Two-loop Feynman integrals for $t\bar{t}\gamma$ hadron production in the leading-colour approximation

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MAX-PLANCK-INSTITUT
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$t\bar{t}$+X production at LHC – crucial tests of the SM

- **pp → $t\bar{t}$+X** (with X=H,j,γ,Z,W,$t\bar{t}$) are crucial processes for the LHC precision program

precision **test of the SM**

- $t\bar{t}\gamma$ production

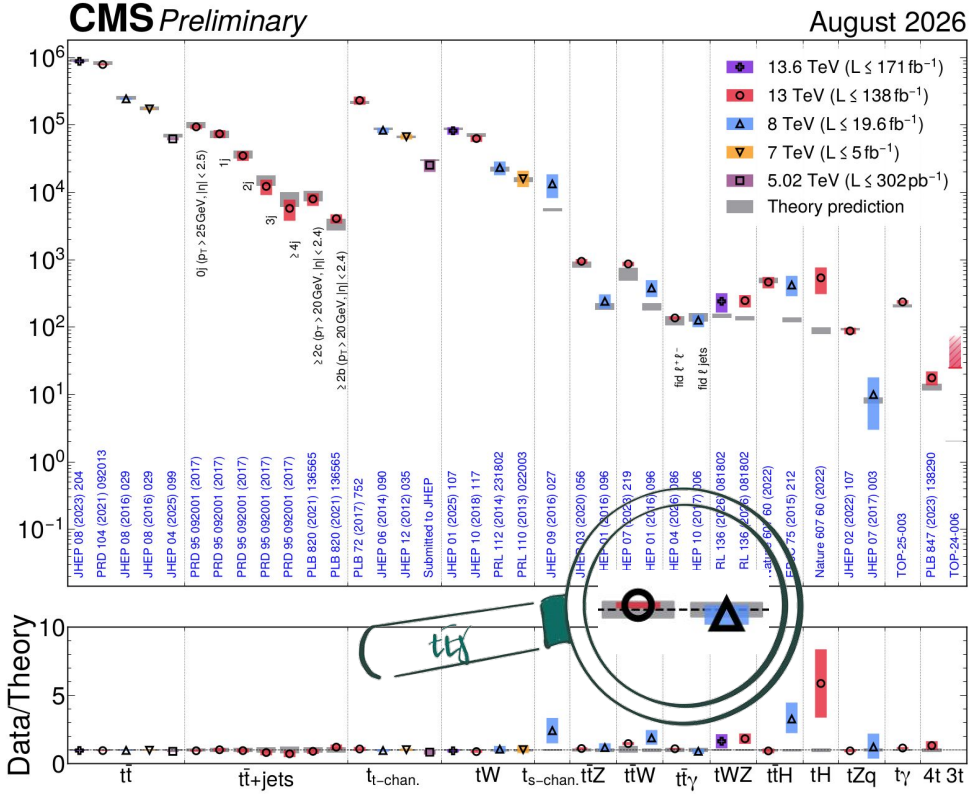


possible hints of **new physics**

$$\mathcal{L}_{eff}^{t\bar{t}\gamma} = e\bar{t}\left(Q_t\gamma_\mu A^\mu + \frac{1}{4m_t}\sigma_{\mu\nu}F^{\mu\nu}(\kappa + i\tilde{\kappa}\gamma_5)\right)t$$

$$A_c = \frac{N(\Delta|y|<0) - N(\Delta|y|>0)}{N(\Delta|y|<0) + N(\Delta|y|>0)}, \quad \Delta|y| = |y_t| - |y_{\bar{t}}|$$

- As **experimental precision grows**, theory predictions must keep pace



The next frontier: NNLO QCD accuracy for pp → $t\bar{t}\gamma$

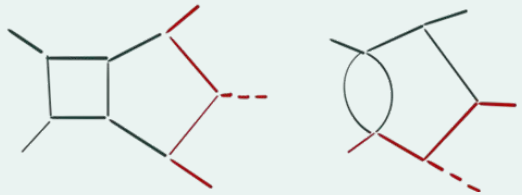
The main bottleneck: two-loop Feynman integrals and scattering amplitudes

Two-loop five-point integrals - massive propagators

- High **algebraic complexity** ⟶ a lot of kinematic scales
- High **analytic complexity** ⟶ presence of complicated (elliptic) geometries

