

A First Look at Bootstrapping QCD Amplitudes

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universe+ is a cooperation of

State of the art Five-particle scattering at NNLO

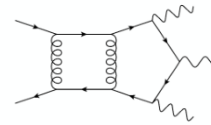
Dramatic progress for massless 5-particle scattering

Analytic results for all Feynman integrals

[Gehrmann, Henn, Lo Presti 2015;
Chicherin, Gehrmann, Henn, Lo Presti, Mitev, Wasser 2018;
Abreu, Page, Zeng 2018; Chicherin, Henn, Mitev 2018;
Abreu, Dixon, Herrmann, Page, Zeng 2018;
Chicherin, Gehrmann, Henn, Wasser, Zhang, **SZ** 2018]

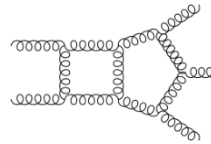
Analytic results for scattering amplitudes

3γ
planar



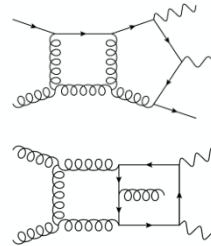
[Abreu, Page, Pascual, Sotnikov 2020;
Chawdhry, Czakon, Mitov, Poncelet 2021]

$3j$
planar



[Abreu, Febres-Cordero, Ita, Page, Sotnikov
2021; Badger, Brönnnum-Hansen, Bayu
Hartanto, Peraro, Moodie, **SZ**, to appear]

$2\gamma + j$
full
colour



[Agarwal, Buccioni, von Manteuffel,
Tancredi 2021 x2; Chawdhry, Czakon,
Mitov, Poncelet 2021]

[Badger, Brönnnum-Hansen, Chicherin,
Gehrmann, B. Hartanto, Henn, Marcoli,
Moodie, Peraro, **SZ** 2021]

Special function basis

[Gehrmann, Henn, Lo Presti '18; Chicherin, Sotnikov '20]

[see M. Marcoli's talk]

$pp \rightarrow 3\gamma$ [Kallweit, Sotnikov, Wiesemann 2020; Chawdhry, Czakon, Mitov, Poncelet 2020]

$d\sigma$ @NNLO QCD: $pp \rightarrow 2\gamma + j$ [Chawdhry, Czakon, Mitov, Poncelet 2021; Badger, Gehrmann, Marcoli, Moodie 2021]

$pp \rightarrow 3j$ [Czakon, Mitov, Poncelet 2021; Chen, Gehrmann, Glover, Huss, Marcoli 2022]

[slide from S. Zoia,
LoopFest 2022]

Perturbative structure in quantum field theory

$$\mathcal{A}_n^{(L)} = \sum_{w=0}^{2L} \sum_{ij} c_{ij}^{(w)} R_i^{(w)} \mathcal{I}_j^{(w)}.$$

Diagram illustrating the perturbative structure of quantum field theory amplitudes:

- Graded Transcendental function** (points to the sum over w)
- Coefficients (rational numbers)** (points to $c_{ij}^{(w)}$)
- Kinematic prefactors** (points to $R_i^{(w)}$)
- Transcendental functions from Canonical Differential Equations (CDE)** (points to $\mathcal{I}_j^{(w)}$)

- Challenges:
- Complicated **multivariate transcendental functions**
- Proliferation of **kinematic prefactors**
- For higher loops/multiplicity amplitudes, both conceptual and practical advances needed
- Today's target: calculating **six-point two-loop QCD amplitudes**

Challenge: Proliferation of kinematic prefactors (1/2)

Challenge

- Rationality of integral coefficients in momenta

$$\mathcal{N}_i(\vec{p}) = \sum_{\vec{\alpha}} n_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots \right), \quad \mathcal{D}_i(\vec{p}) = \sum_{\vec{\alpha}} d_{i,\vec{\alpha}} \left(s_{12}^{\alpha_1} s_{23}^{\alpha_2} \dots \right)$$

$$r_i(\vec{p}) = \frac{\mathcal{N}_i(\vec{p})}{\mathcal{D}_i(\vec{p})}, \quad s_{ij} = (p_i + p_j)^2$$

↪ linear in numerical coefficients $n_{i,\vec{\alpha}} \in \mathbb{Q}$

- Linear systems constructed from multiple numerical computations of $r_i(\vec{p})$

$$\vec{p} \rightarrow \{\vec{p}_1, \vec{p}_2, \dots\} \in \mathbb{Q}$$

↪ linear system for required coefficients $n_{i,\vec{\alpha}}$ and $d_{i,\vec{\alpha}}$

- Bottleneck:

- Run times
- Complexity of coefficients = number of unknowns $n_{i,\vec{\alpha}}$

Five-gluon amplitudes

[De Laurentis, Ita, Klinkert, Sotnikov '23]

Helicity remainder	dim(basis)	ansatz size
$R_{++++}^{(2),(2,0)}$	31	21,910
$R_{+++-}^{(2),(2,0)}$	54	54,148
$R_{++--}^{(2),(1,0)}$	274	163,635
$R_{+-++}^{(2),(1,0)}$	270	241,156
$R_{--++}^{(2),(1,0)}$	203	82,180
$R_{++++}^{(2),(1,1)}$	31	21,910
$R_{+++-}^{(2),(1,1)}$	54	54,148
$R_{++--}^{(2),(0,1)}$	226	118,880
$R_{+-++}^{(2),(0,1)}$	240	209,018
$R_{--++}^{(2),(0,1)}$	157	76,845
$R_{++++}^{(2),(-1,1)}$	25	5,320
$R_{+++-}^{(2),(-1,1)}$	35	9,384

[slide from H. Ita, LoopFest 2025]

