at integrand level

A FLAVOUR OF ALL ORDER
 LOCAL STRUCTURES

The two-loop obstacles

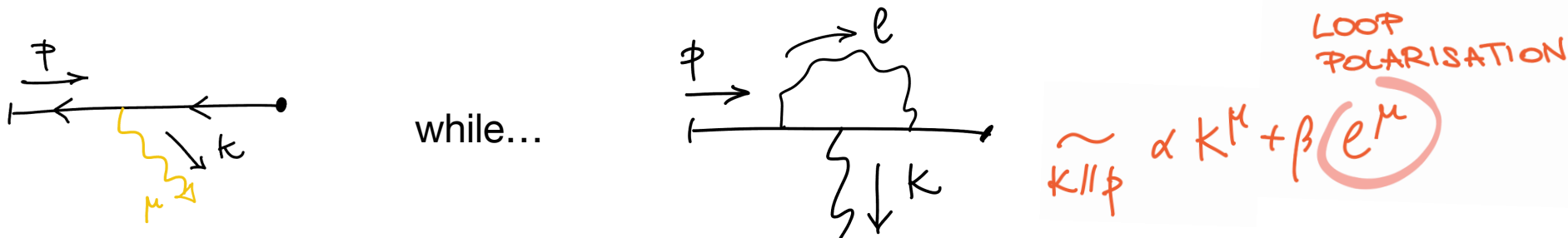
The call for local* counterterms
with vanishing integral value

*at integrand level



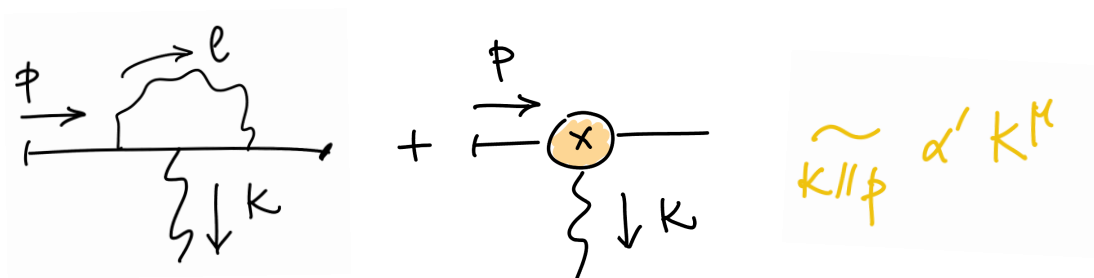
Our nightmare: loop polarisation

Loop corrections to the QED vertex break the polarization properties at integrand level



From tensor reduction, it is clear that this is an integrand artifact. We can remove it with a transient counterterm with zero integral value

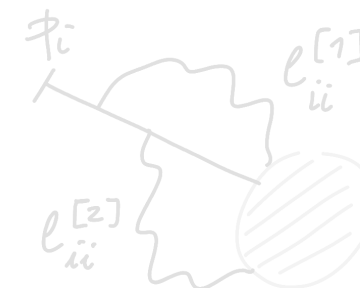
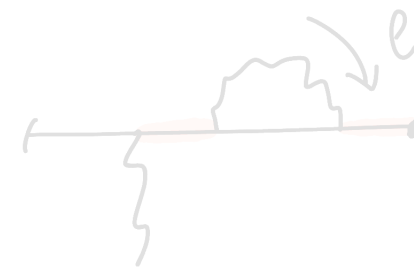
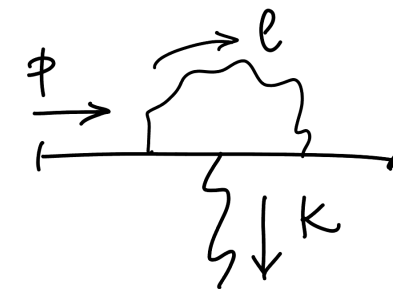
Anastasiou et al.
[2212.12162, 2601.22936, in preparation]



$$ie^2(2-d) \frac{\eta}{2p \cdot \eta} \left[\frac{(p-2l-k)^\mu}{l^2(p-l-k)^2} - \frac{(2p-2l-k)^\mu}{(p-l)^2(p-l-k)^2} \right]$$

$$\eta^2 = 0$$

Transversality will come later in our new approach

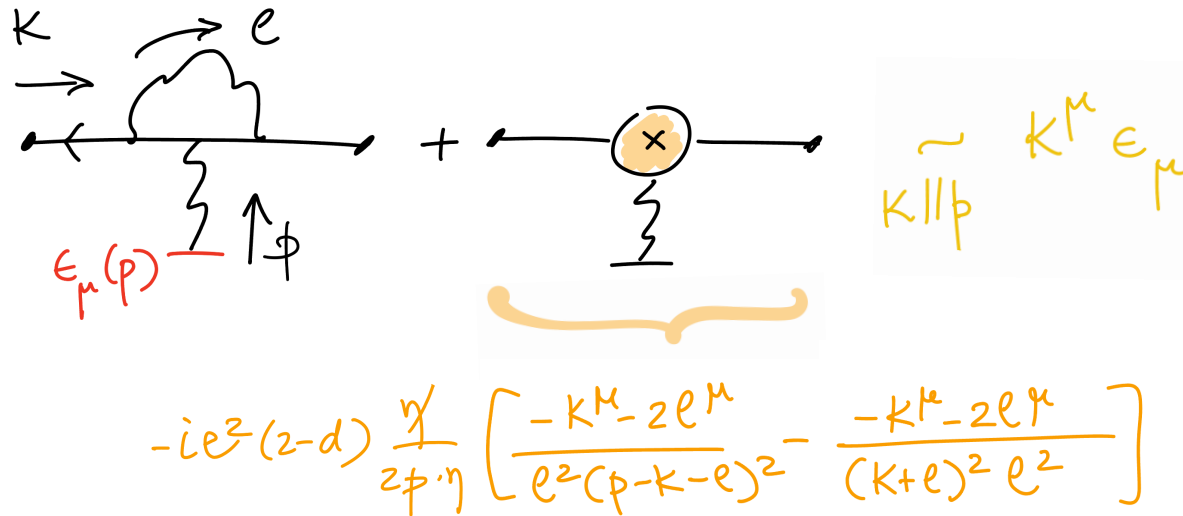


The *external* loop polarisation

At two loops, amplitudes with external on-shell photons are singular at integrand level in the unphysical collinear limit related to the photon momentum

Anastasiou, Karlen, Sahoo, Sterman, Vicini [2509.07805]