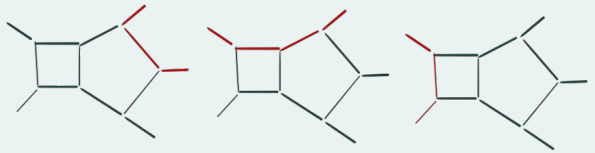
[See S. Zoia's talk for a complete review]

pp → ttH – leading colour, N_f part



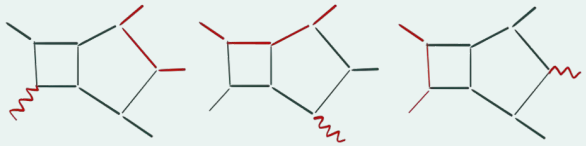
[Febres Cordero, Figueiredo, Kraus, Page, Reina, 2023]

pp → ttj – leading colour



[Badger, Becchetti, Chaubey, Marzucca, 2022; Badger, Becchetti, Giraudo, Zoia, 2024; Becchetti, Dlapa, Zoia, 2025]

pp → ttW – leading colour



[Becchetti, Canko, Chestnov, Peraro, Pozzoli, Zoia, 2025]

NNLO QCD cross-section results for processes involving these topologies have recently become available
Limited to the leading-colour approximation (or to other approximations applied to the double virtual contribution)

[Devoto, Grazzini, Kallweit, Mazzitelli, Savoini, 2025] [Buonocore, Devoto, Grazzini, Kallweit, Mazzitelli, Rottoli, Savoini, 2025] [Biello, Savoini, Signorile-Signorile Wiesemann, 2026]

[Badger, Becchetti, **CB**, Czakon, Hartanto, Poncelet, Zoia, 2025] [Becchetti, Canko, Chen, Chestnov, Delto, Ditsch, Peraro, Pozzoli, Savoini, Zoia, 2026]

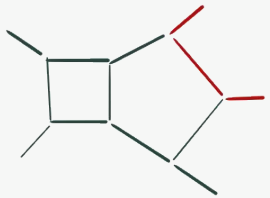
[H. B. Hartanto, M. Pozzoli and X. Chen 's talks]



Image: © Matt Groening / 20th Television

pp → tt̄γ: Integral families contributing at leading-colour

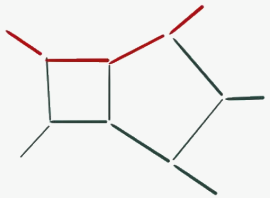
PBA — IN ttj@LC



#MIs = 88

polylogarithmic

PBB — IN ttj@LC



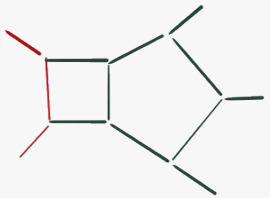
#MIs = 121

non-polylogarithmic

1 elliptic sector

1 nested-square root sector

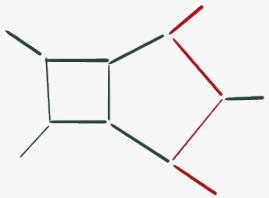
PBC — IN ttj@LC



#MIs = 109

polylogarithmic

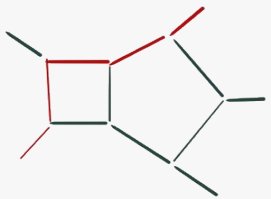
PBD — *This work!*



#MIs = 100

polylogarithmic

PBE — *This work!*



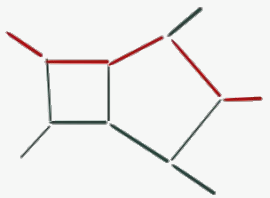
#MIs = 154

non-polylogarithmic

2 elliptic sectors

1 nested-square root sectors

PBF — *This work!*



#MIs = 142

non-polylogarithmic

3 elliptic sectors

2 nested-square root sectors

Differential equations method

1. **IBP relations** allow one to express Feynman integrals of a given family as a linear combination of MIs