Challenge: Proliferation of kinematic prefactors (2/2)

$$\begin{aligned}
 it[\varphi] &= \left\{ R[1] \rightarrow -\frac{5}{2}, \right. \\
 R[2] &\rightarrow -\frac{1}{4 s_{12}^3 (s_{12} - s_{34} - s_{45})^3} \left(6 s_{12}^5 s_{15} + 15 s_{12}^4 s_{15}^2 + 10 s_{12}^3 s_{15}^3 + 6 s_{12}^5 s_{23} + 15 s_{12}^4 s_{23}^2 + 10 s_{12}^3 s_{23}^3 - 12 s_{12}^4 s_{15} s_{34} - 15 s_{12}^3 s_{15}^2 s_{34} - 18 s_{12}^4 s_{23} s_{34} - 45 s_{12}^3 s_{23}^2 s_{34} - 30 s_{12}^2 s_{23}^3 s_{34} + 6 s_{12}^3 s_{15} s_{34}^2 + 18 s_{12}^3 s_{23} s_{34}^2 + \right. \\
 &45 s_{12}^2 s_{23}^2 s_{34}^2 + 30 s_{12} s_{23}^3 s_{34}^2 - 6 s_{12}^2 s_{23} s_{34}^3 - 15 s_{12} s_{23}^2 s_{34}^3 - 10 s_{23}^3 s_{34}^3 - 18 s_{12}^4 s_{15} s_{45} - 45 s_{12}^3 s_{15}^2 s_{45} - 30 s_{12}^2 s_{15}^3 s_{45} - 12 s_{12}^4 s_{23} s_{45} - 15 s_{12}^3 s_{23}^2 s_{45} + 6 s_{12}^4 s_{34} s_{45} + 54 s_{12}^3 s_{15} s_{34} s_{45} + \\
 &60 s_{12}^2 s_{15}^2 s_{34} s_{45} + 54 s_{12}^3 s_{23} s_{34} s_{45} + 60 s_{12}^2 s_{15} s_{23} s_{34} s_{45} + 30 s_{12} s_{15}^2 s_{23} s_{34} s_{45} + 60 s_{12}^2 s_{23}^2 s_{34} s_{45} + 30 s_{12} s_{15} s_{23}^2 s_{34} s_{45} - 12 s_{12}^3 s_{34}^2 s_{45} - 36 s_{12}^2 s_{15} s_{34}^2 s_{45} - 72 s_{12}^2 s_{23} s_{34}^2 s_{45} - \\
 &60 s_{12} s_{15} s_{23} s_{34}^2 s_{45} - 75 s_{12} s_{23}^2 s_{34}^2 s_{45} - 30 s_{15} s_{23}^2 s_{34}^2 s_{45} + 6 s_{12}^2 s_{34}^3 s_{45} + 30 s_{12} s_{23} s_{34}^3 s_{45} + 30 s_{23}^2 s_{34}^3 s_{45} + 18 s_{12}^3 s_{15} s_{45}^2 + 45 s_{12}^2 s_{15}^2 s_{45}^2 + 30 s_{12} s_{15}^3 s_{45}^2 + 6 s_{12}^3 s_{23} s_{45}^2 - 12 s_{12}^3 s_{34} s_{45}^2 - \\
 &72 s_{12}^2 s_{15} s_{34} s_{45}^2 - 75 s_{12} s_{15}^2 s_{34} s_{45}^2 - 36 s_{12}^2 s_{23} s_{34} s_{45}^2 - 60 s_{12} s_{15} s_{23} s_{34} s_{45}^2 - 30 s_{15}^2 s_{23} s_{34} s_{45}^2 + 27 s_{12}^2 s_{34}^2 s_{45}^2 + 60 s_{12} s_{15} s_{34}^2 s_{45}^2 + 60 s_{12} s_{23} s_{34}^2 s_{45}^2 + 60 s_{15} s_{23} s_{34}^2 s_{45}^2 - \\
 &15 s_{12} s_{34}^3 s_{45}^2 - 30 s_{23} s_{34}^3 s_{45}^2 - 6 s_{12}^2 s_{15} s_{45}^3 - 15 s_{12} s_{15}^2 s_{45}^3 - 10 s_{15}^3 s_{45}^3 + 6 s_{12}^2 s_{34} s_{45}^3 + 30 s_{12} s_{15} s_{34} s_{45}^3 + 30 s_{15}^2 s_{34} s_{45}^3 - 15 s_{12} s_{34}^2 s_{45}^3 - 30 s_{15} s_{34}^2 s_{45}^3 + 10 s_{34}^3 s_{45}^3 - 6 s_{12}^4 s_{15} s_{45} - \\
 &15 s_{12}^3 s_{15} s_{45} - 10 s_{12}^2 s_{15}^2 s_{45} - 15 s_{12}^3 s_{23} s_{45} - 10 s_{12}^2 s_{15} s_{23} s_{45} - 10 s_{12}^2 s_{23}^2 s_{45} + 12 s_{12}^3 s_{34} s_{45} + 15 s_{12}^2 s_{15} s_{34} s_{45} + 30 s_{12}^2 s_{23} s_{34} s_{45} + 10 s_{12} s_{15} s_{23} s_{34} s_{45} + 20 s_{12} s_{23}^2 s_{34} s_{45} - \\
 &6 s_{12}^2 s_{34}^2 s_{45} - 15 s_{12} s_{23} s_{34}^2 s_{45} - 10 s_{23}^2 s_{34}^2 s_{45} + 12 s_{12}^3 s_{45} s_{45} + 30 s_{12}^2 s_{15} s_{45} s_{45} + 20 s_{12} s_{15}^2 s_{45} s_{45} + 15 s_{12}^2 s_{23} s_{45} s_{45} + 10 s_{12} s_{15} s_{23} s_{45} s_{45} - 27 s_{12}^2 s_{34} s_{45} s_{45} - 35 s_{12} s_{15} s_{34} s_{45} s_{45} - \\
 &35 s_{12} s_{23} s_{34} s_{45} s_{45} - 20 s_{15} s_{23} s_{34} s_{45} s_{45} + 15 s_{12} s_{34}^2 s_{45} s_{45} + 20 s_{23} s_{34}^2 s_{45} s_{45} - 6 s_{12}^2 s_{45}^2 s_{45} - 15 s_{12} s_{15} s_{45}^2 s_{45} - 10 s_{15}^2 s_{45}^2 s_{45} + 15 s_{12} s_{34} s_{45}^2 s_{45} + 20 s_{15} s_{34} s_{45}^2 s_{45} - 10 s_{34}^2 s_{45}^2 s_{45} \Big), \\
 R[4] &\rightarrow \frac{1}{6 s_{12}^3 s_{34}^3} \left(10 s_{12}^3 s_{15}^3 - 30 s_{12}^3 s_{15}^2 s_{23} + 30 s_{12}^3 s_{15} s_{23}^2 - 10 s_{12}^3 s_{23}^3 - 15 s_{12}^3 s_{15}^2 s_{34} + 30 s_{12}^3 s_{15} s_{23} s_{34} - 15 s_{12}^3 s_{23}^2 s_{34} + 6 s_{12}^3 s_{15} s_{34}^2 - 6 s_{12}^3 s_{23} s_{34}^2 + 3 s_{12}^3 s_{34}^3 - 6 s_{12}^2 s_{23} s_{34}^3 - \right. \\