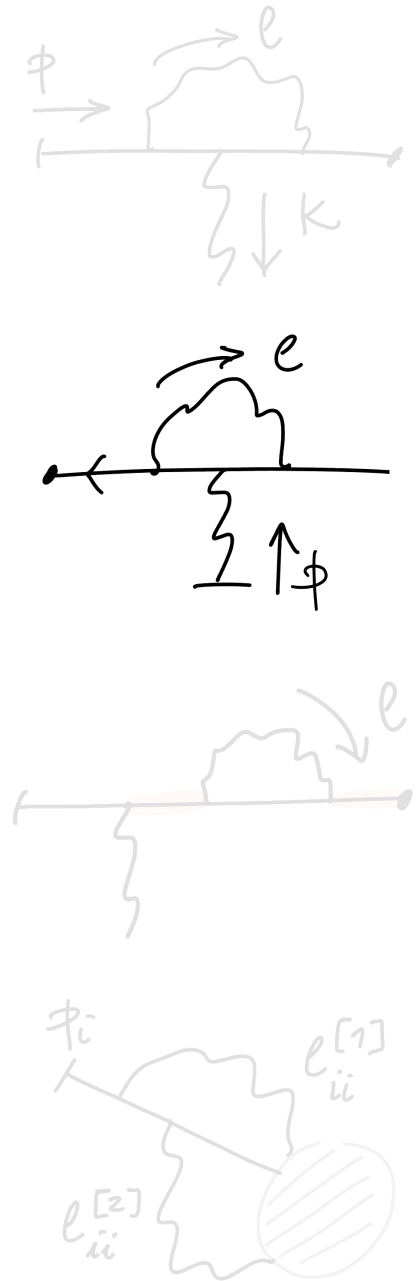
We observe that this transient singularity has the same origin as internal loop polarizations



The diagram shows two Feynman diagrams representing external loop polarization. The first diagram shows a photon line with momentum k and polarization $\epsilon_\mu(p)$ interacting with a loop of electrons (labeled e). The second diagram shows a similar setup but with a different internal structure. The diagrams are summed, and the result is shown to be proportional to $k^\mu \epsilon_\mu$ in the collinear limit $k \parallel p$.

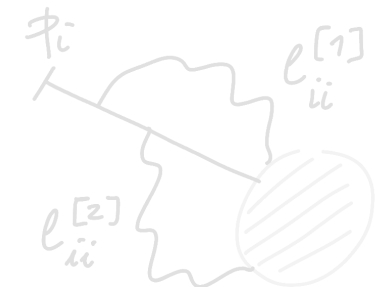
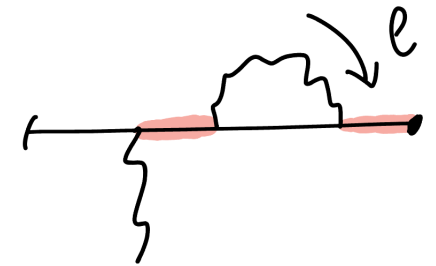
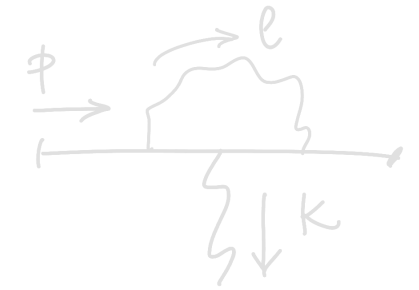
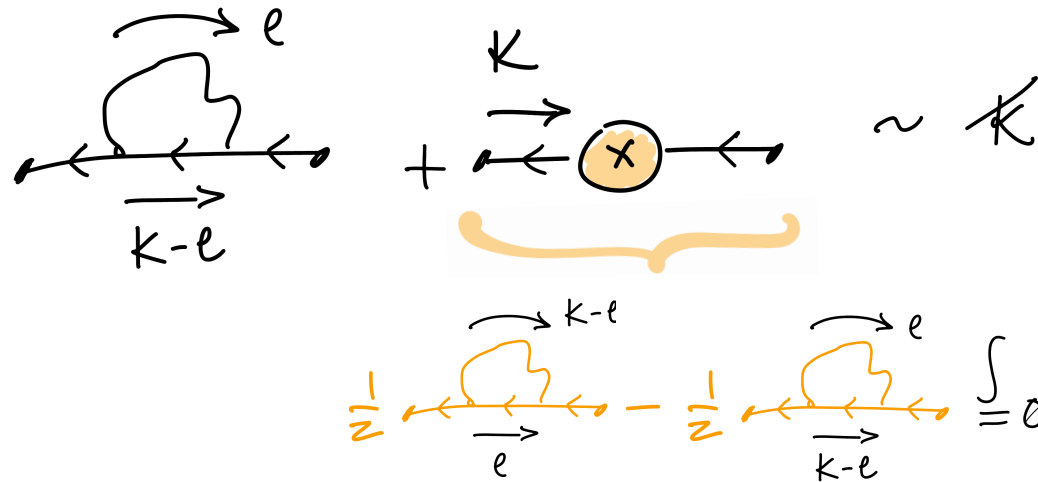
$$\sim k^\mu \epsilon_\mu \quad k \parallel p$$

$$-ie^2(2-d) \frac{\gamma}{2p \cdot \eta} \left[\frac{-k^\mu - 2e^\mu}{e^2(p-k-e)^2} - \frac{-k^\mu - 2e^\mu}{(k+e)^2 e^2} \right]$$



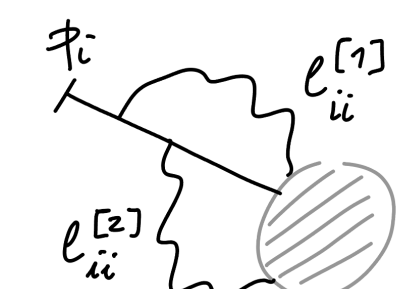
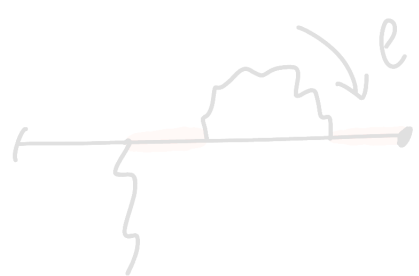
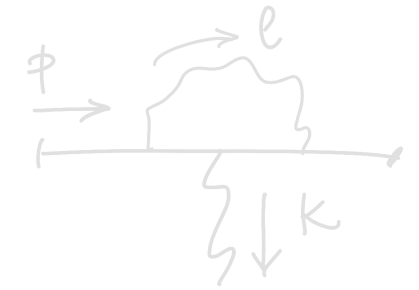
Let's tensor reduce bubbles

The **double propagator** is an integrand artifact that can be removed with a **transient counterterm** that effectively performs the tensor reduction of two-point functions



Triple-collinear little-group rotation

In the two-loop collinear limit, the amplitude seems to not factorize in spin space