[Tkachov, 1981; Chetyrkin, Tkachov, 1981; Laporta, 2000]

$$I^{(\mathrm{F})}_{\vec{\nu}}(\vec{x},\epsilon) \sim \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \prod_{i=1}^{11} \frac{1}{D_{\mathrm{F},i}^{\nu_i}} \quad \xrightarrow{\textbf{IBP reduction}} \quad I^{(\mathrm{F})}_{\vec{\nu}}(\vec{x},\epsilon) = \sum_{i}^{\#\mathrm{MIs}} a_i(\vec{x},\varepsilon)\vec{\mathcal{J}}^{(\mathrm{F})}_i(\vec{x},\varepsilon)$$

Generated with **LiteRed** [Lee, 2013, 2014] and **NeatIBP** [Wu, Boehm, Ma, Xu, Zhang, 2024], solved with **FiniteFlow** [Peraro, 2019]

2. The master integrals satisfy a system of **first-order differential equations** in the kinematic invariants

[Barucchi, Ponzano, 1973; Kotikov, 1991; Bern, Dixon, Kosower, 1994; Gehrmann, Remiddi, 2000]

$$\mathrm{d}\,\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon) = \mathrm{d}\mathcal{A}^{(\mathrm{F})}(\vec{x},\varepsilon)\cdot\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon) \quad \rightarrow \quad \vec{x} = (d_{12},\, d_{23},\, d_{34},\, d_{45},\, d_{15},\, m_t^2)$$

6 kinematic invariants

3. **Canonical form** of the differential equations [Henn, 2013]

$$\mathrm{d}\,\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon) = \varepsilon\,\mathrm{d}\tilde{\mathcal{A}}^{(\mathrm{F})}(\vec{x})\cdot\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon) \quad \textit{where, in the best-understood case} \quad \mathrm{d}\tilde{\mathcal{A}}^{(\mathrm{F})}(\vec{x}) = \sum_i \alpha_i^{(\mathrm{F})}\,\mathrm{dlog}\,W_i(\vec{x})$$

Advantages

- **ε-factorised:** iterative solution, order by order in ε
- **dlog structure:** only simple poles — no essential singularities
- **pure MIs:** powerful mathematical techniques to handle and evaluate efficiently the solution

The polylogarithmic case

No general algorithm for canonical bases — conjecture [Arkani-Hamed, Bourjaily, Cachazo, Trnka, 2012]

integrands with only simple poles and constant leading singularities (LS)

$$\mathcal{J} = \frac{1}{\mathrm{LS}(\vec{x})} \int \mathrm{d}\log(\ldots) \wedge \ldots \wedge \mathrm{d}\log(\ldots) \quad \rightarrow \quad \textbf{Good choice:} \quad \mathrm{LS}(\vec{x}) \times \mathcal{J}$$

Two approaches

- Educated guesses based on the literature + **Magnus exponential** [Argeri, Vita, Mastrolia, Mirabella, Schlenk, Schubert, Tancredi, 2014] [Gehrmann, Manteuffel, Tancredi, Weihs, 2014]
- Integrand analysis in the loop-by-loop Baikov representation with **Baikov.m** [Frellesvig, Papadopoulos, 2017] + **DLogBasis** to analyse it [Henn, Mistlberger, Smirnov, Wasser, 2020]

Baikov representation: propagators as integration variables [Baikov, 1996, 1997]

$$I_{\vec{a}} \propto \int \prod_i \mathrm{d}z_i \, \frac{P(\vec{z})^{\frac{D-L-E-1}{2}}}{\prod_i z_i^{a_i}}$$

z_i = inverse propagators
P = Baikov polynomial (Gram determinant)
L loops, E independent external momenta

studied on the **maximal cut**: $\frac{1}{z_i} \longrightarrow \delta(z_i)$

[See C. Nega's talk]

Deviations from the canonical form of our DEs

matrices of rational numbers

$$\mathrm{d}\mathcal{A}^{(\mathrm{F})}(\vec{x},\varepsilon) = \sum_{i=0}^{2} \varepsilon^i \left(\sum_j \alpha_j^{(\mathrm{F},i)} \,\mathrm{dlog}\left(W_j(\vec{x})\right) + \sum_j \beta_j^{(\mathrm{F},i)}\, \omega_j(\vec{x}) \right)$$

