

A perturbative framework to probe infrared sensitivity in non-Abelian gauge theories

Dennis Horstmann

based on JHEP 06 (2026) 199 [2603.22072] and work in progress
with Duarte Fontes, Kirill Melnikov, and Davide Maria Tagliabue

Institute for Theoretical Particle Physics, KIT, Karlsruhe

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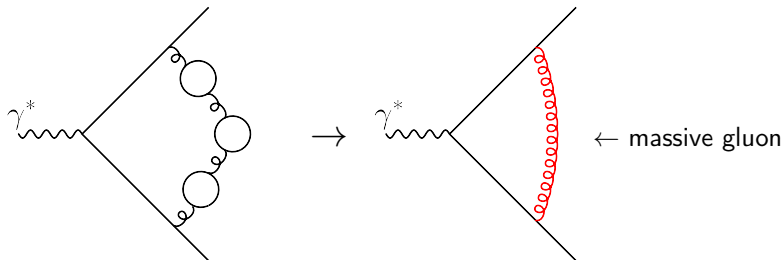


Precision collider measurements

- ▶ Last years & decades have seen enormous progress in perturbative calculations
 - ▶ Perturbative precision is so high that in some cases remaining uncertainties start to become dominated by limited understanding of non-perturbative physics
 - ▶ α_s determinations from event-shapes in e^+e^- annihilation
 - ▶ Are among most precise methods in quoted uncertainty [Benitez et al 24; PDG 25]
 - ▶ In some cases sizable deviations from PDG world average
 - ▶ Large dependence on treatment of non-perturbative power corrections [Luisoni, Monni, Salam 20]
 - ▶ Top mass determinations at the LHC (→ focus of this talk)
 - ▶ Precision is impressive [ATLAS & CMS 24]
- $$m_t = 172.52 \pm 0.14(\text{stat}) \pm 0.30(\text{sys})\text{GeV}$$
- ▶ But what is the Monte-Carlo mass m_t^{MC} ? [Hoang 23]
- ▶ Important to improve understanding of non-perturbative power corrections

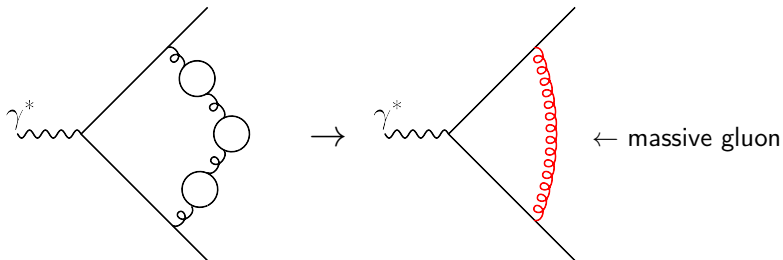
Linear non-perturbative power corrections to date

- ▶ Suppose there is no OPE
- ▶ Study appearance of power corrections via [renormalon calculus](#) [see e.g. review Beneke 98]
- ▶ Relate specific subset of an all-order perturbative calculation to a [one-loop massive gluon analysis](#)



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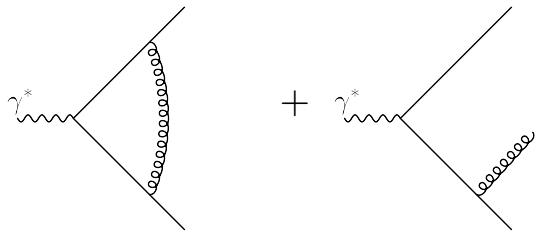
- ▶ Suppose there is no OPE
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- ▶ "Massification" procedure is straightforward [only for Abelian interactions](#)
- ▶ How can this procedure be thought of more pragmatically?