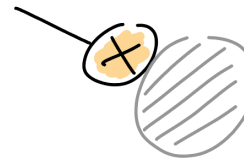
$$\text{Diagram} \rightarrow \bar{u}(\phi) \frac{\ell_{\perp}^{[1]} \ell_{\perp}^{[2]}}{D_1 D_2 D_3 D_4 D_5} (\dots)$$

$$e^{\mu} = \frac{e \cdot \eta}{p \cdot \eta} \not{p}^{\mu} + \frac{e \cdot \not{p}}{p \cdot \eta} \gamma^{\mu} + \underbrace{e_{\perp}^{\mu}}$$

$$\ell_{\perp}^{[1]} \ell_{\perp}^{[2]} = \underbrace{\ell_{\perp}^{[1]} \cdot \ell_{\perp}^{[2]}}_{\text{It contributes to the physical triple-collinear pole}} + \underbrace{\frac{1}{2} [\ell_{\perp}^{[1]}, \ell_{\perp}^{[2]}]}_{\text{Little-group rotation effect, vanishing at integral level}}$$

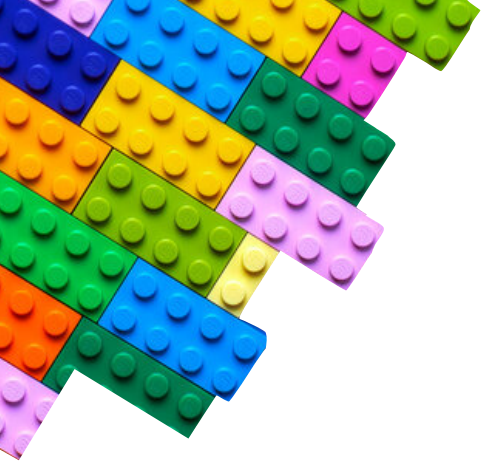
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Little-group rotation effect, vanishing at integral level

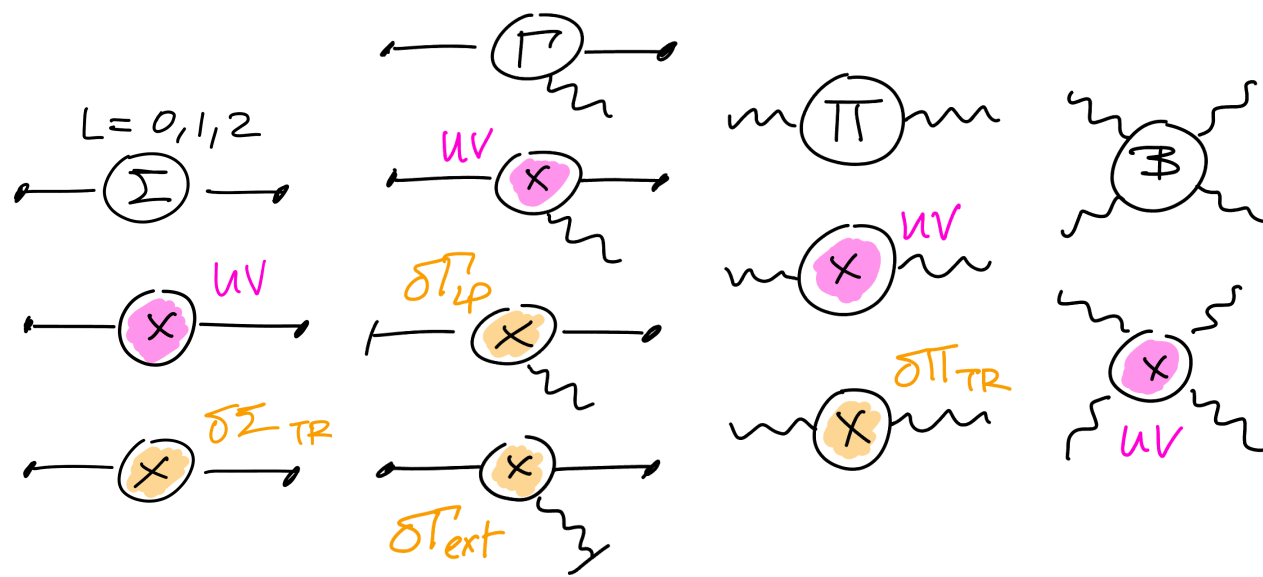


It makes the integrand factorized in scalar form

$$(\ell_{\perp}^{[2]})^{\mu} \rightarrow \frac{2 \ell_{\perp}^{[1]} \cdot \ell_{\perp}^{[2]}}{(\ell_{\perp}^{[1]})^2} (\ell_{\perp}^{[1]})^{\mu} - (\ell_{\perp}^{[2]})^{\mu}$$



*Ward-identity
preserving UV



Building QED from currents

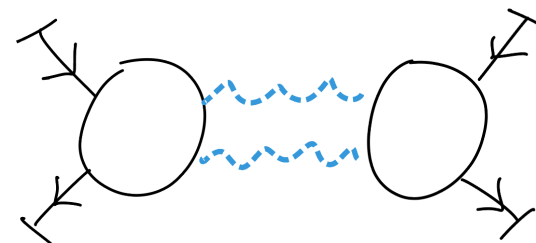


Hand-drawn Feynman diagrams illustrating various particle interactions:

- Top Left:** A circle with Σ inside, with two horizontal lines passing through it. Above it is the text $L = 0, 1, 2$.
- Second Row (Left):** A circle with an 'X' inside, with two horizontal lines passing through it. A pink label uv is to its right.
- Third Row (Left):** A circle with an 'X' inside, with two horizontal lines passing through it. An orange label $\delta \Sigma_{TR}$ is to its right.
- Top Middle:** A circle with Γ inside, with two horizontal lines passing through it. A wavy line extends from the bottom.
- Second Row (Middle):** A circle with an 'X' inside, with two horizontal lines passing through it. A pink label uv is to its left. A wavy line extends from the bottom.
- Third Row (Middle):** A circle with an 'X' inside, with two horizontal lines passing through it. An orange label $\delta \Pi'_4$ is to its left. A wavy line extends from the bottom.
- Bottom Row (Middle):** A circle with an 'X' inside, with two horizontal lines passing through it. An orange label δT_{ext} is below it. A wavy line extends from the bottom.
- Top Right:** A circle with Π inside, with two wavy lines passing through it.
- Second Row (Right):** A circle with an 'X' inside, with two wavy lines passing through it. A pink label uv is to its right.
- Third Row (Right):** A circle with an 'X' inside, with two wavy lines passing through it. An orange label $\delta \Pi_{TR}$ is to its right.
- Far Right (Top):** A circle with Ξ inside, with four wavy lines extending from it.
- Far Right (Bottom):** A circle with an 'X' inside, with four wavy lines extending from it. A pink label uv is below it.