quadratic in ε

non-dlog one-forms

- **For PBD, canonical DEs:** $\alpha_j^{(\mathrm{PB}_D,0)} = \alpha_j^{(\mathrm{PB}_D,2)} = \beta_j^{(\mathrm{PB}_D,i)} = 0$
- **The letters** $W_j(\vec{x})$ are identified with BaikovLetters and SOFIA [Jiang, Liu, Xu, Yang, 2024; Caron-Huot, Correia, Giroux, 2024; Correia, Giroux, Mizera, 2026] + direct analysis of the DE denominators + via the algorithm described in [Matijašić, 2024]
- **Non-dlog one-forms** $\omega_j(\vec{x})$ written as differentials of the variables

$$\omega_i(\vec{x}) = R_i(\vec{x}) \sum_{j=1}^{6} \omega_{ij}(\vec{x})\, \mathrm{d}x_j$$

topology	square roots	dlog letters	one-forms
PBD	6	77	0
PBE	9	113	177
PBF	11	129	222
Total	**13**	**167**	**389**

DIFFERENTIAL EQUATIONS STRUCTURE

Properties of our DEs

“Good” properties of the DEs

- Entries that do not couple to elliptic or nested-square-root sectors are in ε-factorised d log-form
- Non-canonical entries are at worst quadratic in ε and are written in terms of independent one-forms
- The elliptic/nested-square-root MIs start at $O(\varepsilon^4)$ → **analytic cancellation of poles in the amplitude**

STATE OF THE ART

Achieving a canonical form

- We adopt the strategy successfully applied to the computation of $\bar{t}tj$ and $\bar{t}tW$: **retain algebraic DEs while minimising the deviation from the canonical form** [Badger, Becchetti, Giraudo, Zoia, 2024; Becchetti, Canko, Chestnov, Peraro, Pozzoli, Zoia, 2025]

- Significant progress has been made towards extending the notion of a **canonical form beyond d*log***'s

[Görges, Nega, Tancredi, Wagner, 2023; Dlapa, Henn, Wagner, 2023; Frellesvig, Weinzierl, 2024; ε-collaboration, 2025; Duhr, Maggio, Nega, Sauer, Tancredi, Wagner, 2025; Chen, Yang, Zhang, 2025; Forner, Mella, Nega, Tancredi, Wagner, 2026]

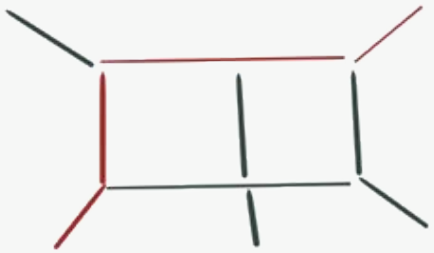
[See V. Sotnikov’s talk]

- PBB has been brought into **canonical form** — the first five-point family with an elliptic curve [Becchetti, Dlapa, Zoia, 2025] — but the complexity notably increases for PBE and PBF

The nested-square-root and the elliptic sectors

NESTED SQUARE-ROOT SECTORS

$I^{(PBE)}_{11101111000}$



#MIs = 7

+ permutation ($I^{(PBF)}_{11101111000}$)

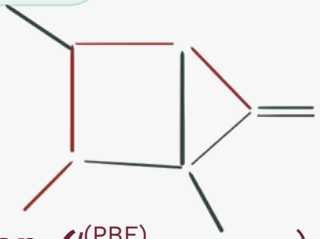
$I^{(PBF)}_{11111101000}$



#MIs = 3

ELLIPTIC SECTORS

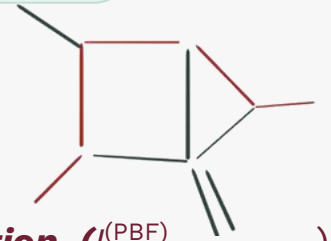
$I^{(PBE)}_{11001111000}$



#MIs = 3

+ permutation ($I^{(PBF)}_{11101101000}$)

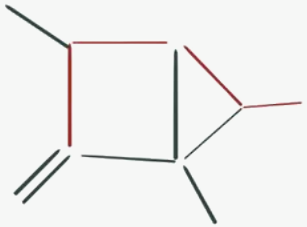
$I^{(PBE)}_{10101111000}$



#MIs = 3

+ permutation ($I^{(PBF)}_{11101011000}$)