 &15 s_{12} s_{23}^2 s_{34}^3 - 10 s_{23}^3 s_{34}^3 - 30 s_{12}^2 s_{15}^3 s_{45} + 60 s_{12}^2 s_{15}^2 s_{23} s_{45} - 30 s_{12}^2 s_{15} s_{23}^2 s_{45} + 60 s_{12}^2 s_{15}^2 s_{34} s_{45} - 60 s_{12}^2 s_{15} s_{23} s_{34} s_{45} + 30 s_{12} s_{15}^2 s_{23} s_{34} s_{45} - 30 s_{12} s_{15} s_{23}^2 s_{34} s_{45} - \\
 &36 s_{12}^2 s_{15} s_{34}^2 s_{45} - 60 s_{12} s_{15} s_{23} s_{34}^2 s_{45} - 30 s_{15} s_{23}^2 s_{34}^2 s_{45} + 6 s_{12}^2 s_{34}^3 s_{45} + 30 s_{12} s_{23} s_{34}^3 s_{45} + 30 s_{23}^2 s_{34}^3 s_{45} + 30 s_{12} s_{15}^3 s_{45}^2 - 30 s_{12} s_{15}^2 s_{23} s_{45}^2 - 75 s_{12} s_{15}^2 s_{34} s_{45}^2 + \\
 &30 s_{12} s_{15} s_{23} s_{34} s_{45}^2 - 30 s_{15}^2 s_{23} s_{34} s_{45}^2 + 60 s_{12} s_{15} s_{34}^2 s_{45}^2 + 60 s_{15} s_{23} s_{34}^2 s_{45}^2 - 15 s_{12} s_{34}^3 s_{45}^2 - 30 s_{23} s_{34}^3 s_{45}^2 - 10 s_{15}^3 s_{45}^3 + 30 s_{15}^2 s_{34} s_{45}^3 - 30 s_{15} s_{34}^2 s_{45}^3 + 10 s_{34}^3 s_{45}^3 - 10 s_{12}^2 s_{15}^2 s_{45} s_{45} + \\
 &20 s_{12}^2 s_{15} s_{23} s_{45} s_{45} - 10 s_{12}^2 s_{23}^2 s_{45} s_{45} + 15 s_{12}^2 s_{15} s_{34} s_{45} s_{45} - 15 s_{12}^2 s_{23} s_{34} s_{45} s_{45} + 10 s_{12} s_{15} s_{23} s_{34} s_{45} s_{45} - 10 s_{12} s_{23}^2 s_{34} s_{45} s_{45} - 6 s_{12}^2 s_{34}^2 s_{45} s_{45} - 15 s_{12} s_{23} s_{34}^2 s_{45} s_{45} - 10 s_{23}^2 s_{34}^2 s_{45} s_{45} + 20 s_{12} s_{15}^2 s_{45} s_{45} - \\
 &20 s_{12} s_{15} s_{23} s_{45} s_{45} - 35 s_{12} s_{15} s_{34} s_{45} s_{45} + 10 s_{12} s_{23} s_{34} s_{45} s_{45} - 20 s_{15} s_{23} s_{34} s_{45} s_{45} + 15 s_{12} s_{34}^2 s_{45} s_{45} + 20 s_{23} s_{34}^2 s_{45} s_{45} - 10 s_{15}^2 s_{45}^2 s_{45} + 20 s_{15} s_{34} s_{45}^2 s_{45} - 10 s_{34}^2 s_{45}^2 s_{45} \Big), \\
 R[5] &\rightarrow -\frac{1}{s_{12}^3 s_{45}^3} \left(10 s_{12}^3 s_{15}^3 - 30 s_{12}^3 s_{15}^2 s_{23} + 30 s_{12}^3 s_{15} s_{23}^2 - 10 s_{12}^3 s_{23}^3 + 30 s_{12}^2 s_{15}^2 s_{23} s_{34} - 60 s_{12}^2 s_{15} s_{23}^2 s_{34} + 30 s_{12}^2 s_{23}^3 s_{34} + 30 s_{12} s_{15} s_{23}^2 s_{34}^2 - 30 s_{12} s_{23}^3 s_{34}^2 + 10 s_{23}^3 s_{34}^3 + \right. \\
 &15 s_{12}^3 s_{15}^2 s_{45} - 30 s_{12}^3 s_{15} s_{23} s_{45} + 15 s_{12}^3 s_{23}^2 s_{45} + 60 s_{12}^2 s_{15} s_{23} s_{34} s_{45} + 30 s_{12} s_{15}^2 s_{23} s_{34} s_{45} - 60 s_{12}^2 s_{23}^2 s_{34} s_{45} - 30 s_{12} s_{15} s_{23}^2 s_{34} s_{45} - 30 s_{12} s_{15} s_{23} s_{34}^2 s_{45} + 75 s_{12} s_{23}^2 s_{34}^2 s_{45} + \\
 &30 s_{15} s_{23}^2 s_{34}^2 s_{45} - 30 s_{23}^2 s_{34}^3 s_{45} + 6 s_{12}^3 s_{15} s_{45}^2 - 6 s_{12}^3 s_{23} s_{45}^2 + 36 s_{12}^2 s_{23} s_{34} s_{45}^2 + 60 s_{12} s_{15} s_{23} s_{34} s_{45}^2 + 30 s_{15}^2 s_{23} s_{34} s_{45}^2 - 60 s_{12} s_{23} s_{34}^2 s_{45}^2 - 60 s_{15} s_{23} s_{34}^2 s_{45}^2 + 30 s_{23} s_{34}^3 s_{45}^2 + \\
 &6 s_{12}^2 s_{15} s_{45}^3 + 15 s_{12} s_{15}^2 s_{45}^3 + 10 s_{15}^3 s_{45}^3 - 6 s_{12}^2 s_{34} s_{45}^3 - 30 s_{12} s_{15} s_{34} s_{45}^3 - 30 s_{15}^2 s_{34} s_{45}^3 + 15 s_{12} s_{34}^2 s_{45}^3 + 30 s_{15} s_{34}^2 s_{45}^3 - 10 s_{34}^3 s_{45}^3 + 10 s_{12}^2 s_{15}^2 s_{45} s_{45} - 20 s_{12}^2 s_{15} s_{23} s_{45} s_{45} + \\
 &10 s_{12}^2 s_{23}^2 s_{45} s_{45} + 20 s_{12} s_{15} s_{23} s_{34} s_{45} s_{45} - 20 s_{12} s_{23}^2 s_{34} s_{45} s_{45} + 10 s_{23}^2 s_{34}^2 s_{45} s_{45} + 15 s_{12}^2 s_{15} s_{45} s_{45} s_{45} + 10 s_{12} s_{15}^2 s_{45} s_{45} s_{45} - 15 s_{12}^2 s_{23} s_{45} s_{45} s_{45} - 10 s_{12} s_{15} s_{23} s_{45} s_{45} s_{45} - 10 s_{12} s_{15} s_{34} s_{45} s_{45} s_{45} + \\
 &35 s_{12} s_{23} s_{34} s_{45} s_{45} s_{45} + 20 s_{15} s_{23} s_{34} s_{45} s_{45} s_{45} - 20 s_{23} s_{34}^2 s_{45} s_{45} s_{45} + 6 s_{12}^2 s_{45}^2 s_{45} s_{45} + 15 s_{12} s_{15} s_{45}^2 s_{45} s_{45} + 10 s_{15}^2 s_{45}^2 s_{45} s_{45} - 15 s_{12} s_{34} s_{45}^2 s_{45} s_{45} - 20 s_{15} s_{34} s_{45}^2 s_{45} s_{45} + 10 s_{34}^2 s_{45}^2 s_{45} s_{45} \Big) \Big\}
 \end{aligned}$$