...we can glue them with a special propagator prescription

A Feynman diagram showing a fermion loop (circle) with two external fermion lines (arrows). A photon (wavy line) is exchanged between two vertices, each represented by a circle. A blue bracket is placed over the photon exchange, and a blue circle with a slash is to the right.



$$\text{dashed line with arrow and } k \text{ above} \equiv \frac{i}{k^2} \left(g^{\mu\nu} - 2 \frac{k^\mu p^\nu + k^\nu p^\mu}{(k+p)^2 - p^2} \right)$$

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Promoting loop subtraction to integrand level in QED

Now that the amplitude is locally factorized, we can define the **finite integrand remainder**

$$\mathcal{F} = \hat{\mathcal{S}} \left[e^{-\hat{\mathcal{K}}} \mathcal{A}_r \right]$$

Diagram illustrating the factorization of the amplitude $\mathcal{F}$ into a loop symmetry operator $\hat{\mathcal{S}}$ (purple box) and a scalar kernel $\hat{\mathcal{K}}$ (orange box) acting on the remainder $\mathcal{A}_r$. The expression is $\mathcal{F} = \hat{\mathcal{S}} \left[e^{-\hat{\mathcal{K}}} \mathcal{A}_r \right]$. Handwritten annotations include "loop sym operator" pointing to $\hat{\mathcal{S}}$ and "scalar kernels" pointing to $\hat{\mathcal{K}}$.

Promoting loop subtraction to integrand level in QED

Now that the amplitude is locally factorized, we can define the **finite integrand remainder**

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loop sym operator (pointing to $\hat{\mathcal{S}}$)

scalar kernels (pointing to $\hat{\mathcal{K}}$)

Perturbatively in the electromagnetic coupling,

$$\mathcal{F}^{(1)} = \mathcal{A}_r^{(1)} - \hat{\mathcal{S}}[\hat{\mathcal{K}}_{[1]}^{(1)} \mathcal{A}^{(0)}]$$

$$\mathcal{F}^{(2)} = \mathcal{A}_r^{(2)} - \hat{\mathcal{S}}[\hat{\mathcal{K}}_{[1]}^{(1)} \mathcal{F}_{[2]}^{(1)}] - \frac{1}{2!} \hat{\mathcal{S}}[\hat{\mathcal{K}}_{[1]}^{(1)} \hat{\mathcal{K}}_{[2]}^{(1)} \mathcal{A}^{(0)}] - \hat{\mathcal{S}}[\hat{\mathcal{K}}_{[1,2]}^{(2)} \mathcal{A}^{(0)}]$$

The local kernels $\hat{\mathcal{K}}^{(1)}$ and $\hat{\mathcal{K}}^{(2)}$ have been derived, once and for all, by taking the IR limits in a nested way from soft to collinear, from single to double limits

$$\hat{\mathcal{K}} = \hat{\mathcal{S}} + \hat{\mathcal{C}}$$

soft dipoles (pointing to $\hat{\mathcal{S}}$)

collinear leftover (pointing to $\hat{\mathcal{C}}$)

Promoting loop subtraction to integrand level in QED

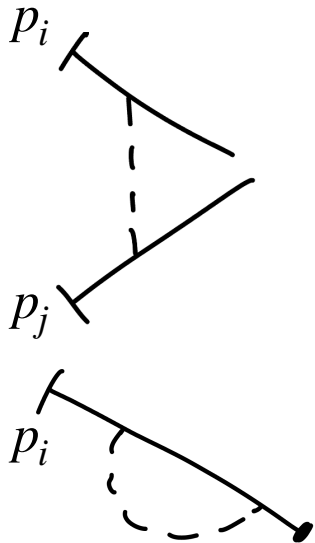
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$$\mathcal{F} = \hat{\mathcal{S}} \left[e^{-\hat{\mathcal{K}}} \mathcal{A}_r \right]$$

loop sym operator $\hat{\mathcal{S}}$
scalar kernels $\hat{\mathcal{K}}$

$$\hat{\mathcal{K}} = \hat{\mathcal{S}} + \hat{\mathcal{C}}$$

soft dipoles $\hat{\mathcal{S}}$
collinear leftover $\hat{\mathcal{C}}$



$$\hat{S}_{ij}^{(1)}(\ell) = e^2 Q_i Q_j \frac{(2\sigma_i p_i) \cdot (2\sigma_j p_j)}{\ell^2 (p_i - \ell)^2 (p_j + \ell)^2}$$

$$\hat{C}_i^{(1)}(\ell) = 2e^2 Q_i^2 \left[\frac{1}{\ell^2 (p_i - \ell)^2} - \frac{1}{(\ell^2 - M^2)^2} \right]$$

Promoting loop subtraction to integrand level in QED

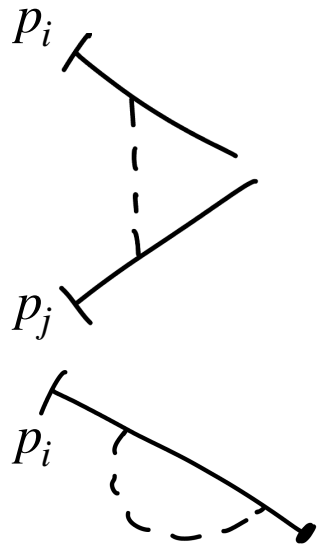
Now that the amplitude is locally factorized, we can define the **finite integrand remainder**

$$\mathcal{F} = \hat{\mathcal{S}} \left[e^{-\hat{\mathcal{K}}} \mathcal{A}_r \right]$$

loop sym operator scalar kernels

$$\hat{\mathcal{K}} = \hat{\mathcal{S}} + \hat{\mathcal{C}}$$

soft dipoles collinear leftover



$$\hat{\mathcal{S}}_{ij}^{(1)}(\ell)$$

$$\hat{\mathcal{C}}_i^{(1)}(\ell)$$

For example,

$$\hat{\mathcal{S}}_{ij}^{(1)}(\ell_{ij}^{[1]}) \left[\ell_{ij}^{[1]} \left\{ \ell_{ij}^{[2]} \right\} + \ell_{ij}^{[1]} \left\{ \ell_{ij}^{[1]} \right\} \right]$$

$$= \ell_{ij}^{[1]} \left[\ell_{ij}^{[2]} \left\{ \ell_{ij}^{[3]} \right\} \right] + \ell_{ij}^{[1]} \left[\ell_{ij}^{[2]} \right]$$

$\ell_{ij}^{[1]} = 0$ $\ell_{ij}^{[1]} = 0$

Promoting loop subtraction to integrand level in QED

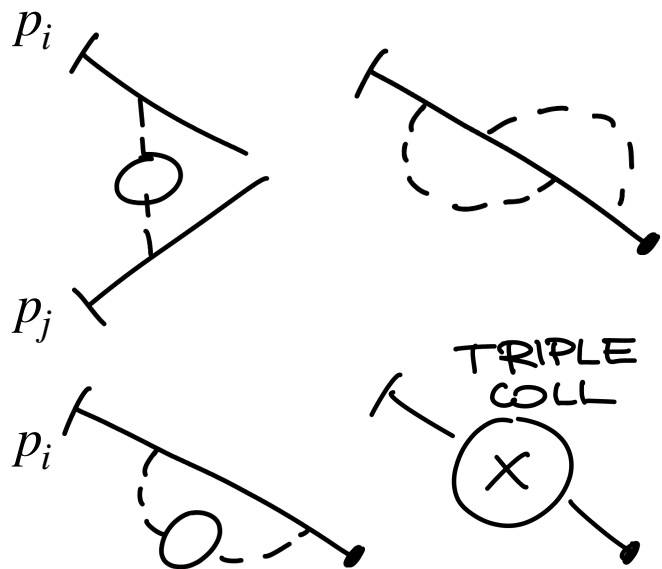
Now that the amplitude is locally factorized, we can define the **finite integrand remainder**