$I^{(PBF)}_{11011101000}$



#MIs = 4

The nested-square-root case: an explicit example

BAIKOV REPRESENTATION OF THE CORNER INTEGRAL

$$I^{(\mathrm{PB}_E)}_{11101111000}\Big|^{\mathrm{M.C.}\setminus\{D_1\}}_{\varepsilon\to 0} \propto \frac{a_1' r_1}{\sqrt{d_{23}}} \int \mathrm{d}z_1\,\mathrm{d}z_4\,\mathrm{d}z_9\,\frac{1}{z_1\sqrt{\mathcal{P}_1}\,\mathcal{P}_3}$$

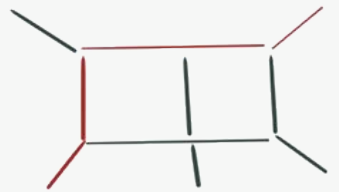
with

$$r_1 = \sqrt{\mathbb{G}(p_1,p_2,p_3,p_4)}$$

$$\mathcal{P}_1 = p_1(\vec{x},z_1,z_4,z_9)\ ,$$

$$\mathcal{P}_3 = \mathcal{P}_2\,\overline{\mathcal{P}_2} = (p_2(\vec{x},z_1,z_4,z_9) + p_3(\vec{x},z_1,z_4,z_9)\cdot r_1)\ (p_2(\vec{x},z_1,z_4,z_9) - p_3(\vec{x},z_1,z_4,z_9)\cdot r_1)$$

$I^{(\mathrm{PBE})}$ 11101111000



#MIs = 7

SECTOR BASIS

$$\mathcal{J}_5^{(\mathrm{PB}_E)} = \frac{\varepsilon^4}{d_{45}} I^{(\mathrm{PB}_E)}_{11101111000}\,[\mathcal{P}_2]$$

$$\mathcal{J}_6^{(\mathrm{PB}_E)} = \frac{\varepsilon^4}{d_{45}} I^{(\mathrm{PB}_E)}_{11101111000}\,[\overline{\mathcal{P}}_2]$$

→ $\mathcal{J}_i = \frac{1}{LS(\vec{x})} \int \mathrm{dlog}\wedge\ldots\wedge\mathrm{dlog}$

LS contains nested-square-roots

$$n_2^{\pm} = \frac{\sqrt{d_{23}(a_2' \mp d_{15}r_1)(a_2' d_{23} - 2a_1' m_t^2 - d_{15}d_{23}r_1)}}{\pm r_1}\,.$$

[Becchetti, Canko, Chestnov, Peraro, Pozzoli, Zoia, 2025]

We choose a basis that does not introduce nested-square-roots in the DEs

$$\mathcal{J}_5^{(\mathrm{PB}_E)} = \frac{\varepsilon^4}{d_{45}} I^{(\mathrm{PB}_E)}_{11101111000}\left[\frac{\mathcal{P}_2 - \overline{\mathcal{P}}_2}{2}\right]$$

$$\mathcal{J}_6^{(\mathrm{PB}_E)} = \frac{\varepsilon^4}{d_{45}} I^{(\mathrm{PB}_E)}_{11101111000}\left[\frac{\mathcal{P}_2 + \overline{\mathcal{P}}_2}{2}\right]$$

The elliptic case: an explicit example

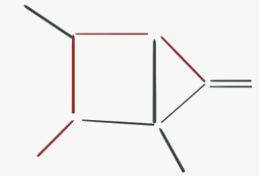
BAIKOV REPRESENTATION OF THE CORNER INTEGRAL

$$\mathcal{J}^{\text{elliptic}}\Big|^{\text{M.C.}}_{\varepsilon\to 0} \propto \int \frac{\mathrm{d}z}{\sqrt{y(z;\vec{x})}}\,,$$

with the elliptic curve:

$$y(z;\vec{x}) = z(z-4m_t^2)d_{23}\left[4d_{13}m_t^2(2d_{45}-z)+d_{23}(4d_{13}^2-4(d_{13}+m_t^2)z+z^2)\right]\,.$$