From Five-Parton NNLO amplitudes
[\[Abreu, Dormans, Febres Cordero, Ita, Page, Sotnikov, 19’\]](#)

Any understanding of analytic
 structures for QCD amplitudes?

Outline

- Introducing the target: **Maximally-transcendental weight part** of the **hard functions**
- Prefactors at maximal weight from **on-shell diagrams** & Bootstrap calculation for six-gluon QCD amplitudes
Example: --++++ helicity sector at two loops, pure YM
- Discussions about the result
Generalized principle of maximal transcendentality

Planar Six-Parton Scattering Amplitudes

- We calculate QCD helicity scattering amplitudes in leading color approximation under dim-reg from bootstrap strategy

$$\mathcal{A}(1_g, 2_g, 3_g, 4_g, 5_g, 6_g)|_{\text{leading order}} = \sum_{\sigma} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(6)}}) \times A(\sigma(1)_g, \dots, \sigma(6)_g)$$

- We keep the leading term in the large limit of number of color and number of flavors, while keeping their ratio fixed

$$A_6^{(2)} = A_6^{(2),0} + \frac{N_f}{N_c} A_6^{(2),1} + \left(\frac{N_f}{N_c}\right)^2 A_6^{(2),2}$$

- Subtraction:
$$I_n^{(1)} = -\frac{1}{\epsilon^2} \sum_{i=1}^n \left(\frac{-s_{i,i+1}}{\mu^2} \right)^{-\epsilon}$$
$$H^{(1)} = A^{(1)} - I^{(1)} A^{(0)},$$
$$H^{(2)} = A^{(2)} - I^{(1)} A^{(1)} - I^{(2)} A^{(0)},$$

Leading singularities and prefactors at maximally-transcendental weight

Maximally-transcendental weight part

$$\mathcal{A}_n^{(L)} = \sum_{w=0}^{2L} \sum_{ij} c_{ij}^{(w)} R_i^{(w)} \mathcal{I}_j^{(w)} .$$

Only focus on $w=2L$ in this talk

- At maximally-transcendental weight part, kinematic prefactors (**Leading singularity**) are related to maximal residues of the integrand! [Henn, Torres Bobadilla,21']

$$\mathcal{A}^{(L)} = \int \omega^{(L)} .$$

integrand

$$\mathcal{I}_i^{(L)} = \sum_j b_{ij} \prod_{k=1}^{4L} d \log \alpha_{ijk} .$$

dlog-form
integrand basis

$$\omega^{(L)} = \sum_{i=1}^m c_i \mathcal{I}_i^{(L)} + \dots .$$

Maximal weight
part + double-
pole part in ...

$$\mathcal{P} \left(\omega^{(L)} \right) = \sum_{i=1}^m c_i \mathcal{I}_i^{(L)} .$$

**Maximal weight
projection**

On-shell diagrams and leading singularity

- All non-trivial leading singularities are calculated by **on-shell diagrams** from gluing tree-level amplitudes [Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka,12']

$$= \frac{\langle 14 \rangle^4 \langle 23 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 14 \rangle \langle 13 \rangle^4}$$

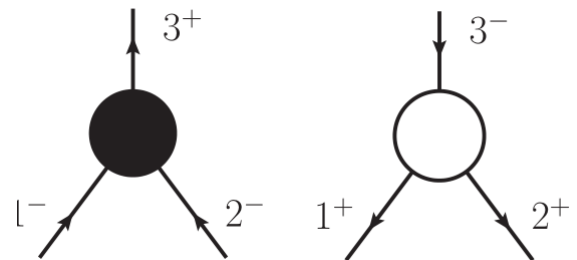
$$= \frac{\langle 12 \rangle^4 \langle 34 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 14 \rangle \langle 13 \rangle^4}$$

All one-loop prefactors at maximal weight part!

$$PT_{n,ij} \times \left(\left(\frac{\langle ik \rangle \langle jl \rangle}{\langle kl \rangle \langle ij \rangle} \right)^4 + \left(\frac{\langle il \rangle \langle jk \rangle}{\langle kl \rangle \langle ij \rangle} \right)^4 \right), \quad 1 \leq i < k < j < l \leq n$$

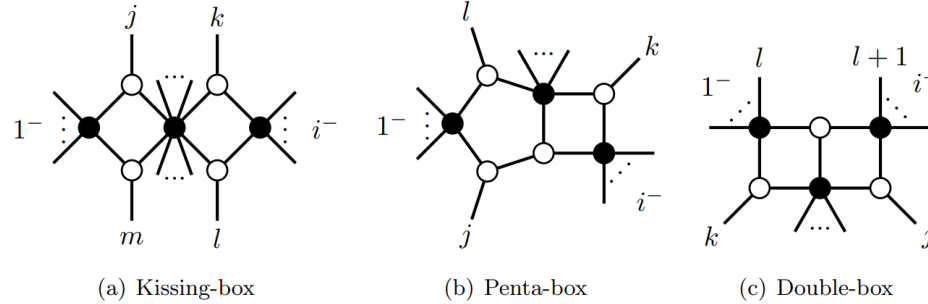
- Conformal invariants** under action of $\sum_{i=1}^n \frac{\partial^2}{\partial \lambda_{\alpha}^i \partial \tilde{\lambda}_{\dot{\alpha}}^i}$