$$\mathcal{F} = \hat{\mathcal{S}} \left[e^{-\hat{\mathcal{K}}} \mathcal{A}_r \right]$$

loop sym operator scalar kernels

$$\hat{\mathcal{K}} = \hat{\mathcal{S}} + \hat{\mathcal{C}}$$

soft dipoles collinear leftover



$$\hat{\mathcal{C}}_a^{(2,sc)}(\ell_1, \ell_2) = -2e^4 Q_a^4 (d-2) \left[\frac{(1-x_1)p_a \cdot \ell_2}{\ell_1^2 \ell_2^2 (p_a - \ell_1)^2 (p_a - \ell_1 - \ell_2)^2 (p_a - \ell_2)^2} - \text{its UV} \right]$$

$$\hat{\mathcal{C}}_a^{(2,dc)}(\ell_1, \ell_2) = -e^4 Q_a^4 (2p_a \cdot \eta)^2 \frac{[(d-2)(x_1^2 + x_2^2) - 4(x_1 + x_2) + 4](\ell_{1,\perp} \cdot \ell_{2,\perp})}{\ell_1^2 \ell_2^2 (p_a - \ell_1)^2 (p_a - \ell_2)^2 (p_a - \ell_1 - \ell_2)^2 (\ell_1 + \eta)^2 (\ell_2 + \eta)^2}$$

$$\eta^2 = 0 \quad \ell_i^\mu = x_i p_a^\mu + y_i \eta^\mu + \ell_{i,\perp}^\mu$$

Matching the IR poles after integration

$$R_\phi^{1/2} = 1 = \underbrace{Z_\phi^{1/2}}_{\text{UV}} \underbrace{J_\phi^{1/2}}_{\text{IR}}$$

UV telescoping cancellation

$$\int \text{---} \textcircled{\times} \text{---} = \int \text{---} \textcircled{\times} \text{---}$$

only $\text{---} \textcircled{\times} \text{---}^{(M)}$ contributes
at integral level

$$\begin{aligned} \mathcal{A} &= \prod_{\phi \text{ ext}} J_\phi^{1/2} \mathcal{A}_r \\ &= \mathbb{S} \prod_{\phi \text{ ext}} \mathbb{J}_\phi \mathcal{F} \\ &\stackrel{!}{=} \mathbb{Z} \mathcal{F}_{\overline{MS}} \end{aligned}$$

our finite remainder

$$\begin{aligned} \mathbb{S} &= \exp(\hat{S}) \\ \mathbb{J}_\phi &= J_\phi^{1/2} \exp(\hat{C}) \end{aligned}$$

our kernels

Becher, Neubert [0901.0722,0903.1126]

Matching the IR poles after integration

$$R_\phi^{1/2} = 1 = \underbrace{Z_\phi^{1/2}}_{\text{UV}} \underbrace{J_\phi^{1/2}}_{\text{IR}}$$

UV telescoping cancellation

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$$\begin{aligned} \mathcal{A} &= \prod_{\phi \text{ ext}} J_\phi^{1/2} \mathcal{A}_r \\ &= \mathbb{S} \prod_{\phi \text{ ext}} \mathbb{J}_\phi \mathcal{F} \\ &\stackrel{!}{=} \mathbb{Z} \mathcal{F}_{\overline{MS}} \end{aligned}$$

our finite remainder

$$\mathbb{S} = \exp(\hat{S})$$

our kernels

$$\mathbb{J}_\phi = J_\phi^{1/2} \exp(\hat{C})$$

Becher, Neubert [0901.0722,0903.1126]

$$\ln \mathbb{Z} \stackrel{\text{poles}}{=} \int \hat{K} + \frac{1}{2} \sum_{\phi \text{ ext}} \ln \mathcal{F}^{(M)}$$

Full agreement with the abelianized pole structure after UV-scheme change

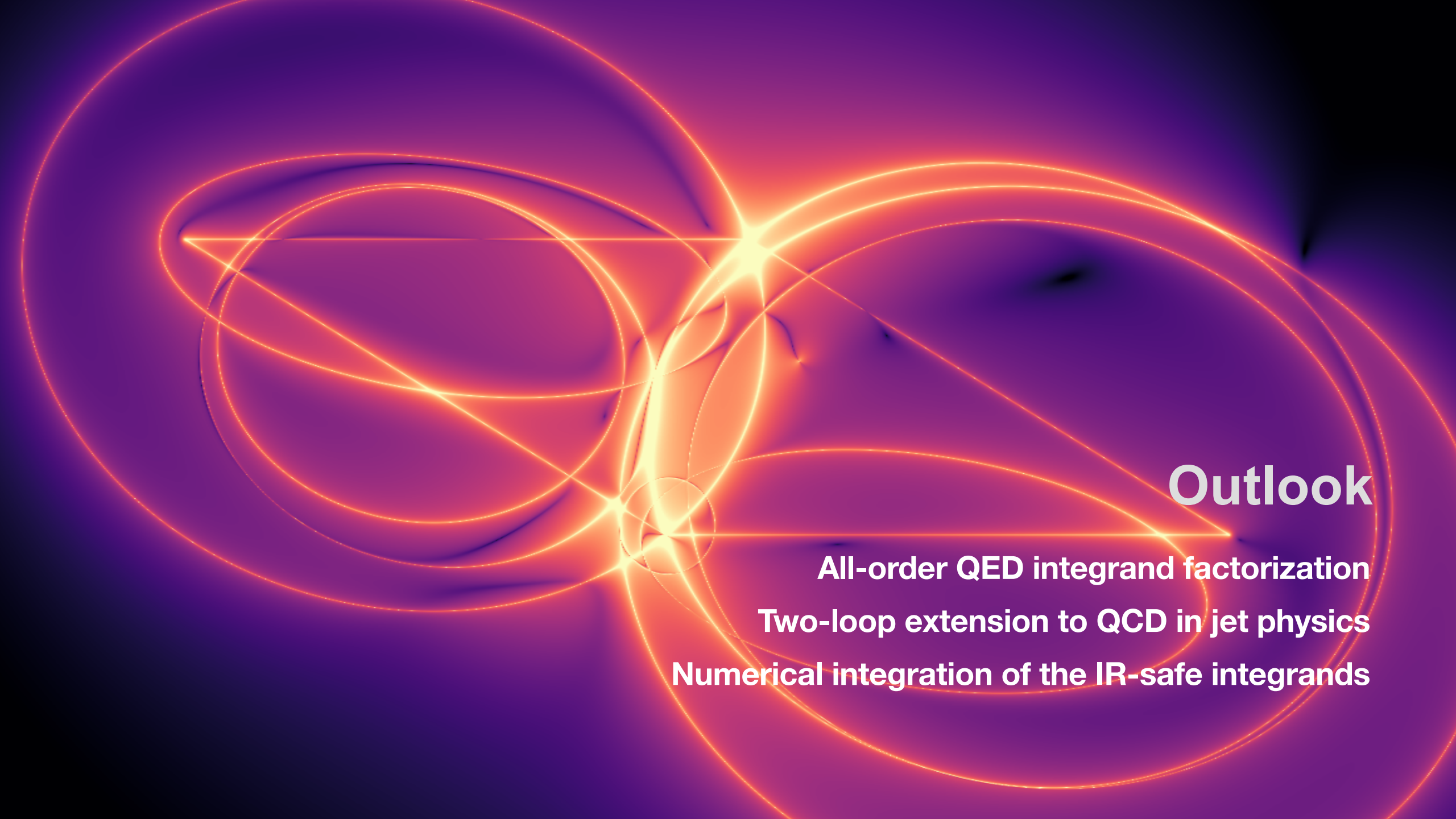
$$\ln \mathbb{Z} = \hat{\alpha}_{\overline{MS}} \left[\frac{\Gamma'_0}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon} \right] + \hat{\alpha}_{\overline{MS}}^2 \left[-\frac{3\beta_0\Gamma'_0}{16\epsilon^3} + \frac{\Gamma'_1 - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon} \right] + \mathcal{O}(\hat{\alpha}^3)$$