Physical picture of renormalon calculus

- Calculate perturbative corrections of some **infrared/collinear safe** observable in dim. reg

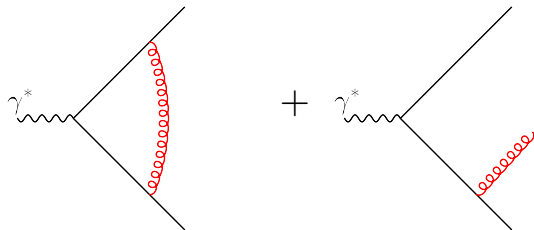


The image shows two Feynman diagrams representing perturbative corrections. The first diagram on the left shows a wavy line labeled γ^* entering a vertex from which two straight lines diverge. A gluon loop, represented by a series of small circles, is attached to the upper straight line. The second diagram on the right shows a similar setup, but the gluon loop is attached to the lower straight line. A plus sign is placed between the two diagrams.

$$\frac{\#}{\epsilon} + \dots \quad + \quad -\frac{\#}{\epsilon} + \dots = \mathcal{O}(\epsilon^0)$$

Physical picture of renormalon calculus

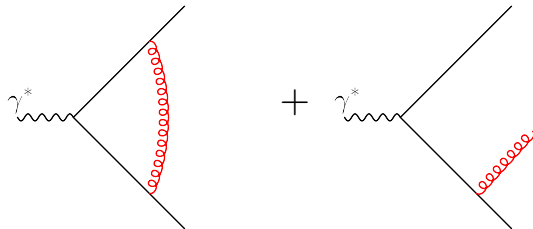
- Calculate perturbative corrections of some **infrared/collinear safe** observable



$$\# \ln(m_g) + \dots \quad + \quad -\# \ln(m_g) + \dots = \mathcal{O}(\ln^0(m_g))$$

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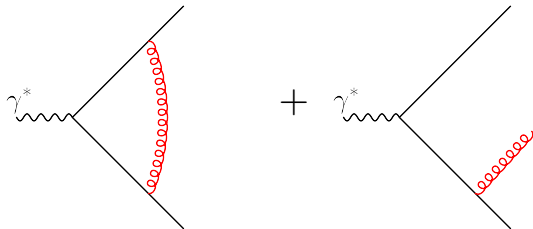
- Calculate perturbative corrections of some ~~infrared/collinear~~ safe observable



$$\#_1 \ln(m_g) + \dots + -\#_2 \ln(m_g) + \dots = \tilde{\#} \ln(m_g) + \dots$$

Physical picture of renormalon calculus

- Calculate perturbative corrections of some ~~infrared/collinear~~ **safe** observable



$$\#_1 \ln(m_g) + \dots + -\#_2 \ln(m_g) + \dots = \tilde{\#} \ln(m_g) + \dots$$

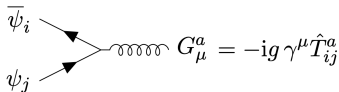
- Observables become sensitive to non-perturbative physics
- We aim to investigate terms with linear sensitivity $\frac{m_g}{Q} (\#_1 + \#_2 \ln(m_g)) \leftrightarrow \frac{\Lambda_{\text{QCD}}}{Q}$
- One difficulty is **gauge-invariant implementation of gluon mass**

Gauge invariant implementation of gluon mass

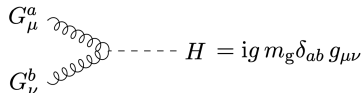
- ▶ Need small parameter as scale to render relevant regions non-scaleless
- ▶ Implement mass via **spontaneous symmetry breaking** in analogy to Standard Model

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}^a G^{a,\mu\nu} + \bar{\psi}_t(i\not{D} - m_t)\psi_t + \sum_{k=1}^{n_f} \bar{\psi}_k i\not{D}\psi_k \\ + (D_\mu\Phi)^\dagger(D^\mu\Phi) + \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{Ghost}}$$

$$\Phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\varphi_2(x) - i\varphi_1(x) \\ v + H(x) + i\varphi_3(x) \end{pmatrix}$$



$$G_\mu^a = -ig \gamma^\mu \hat{T}_{ij}^a$$



$$H = ig m_g \delta_{ab} g_{\mu\nu}$$

- ▶ Interactions are **not chiral** $\rightarrow$ Higgs does not need to give mass to top
- ▶ Fine tune parameters to obtain hierarchy of scales: $m_g \ll m_H \sim m_t$

First application: infrared sensitivity of $t\bar{t}$ production

- ▶ Study $t\bar{t}$ production from a neutral vector current. LO matrix element is

$$\mathcal{M}_0^\mu = \bar{v}(p_{\bar{t}})\gamma^\mu u(p_t)$$

- ▶ Consider $m_g \ll \sqrt{(p_t + p_{\bar{t}})^2} = \sqrt{s} \sim m_t \sim m_H$
- ▶ Up to kinematic constraints we do not integrate over final state momenta of top quarks
- ▶ Focus on **two-loop $\mathcal{O}(m_g)$ terms**
- ▶ Overview of calculation
 - ▶ Calculate virtual and real emission contributions
 - ▶ Real and virtual integrals must be related to one another
 - ▶ Naive calculation does not lead to cancellation
 - ▶ Realize that σ is a function of the pole mass which has linear infrared sensitivity
 - ▶ Change to short-distance mass scheme