[Witten 03';Henn, Power, Zoia, 19']



Two-loop MHV on-shell diagrams and prefactors

- At two loops, we summarize all non-trivial two-loop pure YM MHV on-shell diagrams as three types (Also applicable to general QCD MHV amplitudes!):



- Naturally simplified by spinor-helicity variables in this calculation!

$$\begin{aligned}
 R(1) &\rightarrow -\frac{5}{2}, \\
 R(2) &\rightarrow -\frac{1}{4s_{12}^2(s_{12}-s_{34}-s_{45})^3} (6s_{12}^5s_{15}+15s_{12}^4s_{15}^2+10s_{12}^3s_{15}^3+6s_{12}^2s_{23}+15s_{12}^2s_{23}^2+10s_{12}^2s_{23}^3-12s_{12}^2s_{15}s_{34}-15s_{12}^2s_{15}^2s_{34}-18s_{12}^2s_{23}s_{34}-45s_{12}^2s_{23}^2s_{34}-30s_{12}^2s_{23}^3s_{34}+6s_{12}^2s_{15}s_{34}^2+18s_{12}^2s_{23}s_{34}^2+ \\
 &45s_{12}^2s_{23}^2s_{34}^2-30s_{12}^2s_{23}^3s_{34}^2-6s_{12}^2s_{23}^3s_{34}^3-15s_{12}^2s_{23}^3s_{34}^4-10s_{23}^3s_{34}^4-18s_{12}^4s_{15}s_{45}-45s_{12}^4s_{15}^2s_{45}-30s_{12}^4s_{15}^3s_{45}-12s_{12}^4s_{23}s_{45}-15s_{12}^4s_{23}^2s_{45}+6s_{12}^4s_{34}s_{45}+54s_{12}^4s_{15}s_{34}s_{45}+ \\
 &60s_{12}^4s_{15}^2s_{34}s_{45}+54s_{12}^4s_{23}s_{34}s_{45}+60s_{12}^4s_{15}s_{23}s_{34}s_{45}+30s_{12}^4s_{15}^2s_{23}s_{34}s_{45}+60s_{12}^4s_{15}^3s_{23}s_{34}s_{45}-12s_{12}^4s_{34}^2s_{45}-36s_{12}^4s_{15}s_{34}^2s_{45}-72s_{12}^4s_{23}s_{34}^2s_{45}- \\
 &60s_{12}^4s_{15}s_{23}s_{34}^2s_{45}-75s_{12}^4s_{23}^2s_{34}^2s_{45}-30s_{15}s_{23}^3s_{34}^2s_{45}+6s_{12}^2s_{34}^3s_{45}+30s_{23}^3s_{34}^3s_{45}+18s_{12}^2s_{15}s_{45}^2+45s_{12}^2s_{15}^2s_{45}^2+30s_{12}^2s_{15}^3s_{45}^2+6s_{12}^2s_{23}s_{45}^2-12s_{12}^2s_{34}s_{45}^2- \\
 &72s_{12}^2s_{15}s_{34}s_{45}^2-75s_{12}^2s_{15}^2s_{34}s_{45}^2-36s_{12}^2s_{23}s_{34}s_{45}^2-60s_{12}^2s_{15}s_{23}s_{34}s_{45}^2-30s_{15}^2s_{23}s_{34}s_{45}^2+27s_{12}^2s_{34}^2s_{45}^2+60s_{12}^2s_{15}s_{34}^2s_{45}^2+60s_{12}^2s_{23}s_{34}^2s_{45}^2+60s_{15}s_{23}s_{34}^2s_{45}^2- \\
 &15s_{12}^3s_{34}^2s_{45}^2-30s_{23}s_{34}^2s_{45}^2-6s_{12}^2s_{15}s_{45}^3-15s_{12}^2s_{15}^2s_{45}^3-10s_{15}^3s_{45}^3+6s_{12}^2s_{34}s_{45}^3+30s_{12}^2s_{15}s_{34}s_{45}^3+30s_{15}^2s_{34}s_{45}^3-15s_{12}^3s_{34}^2s_{45}^3-30s_{15}s_{34}^2s_{45}^3-10s_{34}^3s_{45}^3-6s_{12}^4tr5- \\
 &15s_{12}^3s_{15}tr5-10s_{12}^2s_{15}^2tr5-15s_{12}^3s_{23}tr5-10s_{12}^2s_{15}s_{23}tr5-10s_{12}^2s_{23}^2tr5-12s_{12}^2s_{34}tr5+15s_{12}^2s_{15}s_{34}tr5+30s_{12}^2s_{15}s_{23}s_{34}tr5+10s_{12}^2s_{15}s_{23}s_{34}tr5+20s_{12}^2s_{23}^2s_{34}tr5- \\
 &6s_{12}^2s_{34}^2tr5-15s_{12}^2s_{23}s_{34}^2tr5-10s_{23}^3s_{34}^2tr5+12s_{12}^2s_{45}tr5+12s_{12}^2s_{15}s_{45}tr5+20s_{12}^2s_{15}s_{23}s_{45}tr5+15s_{12}^2s_{23}s_{45}tr5-27s_{12}^2s_{34}s_{45}tr5-35s_{12}^2s_{15}s_{34}s_{45}tr5- \\
 &35s_{12}^2s_{23}s_{34}s_{45}tr5-20s_{15}s_{23}s_{34}s_{45}tr5+15s_{12}^3s_{34}^2s_{45}tr5+20s_{23}s_{34}^2s_{45}tr5-6s_{12}^2s_{45}^2tr5-15s_{12}^2s_{15}s_{45}^2tr5-10s_{15}^2s_{45}^2tr5+15s_{12}^3s_{34}s_{45}^2tr5+20s_{15}s_{34}s_{45}^2tr5-10s_{34}^2s_{45}^2tr5), \\
 R(4) &\rightarrow \frac{1}{6s_{12}^3s_{34}^3} (10s_{12}^3s_{15}^3-30s_{12}^3s_{15}^2s_{23}+30s_{12}^3s_{15}s_{23}^2-10s_{12}^3s_{23}^3-15s_{12}^3s_{15}^2s_{34}+30s_{12}^3s_{15}s_{23}s_{34}-15s_{12}^3s_{23}^2s_{34}+6s_{12}^3s_{15}s_{34}^2-6s_{12}^3s_{23}s_{34}^2-3s_{12}^3s_{34}^3-6s_{12}^2s_{23}s_{34}^3- \\
 &15s_{12}^2s_{23}^2s_{34}^3-10s_{23}^3s_{34}^3-30s_{12}^2s_{15}^3s_{45}+60s_{12}^2s_{15}^2s_{23}s_{45}-30s_{12}^2s_{15}s_{23}^2s_{45}-60s_{12}^2s_{15}^2s_{34}s_{45}-60s_{12}^2s_{15}^3s_{34}s_{45}-60s_{12}^2s_{15}s_{23}s_{34}s_{45}-30s_{12}^2s_{15}^2s_{23}s_{34}s_{45}-30s_{12}^2s_{15}^3s_{23}s_{34}s_{45}- \\
 &30s_{12}^2s_{15}s_{23}s_{34}^2s_{45}-30s_{15}^2s_{23}s_{34}^2s_{45}+60s_{15}s_{23}s_{34}^2s_{45}^2-15s_{12}^3s_{34}^2s_{45}^2-30s_{23}^3s_{34}^2s_{45}^2-10s_{15}^3s_{45}^2+30s_{15}^2s_{34}s_{45}^2-30s_{15}s_{34}^2s_{45}^2+10s_{34}^3s_{45}^2-10s_{12}^2s_{15}^2tr5+ \\
 &20s_{12}^2s_{15}s_{23}tr5-10s_{12}^2s_{23}^2tr5+15s_{12}^2s_{15}s_{34}tr5-15s_{12}^2s_{23}s_{34}tr5+10s_{12}^2s_{15}s_{23}s_{34}tr5-10s_{12}^2s_{23}^2s_{34}tr5-6s_{12}^2s_{34}^2tr5-10s_{23}^3s_{34}^2tr5-20s_{12}^2s_{15}s_{34}tr5- \\
 &20s_{12}^2s_{15}s_{23}s_{45}tr5-35s_{12}^2s_{15}s_{34}s_{45}tr5-10s_{12}^2s_{23}s_{34}s_{45}tr5-20s_{15}s_{23}s_{34}s_{45}tr5+15s_{12}^3s_{34}^2s_{45}tr5+20s_{23}s_{34}^2s_{45}tr5-10s_{15}^2s_{45}^2tr5+20s_{15}s_{34}s_{45}^2tr5-10s_{34}^2s_{45}^2tr5), \\
 R(5) &\rightarrow \frac{1}{s_{12}^3s_{45}^3} (10s_{12}^3s_{15}^3-30s_{12}^3s_{15}^2s_{23}+30s_{12}^3s_{15}s_{23}^2-10s_{12}^3s_{23}^3-30s_{12}^3s_{15}^2s_{34}+60s_{12}^3s_{15}s_{23}s_{34}-30s_{12}^3s_{23}^2s_{34}+30s_{12}^3s_{15}s_{34}^2-30s_{12}^3s_{23}s_{34}^2-10s_{23}^3s_{34}^2+ \\
 &15s_{12}^3s_{15}^2s_{45}+30s_{12}^3s_{15}s_{23}s_{45}+15s_{12}^3s_{23}^2s_{45}+60s_{12}^3s_{15}s_{23}s_{34}s_{45}+30s_{12}^3s_{15}^2s_{23}s_{34}s_{45}-60s_{12}^3s_{23}^2s_{34}s_{45}-30s_{12}^3s_{15}s_{23}s_{34}^2s_{45}-30s_{12}^3s_{15}^2s_{23}s_{34}^2s_{45}-75s_{12}^3s_{15}^3s_{23}s_{34}^2s_{45}+ \\
 &30s_{15}s_{23}s_{34}^2s_{45}-30s_{23}^3s_{34}^2s_{45}+6s_{12}^2s_{15}s_{45}^2+6s_{12}^2s_{23}s_{45}^2+36s_{12}^2s_{23}s_{34}s_{45}^2-60s_{12}^2s_{15}s_{23}s_{34}s_{45}^2-30s_{15}^2s_{23}s_{34}s_{45}^2-60s_{12}^3s_{34}^2s_{45}^2-60s_{15}s_{23}s_{34}^2s_{45}^2+30s_{23}s_{34}^2s_{45}^2+ \\
 &6s_{12}^2s_{15}s_{45}^3+15s_{12}^2s_{15}^2s_{45}^3-10s_{15}^3s_{45}^3-6s_{12}^2s_{34}s_{45}^3-30s_{12}^2s_{15}s_{34}s_{45}^3-30s_{15}^2s_{34}s_{45}^3+15s_{12}^3s_{34}^2s_{45}^3+30s_{15}s_{34}^2s_{45}^3-10s_{34}^3s_{45}^3+10s_{12}^2s_{15}^2tr5-20s_{12}^2s_{15}s_{23}tr5+ \\
 &10s_{12}^2s_{23}^2tr5-20s_{12}^2s_{15}s_{34}tr5-20s_{12}^2s_{23}s_{34}tr5+10s_{23}^3s_{34}tr5+15s_{12}^2s_{15}s_{45}tr5+10s_{12}^2s_{15}^2s_{45}tr5-15s_{12}^2s_{23}s_{45}tr5-10s_{12}^2s_{15}s_{23}s_{45}tr5-10s_{12}^2s_{15}s_{34}s_{45}tr5- \\
 &35s_{12}^2s_{23}s_{34}s_{45}tr5-20s_{15}s_{23}s_{34}s_{45}tr5-20s_{23}s_{34}^2s_{45}tr5-6s_{12}^2s_{45}^2tr5+15s_{12}^2s_{15}s_{45}^2tr5+10s_{15}^2s_{45}^2tr5-15s_{12}^3s_{34}s_{45}^2tr5-20s_{15}s_{34}s_{45}^2tr5+10s_{34}^2s_{45}^2tr5)
 \end{aligned}$$