Conclusions

We have constructed **two-loop integrand representations** that are **free of IR and UV singularities** for **generic QED processes**

We propose a **scalar integrand IR subtraction** that matches Catani poles after integration

We built a **code** that **constructs the integrands and efficiently checks** their UV and IR safety using momentum-twistor kinematics and exact rational algebra

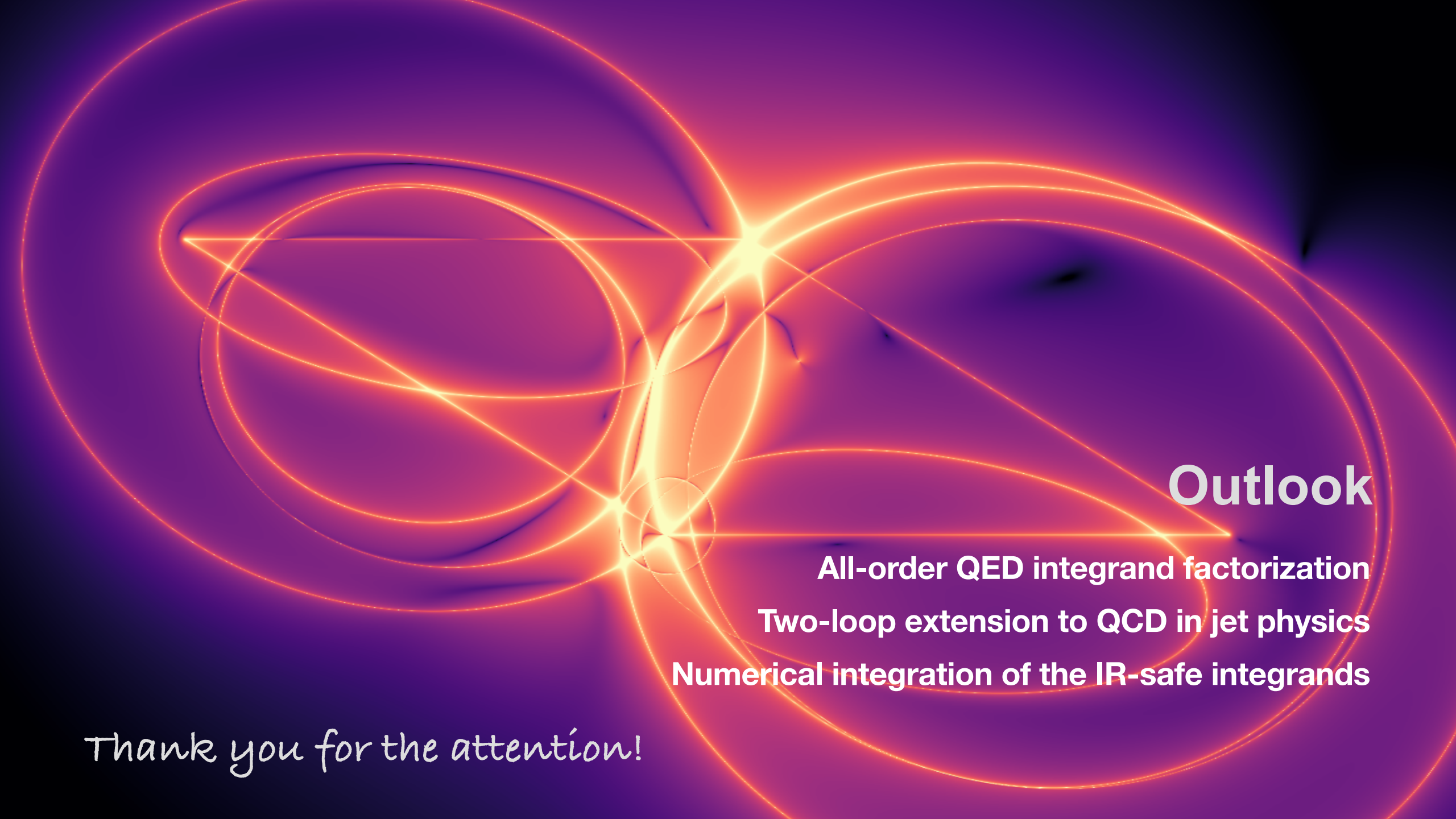


Outlook

All-order QED integrand factorization

Two-loop extension to QCD in jet physics

Numerical integration of the IR-safe integrands



Outlook

All-order QED integrand factorization

Two-loop extension to QCD in jet physics

Numerical integration of the IR-safe integrands

Thank you for the attention!