Infrared sensitivity of $t\bar{t}$ production at NLO

- ▶ tree-level cross section

$$\frac{d\sigma_{t\bar{t}}^{(0)}}{d\Omega_t} = \sigma_0 \beta^{1-2\epsilon} (3 - 2\epsilon - \beta^2), \quad \beta = \sqrt{1 - 4m_t^2/s}$$

- ▶ one virtual and one real integral appear at NLO, relate them via contour integration

$$I_V \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m_g^2 + i0} \frac{1}{2k \cdot p + i0}, \quad I_R = \int \frac{d^{d-1} \vec{k}}{(2\pi)^{d-1} 2\omega_k} \frac{1}{|2k \cdot p|} \Big|_{k^0=\omega_k}, \quad I_V = iI_R$$

- ▶ Relate real emission to Born phase space

[Makarov et al. 23]

$$q_t = p_t - \alpha k + A(k)p_t + B(k)p_{\bar{t}}$$

$$q_{\bar{t}} = p_{\bar{t}} - \xi k - A(k)p_t - B(k)p_{\bar{t}}$$

$$d\text{Lips}(P; q_t, q_{\bar{t}}, k) = d\text{Lips}_{\text{LO}}(P; p_t, p_{\bar{t}}) \frac{d^d k}{(2\pi)^d} 2\pi \delta(k^2 - m_g^2) \left[1 + \mathcal{O}(k) \right]$$

- ▶ Result for tree-level single real emission

$$\mathcal{T}_{m_g} \left[\frac{d\sigma_{t\bar{t},R}^{(1)}}{d\Omega_t} \right] = 4C_F \beta^{1-2\epsilon} (3 - 2\epsilon) (-3 + 2\epsilon + \beta^2) I_R$$

Infrared sensitivity of $t\bar{t}$ production at NLO

- Result for one-loop virtual (given purely by soft region $k \sim m_g$)

$$\mathcal{T}_{m_g} \left[\frac{d\sigma_{t\bar{t},V}^{(1)}}{d\Omega_t} \right] = 4C_F \beta^{-1-2\epsilon} (3-2\epsilon) (1-2\epsilon + 2\beta^2(2-\epsilon) - \beta^4) I_R.$$

- Wave function renormalization of the tree-level

$$\mathcal{T}_{m_g} \left[\frac{d\sigma_{t\bar{t},\text{Ren}}^{(1)}}{d\Omega_t} \right] = \mathcal{T}_{m_g} \left[2Z_t^{(1)} \frac{d\sigma_{t\bar{t}}^{(0)}}{d\Omega_t} \right] = 4C_F \beta^{1-2\epsilon} (3-2\epsilon) (-3+2\epsilon+\beta^2) I_R$$

- Sum all contributions:

$$\mathcal{T}_\lambda \left[\frac{d\sigma_{t\bar{t},V}^{(1)}}{d\Omega_t} + \frac{d\sigma_{t\bar{t},R}^{(1)}}{d\Omega_t} + \frac{d\sigma_{t\bar{t},\text{Ren}}^{(1)}}{d\Omega_t} \right] = 4C_F \beta^{-1-2\epsilon} (3-2\epsilon)(1-\beta^2) (1-2\epsilon-\beta^2) I_R$$

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Infrared sensitivity of $t\bar{t}$ production at NLO

- Replace pole mass with some short-distance mass $m_t = \tilde{m}_t + \delta m_t$

$$\mathcal{T}_{m_g} \left[\frac{d\sigma_{t\bar{t},\text{mass}}^{(1)}}{d\Omega_t} \right] = -4C_F \tilde{\beta}^{-1-2\epsilon} (3-2\epsilon)(1-\tilde{\beta}^2) (1-2\epsilon-\tilde{\beta}^2) I_R,$$

- Again summing all contributions

$$\mathcal{T}_{m_g} \left[\frac{d\sigma_{t\bar{t},V}^{(1)}}{d\Omega_t} + \frac{d\sigma_{t\bar{t},R}^{(1)}}{d\Omega_t} + \frac{d\sigma_{t\bar{t},\text{Ren}}^{(1)}}{d\Omega_t} + \frac{d\sigma_{t\bar{t},\text{mass}}^{(1)}}{d\Omega_t} \right] = 0$$

- Terms linear in m_g vanish!