$$\begin{aligned}
 &\left\{ 1, -\frac{(-2ab[1, 3] \times ab[2, 4] + ab[1, 2] \times ab[3, 4]) (10ab[1, 3]^2 ab[2, 4]^2 - 10ab[1, 2] \times ab[1, 3] \times ab[2, 4] \times ab[3, 4] + ab[1, 2]^2 ab[3, 4]^2)}{ab[1, 2]^3 ab[3, 4]^3}, \right. \\
 &- \frac{(-2ab[1, 3] \times ab[2, 5] + ab[1, 2] \times ab[3, 5]) (10ab[1, 3]^2 ab[2, 5]^2 - 10ab[1, 2] \times ab[1, 3] \times ab[2, 5] \times ab[3, 5] + ab[1, 2]^2 ab[3, 5]^2)}{ab[1, 2]^3 ab[3, 5]^3}, \\
 &- \left. \frac{(-2ab[1, 4] \times ab[2, 5] + ab[1, 2] \times ab[4, 5]) (10ab[1, 4]^2 ab[2, 5]^2 - 10ab[1, 2] \times ab[1, 4] \times ab[2, 5] \times ab[4, 5] + ab[1, 2]^2 ab[4, 5]^2)}{ab[1, 2]^3 ab[4, 5]^3} \right\}
 \end{aligned}$$