Backup slides

UV renormalisation at loop integrand level

We perform a proper UV renormalisation at local level for any building block by **removing sub-divergences** using previous order results and the **global divergence** with all large loops

$$X^{(L)} \longrightarrow \overline{R}X^{(L)} - \text{UV} \left[X^{(L)} \right]$$

$$\int_{\ell_1 \dots \ell_L} \text{UV} \left[\not{Z}_i^{(L)}(p) \right] = \not{p} \delta_{2,i}^{(L)}, \quad \int_{\ell_1 \dots \ell_L} \text{UV} \left[\Gamma_i^{(L)\mu} \right] = \gamma^\mu \delta_{1,i}^{(L)}$$

$$\int_{\ell_1 \dots \ell_L} \text{UV} \left[\Pi^{(L)\mu\nu}(p) \right] = (p^2 g^{\mu\nu} - p^\mu p^\nu) \delta_3^{(L)}$$

The latter defines the UV scheme and affect the pure coupling renormalisation

$$\text{UV} \left[\not{Z}_i^{(L)}(p) \right] = \mathcal{M} \left\{ \overline{R}\not{Z}_i^{(L)}(0) + p_\mu \left[\frac{\partial}{\partial p_\mu} \overline{R}\not{Z}_i^{(L)}(p) \right]_{p=0} \right\}$$

The idea is to capture the global limit via the dual **expansion in the small external momentum**

$$\text{UV} \left[\Gamma_i^{(L)\mu} \right] = \mathcal{M} \left[\frac{\partial}{\partial p_\mu} \overline{R}\not{Z}_i^{(L)}(p) \right]_{p=0}$$

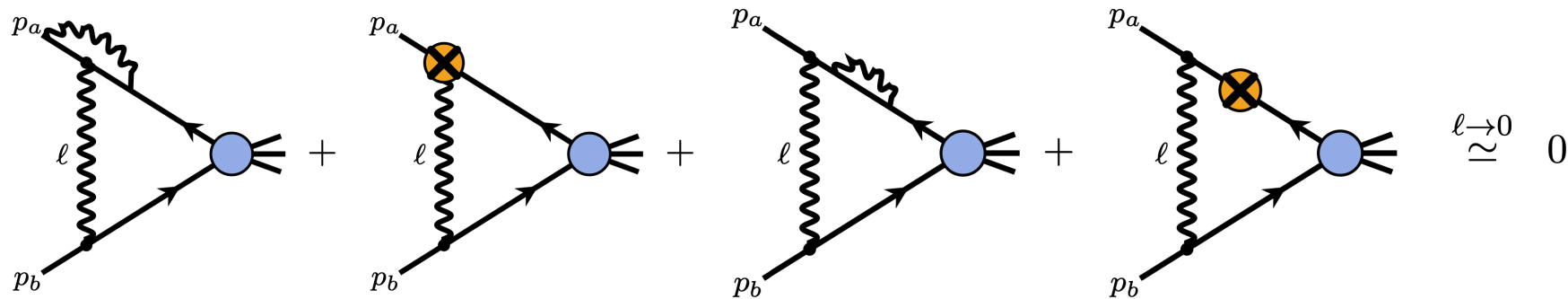
Fixed from the two-point UV construction in order to **satisfy the Ward identity**, order by order, $\delta_2^{(L)} = \delta_1^{(L)}$

Massification operator

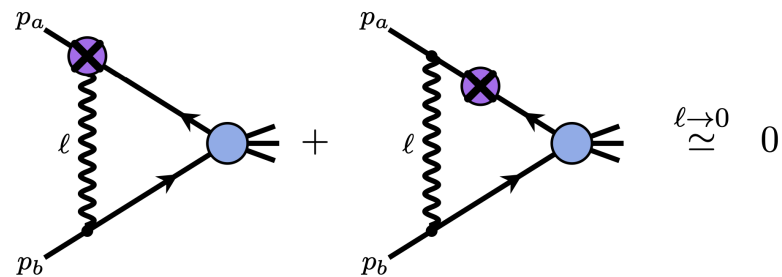
for propagators: $q^2 \rightarrow q^2 - M^2$

Integrand exactness of one-loop dipole

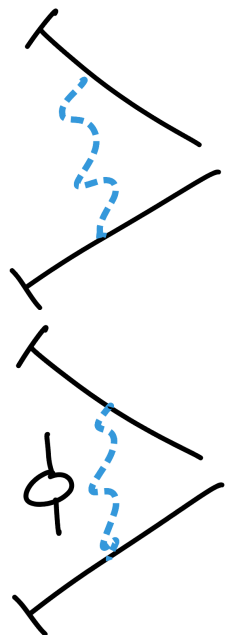
The loop polarisation counterterm, introduced for the factorisation in the collinear region, plays also a crucial role to prove the absence of radiative corrections to the one-loop dipole



The cancellation happens also for the UV counterterms

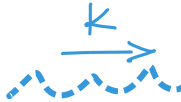


Gluing propagators do not modify the soft structure



$$\mathcal{S}_{ab}^{(1),\chi}(k) = e_a e_b \frac{(2\sigma_a p_a^\mu) \cdot (2\sigma_b p_b^\nu)}{k^2 (p_a - k)^2 (p_b + k)^2} \left(g_{\mu\nu} - 2 \frac{k^\mu \chi^\nu + \chi^\mu k^\nu}{(k + \chi)^2 - \chi^2} \right)$$

$$\mathcal{S}_{ab,f}^{(2),\chi} = -2e_f^2 \frac{d-2}{d-1} \left[\frac{1}{\ell_f^2 (\ell_f - k)^2} - \frac{1}{(\ell_f^2 - M^2)^2} \right] \mathcal{S}_{ab}^{(1),\chi}(k)$$


 $\equiv \frac{i}{k^2} \left(g^{\mu\nu} - 2 \frac{k^\mu \chi^\nu + k^\nu \chi^\mu}{(k + \chi)^2 - \chi^2} \right)$

If we define the kernel sum of two fermion currents...

$$\hat{\mathcal{K}}_{ij}^{(1),\chi}(\ell_{ij}^{[1]}) = \hat{\mathcal{S}}_{ij}^{(1),\chi}(\ell_{ij}^{[1]}) + \hat{\mathcal{S}}_{i\bar{j}}^{(1),\chi}(\ell_{ij}^{[1]}) + \hat{\mathcal{S}}_{\bar{i}j}^{(1),\chi}(\ell_{ij}^{[1]}) + \hat{\mathcal{S}}_{i\bar{j}}^{(1),\chi}(\ell_{ij}^{[1]}) ,$$

$$\hat{\mathcal{K}}_{ij}^{(2),\chi}(\ell_{ij}^{[1]}) = \sum_f \hat{\mathcal{S}}_{ij,f}^{(2),\chi}(\ell_{ij}^{[1]}, k) + \hat{\mathcal{S}}_{i\bar{j},f}^{(2),\chi}(\ell_{ij}^{[1]}, k) + \hat{\mathcal{S}}_{\bar{i}j,f}^{(2),\chi}(\ell_{ij}^{[1]}, k) + \hat{\mathcal{S}}_{i\bar{j},f}^{(2),\chi}(\ell_{ij}^{[1]}, k)$$

...you can show the independence on χ

$$\hat{\mathcal{K}}_{ij}^{(L),\chi} = \hat{\mathcal{K}}_{ij}^{(L)}$$

It corresponds to the cancellation of collinear transient singularities between dipoles in ladder diagrams.