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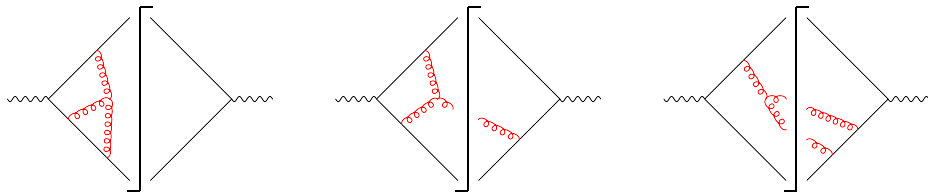
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- Terms linear in m_g vanish!
- No expansion in dimensional regulator ϵ
- At no point we need to evaluate any integral explicitly
- NLO is essentially Abelian $\rightarrow$ need to go to NNLO to study genuinely non-Abelian effects

Genuine two-loop contributions



- ▶ At NNLO purely soft and mixed soft-hard regions contribute at $\mathcal{O}(m_g)$ (soft $k \sim m_g$; hard $k \sim m_t$)
- ▶ Mixed regions factorize while soft-soft regions are genuinely two-loop
- ▶ VV, RV and RR each generate $\mathcal{O}(10)$ genuine two-loop master integrals
- ▶ Some of the most complicated master integrals are

$$I_{\text{VV}}^{(1)} = \langle G(k_1)G(k_2)G(k_1 - k_2)F_{p_t}(k_1)F_{p_{\bar{t}}}(-k_1)F_{p_{\bar{t}}}(-k_2) \rangle$$

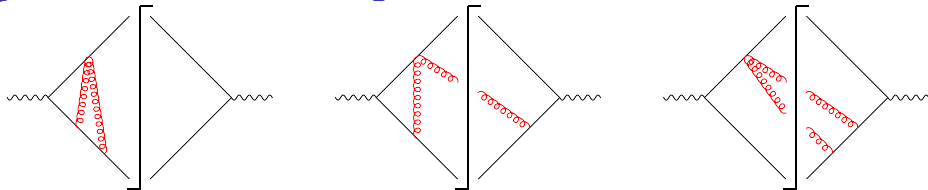
$$I_{\text{RV}}^{(1)} = \langle G(k_2)G(k_1 + k_2)F_{p_t}(k_1)F_{p_{\bar{t}}}(k_2)F_{p_{\bar{t}}}(-k_2) \rangle_1$$

$$I_{\text{RR}}^{(1)} = \langle G(k_1 + k_2)F_{p_{\bar{t}}}(k_1)F_{p_t}(k_2)F_{p_{\bar{t}}}(k_1 + k_2) \rangle_{12}$$

where $G(k) = \frac{1}{k^2 - m_g^2 + i0}$,

$$F_p(k) = \frac{1}{2k \cdot p + i0}.$$

Relating VV to RV and RR integrals



- ▶ Again we need to map VV to RV and RR integrals
- ▶ Poles from eikonal propagators are in both half planes
- ▶ Second energy contour integral inherits $i0$ from first contour integral

$$G(k_1 - k_2^c) \Big|_{k_{2,0}^c = \omega_2 - i0}^{(k_2^c)^2 = m_g^2 - i0} = \frac{1}{(k_{12}^-)^2 - m_g^2 + i0 \operatorname{sign}(k_1^0)} \Big|_{k_2^2 = m_g^2} = \left[G(k_{12}^-) + 2\pi i \theta(-k_1^0) \delta[(k_{12}^-)^2 - m_g^2] \right]_{k_2^2 = m_g^2}$$

- ▶ An example of a relatively simple relation

$$\operatorname{Re} I_{\text{VV}}^{(4)} = \operatorname{Re} I_{\text{RV}}^{(8)} - I_{\text{RR}}^{(7)}.$$