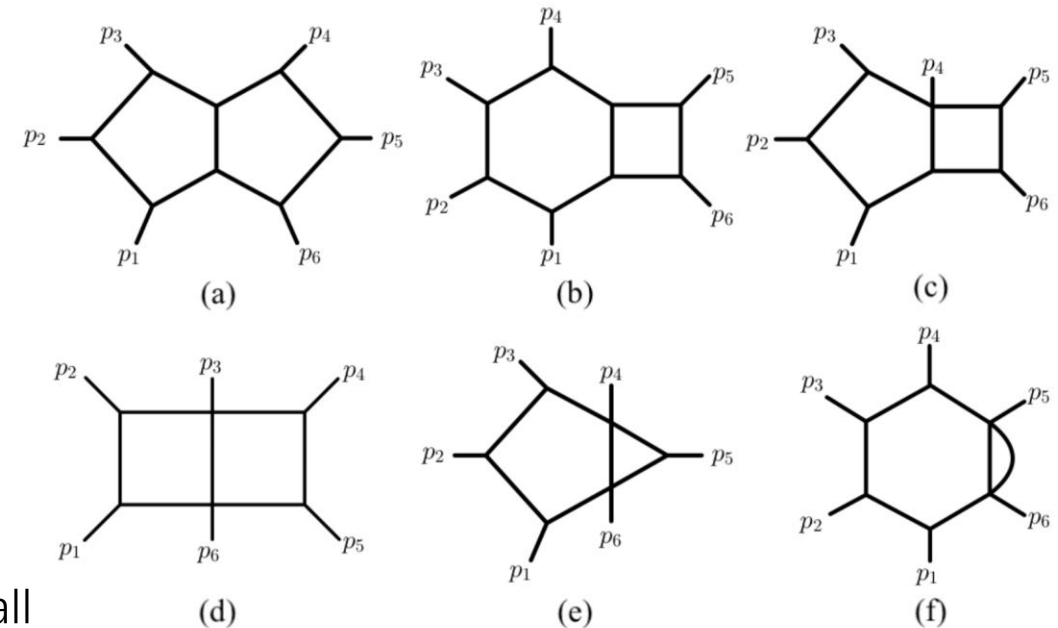
Conformal invariance!

Hexagon functions at two loops [Abreu, Monni, Page, Usovitsch, 24'; Henn, Matijašić, Miczajka, Peraro, Xu, Zhang, 25']

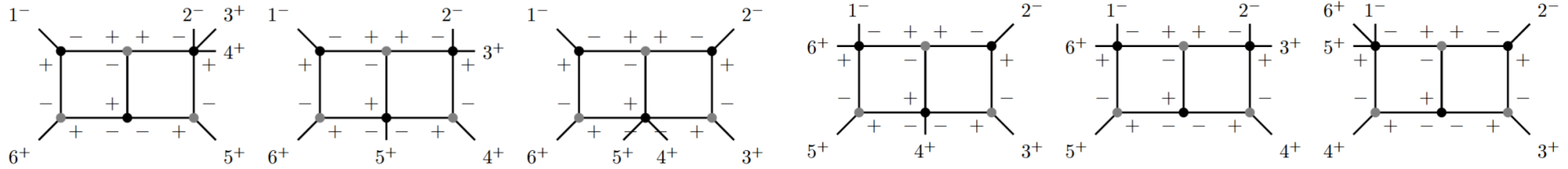
- **Two-loop massless scattering processes with six particles**, State-of-art for Feynman integral calculation
- Up to weight four, **hexagon function space** can be constructed
- **dlog-iterative integrals (4D external legs)**

Transcendental weight	1	2	3	4
# All symbols	9	62	319	945
# Two-loop six-point symbols	9	62	266	639
# Two-loop five-point one-mass symbols	9	59	263	594
# One-loop squared symbols	9	59	221	428
# Genuine two-loop six-point symbols	0	0	3	45

Basis for all planar six-point two-loop observables, at maximal weight, symbol level



A bootstrap example: pure YM amplitudes with $--++++(1/2)$



- In this sector, there are **7 linear-independent leading singularities** from on-shell diagrams and **945 hexagon functions (at symbol level)** from CDE

$$H_6^{(2)} = \sum_{i=1}^7 \sum_{j=1}^{945} c_{i,j} R_i \mathcal{I}_j^{(2)}$$

Coefficients
(to be fixed)
Independent
prefactors
(from unitarity
input)
Master integral
basis (from CDE
input)

$$R_1 = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}$$

$$R_{i,j}/R_1 = -1 + 12u_{i,j} - 30u_{i,j}^2 + 20u_{i,j}^3$$

$$u_{i,j} := \langle 1i \rangle \langle 2j \rangle / (\langle 12 \rangle \langle ij \rangle)$$

- 2412** genuine unknown coefficients by reflection symmetry and ruling out unnecessary products of one-loop functions

A bootstrap example: pure YM amplitudes with --++++(2/2)

$$\mathcal{H}_{\text{YM}}^{(2)} = R_1 G_1 + \sum_{2 < i < j \leq 6} R_{i,j} G_{i,j} \, .$$

We impose cancellations of spurious poles, as well as consistency with soft and collinear limits.

$$\begin{aligned} & \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \text{---} \bigcirc \mathcal{A}_6 \text{---} \begin{array}{c} p_6 \\ p_5 \\ p_4 \end{array} \xrightarrow{p_6 || p_5} \left(\begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \text{---} \bigcirc \mathcal{A}_5 \text{---} \begin{array}{c} p \\ p_4 \end{array} \right) \times p \text{---} \bigcirc \mathcal{S}p \text{---} \begin{array}{c} p_5 \\ p_6 \end{array} . \\ & \begin{array}{c} p_1 \\ p_2 \\ p_3 \end{array} \text{---} \bigcirc \mathcal{A}_6 \text{---} \begin{array}{c} p_6 \\ p_5 \\ p_4 \end{array} \xrightarrow{p_4 || p_5 || p_6} \left(\begin{array}{c} p_1 \\ p_2 \end{array} \text{---} \bigcirc \mathcal{A}_4 \text{---} \begin{array}{c} p \\ p_3 \end{array} \right) \times p \text{---} \bigcirc \mathcal{S}p \text{---} \begin{array}{c} p_4 \\ p_5 \\ p_6 \end{array} . \end{aligned}$$

The answer is fixed uniquely!