$I^{(\text{PBE})}_{11001111000}$



#MIs = 3

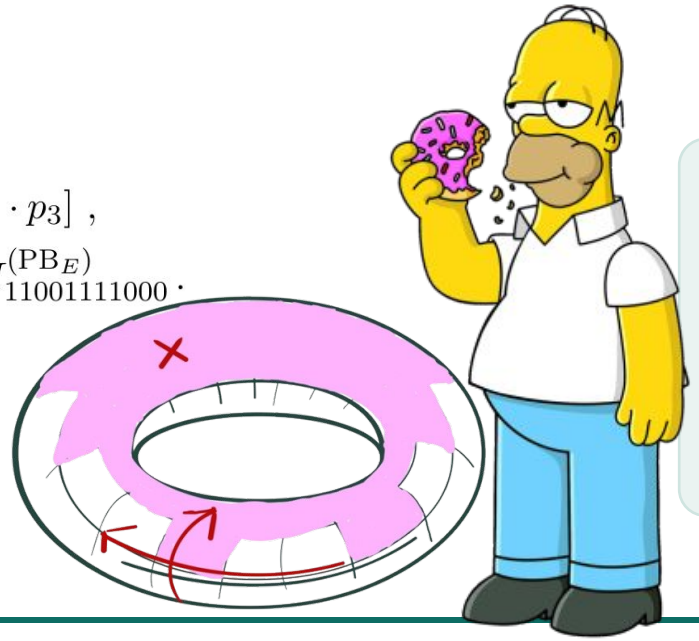
SECTOR BASIS

$$\mathcal{J}^{(\text{PB}_E)}_{39} = \varepsilon^4 d_{23} I^{(\text{PB}_E)}_{11001111000}\left[2\,k_2\cdot p_1\right]\,,$$
$$\mathcal{J}^{(\text{PB}_E)}_{40} = \varepsilon^4 (1-2\varepsilon) d_{23} I^{(\text{PB}_E)}_{11001111000}\left[2\,k_1\cdot p_3\right]\,,$$
$$\mathcal{J}^{(\text{PB}_E)}_{41} = \varepsilon^4 d_{23}(d_{12}+d_{23}-d_{45}+m_t^2)\, I^{(\text{PB}_E)}_{11001111000}\,.$$

↓

vanish up to order ε^3 (numerical check)

[Badger, Becchetti, Giraudo, Zoia, 2024]



ε-dependence of our connection matrix on the maximal cut:

$$\begin{pmatrix} *+*\varepsilon & * & *\varepsilon \\ (2\varepsilon-1)(*+*\varepsilon) & *+*\varepsilon & (2\varepsilon-1)*\varepsilon \\ *+*\varepsilon & * & *\varepsilon \end{pmatrix}$$

Image: © Matt Groening / 20th Television

A potentially over-complete basis of special functions

$$\vec{\mathcal{J}}^{(\mathrm{F})}(\vec{x},\varepsilon) = \sum_{k=0}^{4} \varepsilon^k \, \vec{\mathcal{J}}^{(\mathrm{F},k)}(\vec{x}) + \mathcal{O}\left(\varepsilon^5\right)$$

linearly independent by construction

REDUNDANT

Enough to get the **finite remainder** of the amplitude

A big thanks for this check to H. B. Hartanto!

Construction of a **generating set of algebraically independent special functions:** $f_j^{(\mathrm{F},k)}(\vec{x})$ – following the procedure of [Badger, Becchetti, **CB**, Hartanto, Zoia, 2025] – in terms of which we can write the MIs.

- MIs satisfying canonical DEs can be expressed in terms of **Chen iterated integrals**, enabling an algorithmic special functions basis derivation

[Gehrmann, Henn, Lo Presti, 2018] [Chicherin, Sotnikov, 2020] [Chicherin, Sotnikov, Zoia 2022] [Abreu, Chicherin, Ita, Page, Sotnikov, Tschernow, Zoia, 2024]

- **Non-polylogarithmic functions** $f_i^{(\mathrm{F},4^*)}(\vec{x})$ are pushed in the **finite part** of the amplitude