- ▶ Eventually we **do not need to evaluate any integral explicitly**

Renormalization and mass shift at NNLO

- ▶ Reminder: At NLO the tree-level contributions via $\mathcal{O}(m_g)$ wave function renormalization and $\mathcal{O}(m_g)$ mass shift of the top quark
- ▶ At NNLO both the renormalization and mass shift of the tree-level have two loop contributions and they mix among each other
- ▶ Coupling renormalization and concrete short-distance mass scheme definition start to enter

$$g_{(0)} = \left(1 + \frac{\alpha_s}{2\pi} Z_g^{(1)} + \mathcal{O}(\alpha_s^2)\right) \mu^\epsilon g, \quad m_{t(0)} = \left(1 + \frac{\alpha_s}{2\pi} Z_{\tilde{m}_t}^{(1)} + \mathcal{O}(\alpha_s^2)\right) \tilde{m}_t$$

- ▶ The one-loop virtual amplitude is subject to renormalization and mass shift
- ▶ The tree-level single real emission is subject to renormalization, real-to-Born transformation and mass shift

Linear independent classes of contributions and their cancellations

- ▶ Class A: genuine two-loop integrals that depend on $(p_t \cdot p_{\bar{t}})$
 - ▶ VV, RV and RR amplitudes
- ▶ Class B: genuine two-loop integrals that do not depend on $(p_t \cdot p_{\bar{t}})$
 - ▶ VV, RV and RR amplitudes and renormalization and mass shift of the tree-level
- ▶ Class C: one-loop squared soft integrals
 - ▶ VV, RV and RR amplitudes and renormalization and mass shift of the tree-level and one-loop soft amplitudes
- ▶ Class D: one-loop squared mixed hard-soft integrals
 - ▶ VV and RV amplitudes and renormalization and mass shift of the tree-level and one-loop soft and hard amplitudes
- ▶ Class E: Coefficients of $Z_g^{(1)}$ and $Z_{\tilde{m}_t}^{(1)}$
 - ▶ renormalization and mass shift of the tree-level and one-loop soft and hard amplitudes

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Conclusions and outlook

- ▶ Set up with framework to test the linear infrared sensitivity beyond the limitations of renormalon calculus
- ▶ Implementation of small gluon mass through spontaneous symmetry breaking
- ▶ Enables us to test genuinely non-Abelian contributions
- ▶ First application to $t\bar{t}$ production
- ▶ Extended existing NLO calculation to NNLO with non-Abelian contributions
- ▶ Have shown explicit cancellation of $\mathcal{O}(m_g)$ terms
- ▶ Future tasks:
 - ▶ Study concrete observables
 - ▶ Study other observable like thrust at e^+e^- colliders (relevant for α_s determinations!)
 - ▶ Make this statement explicit to all perturbative orders by means of OPE/EFT analysis

Backup

Real-to-Born transformation

$$q_t = p_t - \alpha k + A(k)p_t + B(k)p_{\bar{t}} ,$$

$$q_{\bar{t}} = p_{\bar{t}} - \xi k - A(k)p_t - B(k)p_{\bar{t}} ,$$

$$1 = \alpha + \xi ,$$

$$A = -\frac{\alpha m_t^2 (p_t k) + \xi (p_t \cdot p_{\bar{t}}) (p_{\bar{t}} k)}{(p_t \cdot p_{\bar{t}})^2 - m_t^4} ,$$

$$B = -A(\alpha \leftrightarrow \xi, p_t \leftrightarrow p_{\bar{t}}) .$$

$$\begin{aligned} \text{dLips}(P; q_t, q_{\bar{t}}, k) &= \text{dLips}_{\text{LO}}(P; p_t, p_{\bar{t}}) \frac{\text{d}^d k}{(2\pi)^d} 2\pi \delta(k^2 - \lambda^2) \left[1 + \frac{1}{(p_t p_{\bar{t}})^2 - m_t^4} \right. \\ &\quad \left\{ m_t^2 [2\alpha(\epsilon - 1)(p_t k) + 2\xi(\epsilon - 1)(p_{\bar{t}} k)] + [(-\alpha + \xi + 2\alpha\epsilon)(p_t k) + \right. \\ &\quad \left. \left. (\alpha - \xi + 2\xi\epsilon)(p_{\bar{t}} k)](p_t p_{\bar{t}}) \right\} \right] + \mathcal{O}(k^2) . \end{aligned}$$