Details of the pole matching: one-loop

$$L_{ij} = \ln \frac{\mu^2}{-(p_i + p_j)^2 - i0^+}$$

$$L_M = \ln \frac{\mu^2}{M^2}$$

The integrated kernels

$$\int_k \mathcal{S}_{ab}^{(1)}(k) = \hat{\alpha} Q_a Q_b \left[\frac{2}{\epsilon^2} + \frac{2L_{ab}}{\epsilon} + L_{ab}^2 - \frac{\pi^2}{6} \right]$$

$$\int_k \mathcal{C}_a^{(1)}(k) = \hat{\alpha} Q_a^2 \left[-\frac{2}{\epsilon} - 2L_M \right] .$$

The abelianized IR operator

$$[\ln Z]^{(1)} = \frac{1}{\epsilon_{\text{IR}}^2} \sum_{i \neq j} Q_i Q_j + \frac{1}{\epsilon_{\text{IR}}} \left[\sum_{i \neq j} Q_i Q_j L_{ij} - \frac{3}{2} \sum_i Q_i^2 + \frac{2}{3} n_f^{(2)} n_\gamma \right]$$

The external integrated constants

$$\delta_{2,i}^{(1)}(M) = \hat{\alpha} Q_i^2 \int_k \frac{2-d}{2} \frac{1}{(k^2 - M^2)^2} = -\hat{\alpha} Q_i^2 \left[\frac{1}{\epsilon_{\text{IR}}} + L_M - 1 \right] + \mathcal{O}(\epsilon_{\text{IR}})$$

$$\delta_{3,k}^{(1)}(M) = -\frac{4}{3} \hat{\alpha} n_f^{(2)} \left[\frac{1}{\epsilon_{\text{IR}}} + L_M - \frac{1}{3} \right] + \mathcal{O}(\epsilon_{\text{IR}}) ,$$

$$\hat{\alpha} [\ln Z]^{(1)} \stackrel{\text{poles}}{=} \int_\ell \hat{\mathcal{K}}^{(1)}(\ell) - \frac{1}{2} \sum_\phi \delta_\phi^{(1)}(M)$$



Details of the pole matching: two-loop

The integrated kernels

$$\int_{\ell_1 \ell_2} \mathcal{S}_{ab;f}^{(2)}(\ell_1, \ell_2) = \hat{\alpha}^2 Q_a Q_b Q_f^2 \left[\frac{2}{\varepsilon_{\text{IR}}^3} + \frac{2(2L_{ab} + 4L_M - 3)}{3\varepsilon_{\text{IR}}^2} + \frac{4L_{ab}(6L_M - 7) + 12L_M^2 - 8L_M - 24 - \pi^2}{9\varepsilon_{\text{IR}}} + \mathcal{O}(\varepsilon_{\text{IR}}^0) \right]$$

$$\int_{\ell_1 \ell_2} \mathcal{C}_{a;f}^{(2)}(\ell_1, \ell_2) = \hat{\alpha}^2 Q_a^2 Q_f^2 \left[-\frac{4}{3\varepsilon_{\text{IR}}^2} - \frac{8(3L_M - 2)}{9\varepsilon_{\text{IR}}} + \mathcal{O}(\varepsilon_{\text{IR}}^0) \right]$$

$$\int_{\ell_1 \ell_2} \mathcal{C}_a^{(2,sc)}(\ell_1, \ell_2) = \hat{\alpha}^2 Q_a^4 \left[\frac{1}{2\varepsilon_{\text{IR}}} + \mathcal{O}(\varepsilon_{\text{IR}}^0) \right]$$

$$\int_{\ell_1 \ell_2} \delta \mathcal{C}_a^{(2,dc)}(\ell_1, \ell_2) = -\hat{\alpha}^2 Q_a^4 \left[\frac{12\zeta_3 - \pi^2 + 1}{2\varepsilon_{\text{IR}}} + \mathcal{O}(\varepsilon_{\text{IR}}^0) \right]$$

The external integrated insertions

$$\delta_{2,i}^{(2)}(M) = \hat{\alpha}^2 Q_i^4 \left[\frac{1}{\varepsilon_{\text{IR}}^2} \left(\frac{1}{2} - \frac{2}{3} n_f^{(2)} \right) + \frac{1}{\varepsilon_{\text{IR}}} \left(-\frac{1}{4} + L_M + \frac{1}{9} n_f^{(2)} (17 - 12L_M) \right) + \mathcal{O}(\varepsilon_{\text{IR}}^0) \right]$$

$$\delta_{3,k}^{(2)}(M) = \hat{\alpha}^2 n_f^{(4)} \left[-\frac{2}{\varepsilon_{\text{IR}}} + \mathcal{O}(\varepsilon_{\text{IR}}^0) \right]$$

Details of the pole matching: two-loop

The abelianized IR operator

$$[\ln Z]_M^{(2)} = [\ln Z]^{(2)} - n_f^{(2)} s(\varepsilon_{\text{IR}}) [\ln Z]^{(1)} = -\frac{3\beta_0}{16\varepsilon_{\text{IR}}^3} \Gamma'_0 + \frac{\Gamma'_1 - 4\beta_0 \Gamma_0}{16\varepsilon_{\text{IR}}^2} + \frac{\Gamma_1}{4\varepsilon_{\text{IR}}} - \sum_f Q_f^2 s(\varepsilon_{\text{IR}}) \left[\frac{\Gamma'_0}{4\varepsilon_{\text{IR}}^2} + \frac{\Gamma_0}{2\varepsilon_{\text{IR}}} \right]$$

$$\Gamma'_0 = -4 \sum_i Q_i^2,$$

$$s(\varepsilon_{\text{IR}}) = \frac{4}{9}(1 - 3L_M) + \frac{1}{27}\varepsilon_{\text{IR}}(8 + 12L_M - 18L_M^2 - 3\pi^2) + \mathcal{O}(\varepsilon_{\text{IR}}^2)$$

$$\Gamma_0 = 2 \sum_{i,j \neq i} Q_i Q_j L_{ij} - 3 \sum_i Q_i^2 + \frac{4}{3} n_f^{(2)} n_\gamma$$

in order to match the UV-coupling scheme

$$\Gamma'_1 = \frac{80}{9} n_f^{(2)} \sum_i Q_i^2,$$

$$\Gamma_1 = -\frac{40}{9} n_f^{(2)} \sum_{i \neq j} Q_i Q_j L_{ij} + \sum_i Q_i^4 \left(-\frac{3}{2} + 2\pi^2 - 24\zeta_3 \right) + n_f^{(2)} \sum_i Q_i^2 \left(\frac{130}{27} + \frac{2\pi^2}{3} \right) + 4n_f^{(4)} n_\gamma.$$

$$\hat{\alpha}^2 [\ln Z]_M^{(2)} \stackrel{\text{poles}}{=} \int_{\ell_1, \ell_2} \hat{\mathcal{K}}^{(2)}(\ell_1, \ell_2) - \frac{1}{2} \sum_i \delta_{2,i}^{(2)} + \frac{1}{4} \sum_i (\delta_{2,i}^{(1)})^2 - \frac{1}{2} n_\gamma \delta_3^{(2)} + \frac{1}{4} n_\gamma (\delta_3^{(1)})^2$$