Similar calculations are performed for other two MHV helicity sectors, as well as Nf terms of gluonic amplitudes in QCD

Table I. Two-loop bootstrap constraints for $\mathcal{H}_{\text{YM}}^{(2)}$	
Condition on $\mathcal{H}_{\text{YM}}^{(2)}$	No. of constraints
dimensionless symbols G	996
no spurious/high-order poles	
$\langle 36 \rangle = 0$	333
$\langle 35 \rangle = 0$	614
$\langle 34 \rangle = 0$	629
$\langle 45 \rangle = 0$	343
collinear limit $p_5 p_6$	
--++++	1785
+---++	1307
++--++	1785
+++--+	1646
-++++-	1646
triple collinear limit $p_4 p_5 p_6$	
--++++	1836
++--++	724
Total	2412

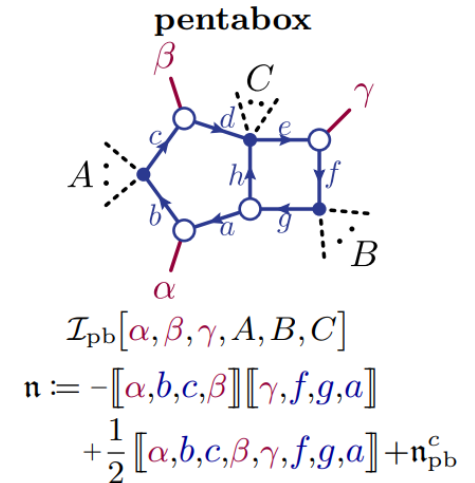
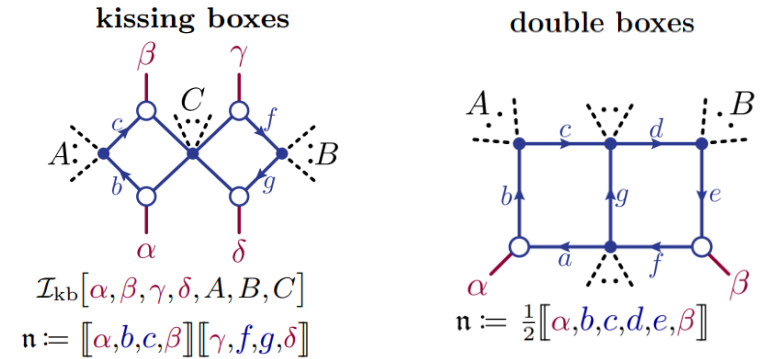
QCD amplitudes and prescriptive unitarity (1/2)

- In the bootstrap result, we find a same set of integrals appear in different helicity sectors
- Prescriptive unitarity** approach! [Bourjaily,Herrmann,Trnka]

$$H_{--++++}^{(2)} = \text{PT}_{6,12} f_0 + \sum_{2 < i < j \leq 6} R_{12,ij1}^{(2),\text{db}} I_{6,ij1}^{\text{db}}, \quad (6.4)$$

$$H_{-+--++}^{(2)} = \text{PT}_{6,13} g_0 + R_{13,24}^{(1)} g_{2,4} + R_{13,25}^{(1)} g_{2,5} + R_{13,26}^{(1)} g_{2,6} \\ + \sum_{\substack{l=1,2 \\ 3 < j < k \leq 6}} R_{13,jkl}^{(2),\text{db}} I_{6,jkl}^{\text{db}} + \sum_{\{j,k,l\} \in \sigma_1} R_{13,jkl}^{(2),\text{pb}} (I_{6,jkl}^{\text{pb}} + I_{6,jlk}^{\text{pb}}), \quad (6.5)$$

$$H_{-++--+}^{(2)} = \text{PT}_{6,14} h_0 + R_{14,25}^{(1)} h_{2,5} + R_{14,26}^{(1)} h_{2,6} + R_{14,35}^{(1)} h_{3,5} + R_{14,36}^{(1)} h_{3,6} \\ + R_{14,2356}^{(2),\text{kb}} I_{6,2356}^{\text{kb}} + \sum_{1 \leq l \leq 3} R_{14,56l}^{(2),\text{db}} I_{6,56l}^{\text{db}} + \sum_{4 \leq l \leq 6} R_{41,23l}^{(2),\text{db}} I_{6,23l}^{\text{db}} \\ + \sum_{\{j,k,l\} \in \sigma_2} R_{14,jkl}^{(2),\text{pb}} (I_{6,jkl}^{\text{pb}} + I_{6,jlk}^{\text{pb}}), \quad (6.9)$$



[Bourjaily, Herrmann, Langer, McLeod, Trnka, 19']

Analytic structure at ALL multiplicity for two-loop MHV amplitudes!

QCD amplitudes and prescriptive unitarity (2/2)

- In the bootstrap result, if we replace all prefactors by 1 or 0 directly, we recover the six-point two-loop hard function for $\mathcal{N}=4$ sYM

$$H_{h_1 \dots h_6}^{(2)} \xrightarrow[R^{(2), \text{db} \rightarrow 0}]{\text{PT}, R^{(1)}, R^{(2), \text{pb}}, R^{(2), \text{kb}} \rightarrow 1} \left(H_6^{(2)} \right)^{\mathcal{N}=4 \text{ sYM}}. \quad (6.21)$$

- For different helicity sectors, essentially, pure YM and $\mathcal{N}=4$ sYM are formed by the same set of function basis, but with differential coefficients

Generalization of the principle of maximal transcendentality!

Summary

- Two key simplification in our cases are
 1. Symbol bootstrap based on hexagon function basis instead of IBP
(Six-point bootstrap is simpler than five-point case due to various strong physical limits)
 2. Leading singularities classified by calculation from on-shell methods
(Nice properties for prefactors are always expected at maximal transcendental weight part)
- Calculation for more observables in this function space. Letters for NMHV ?
- How to determine the prefactors at lower weight parts & rational terms?
- Uplifting to function level, which allows us to explore various properties (positivity, complete monotonicity [\[Henn, Raman\]](#) ..)

Thanks!

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