  ↓

  analytic UV/IR pole subtraction

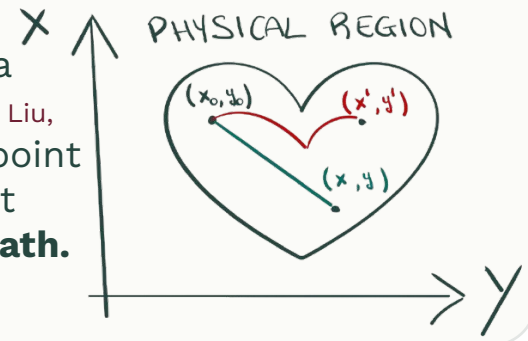
- Special functions basis derivation for each family

family	MI	$f^{(\mathrm{F},1)}$	$f^{(\mathrm{F},2)}$	$f^{(\mathrm{F},3)}$	$f^{(\mathrm{F},4)}$	$f^{(\mathrm{F},4^*)}$	all $f^{(\mathrm{F},k)}$
PB_D	100	6	10	46	85	0	147
PB_E	154	6	10	53	110	13	192
PB_F	142	6	10	51	102	16	185

Numerical evaluation strategy

NUMERICAL INTEGRATION CODE

- **Numerical integration** of the DEs derived for the special functions of each family;
- Usage of **Bulirsch–Stoer** [Hairer, Nørsett, Wanner, 2000], as implemented in the Odeint library of Boost [Boost C++ Libraries], already used for similar applications [Czakon, Niggetiedt, 2020; Petit Rosàs, Torres Bobadilla, 2025; Badger, Becchetti, C.B., Czakon, Hartanto, Poncelet, Zoia, 2025; Pozzoli, Torres Bobadilla, 2026; Götzfried, von Manteuffel, 2026; Czakon, Tancredi, 2026];
- We integrate from a boundary $\vec{x}_0$ [AMFlow, Liu, Ma, 2022] to a target point $\vec{x}$, through straight **line or** a **physical path.**



OPTIMISING THE NUMERICAL EVALUATION OF THE DERIVATIVES

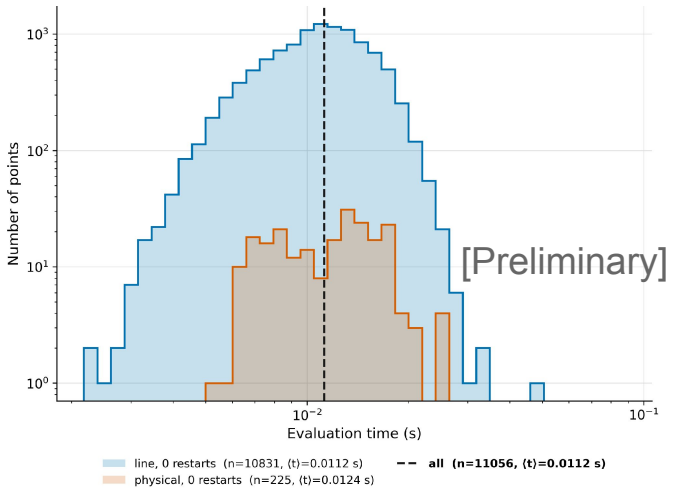
- We set $\boxed{m_t^2 = 1}$;
- We choose the variables

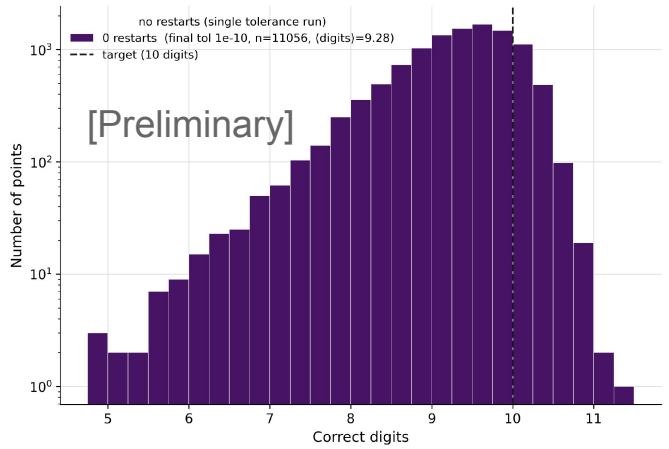
$$\boxed{\vec{y} = (d_{13}, d_{23}, d_{24}, d_{45}, d_{15})};$$

- We decompose the one-form coefficients into **multivariate partial fractions** using MultivariateApart [Heller, von Manteuffel, 2022]
- We write the partial fractions in terms of a global set of **multiplicatively independent polynomials**;
- We write the right-hand side of the DEs in terms of a set of **independent polynomials in the special functions.**

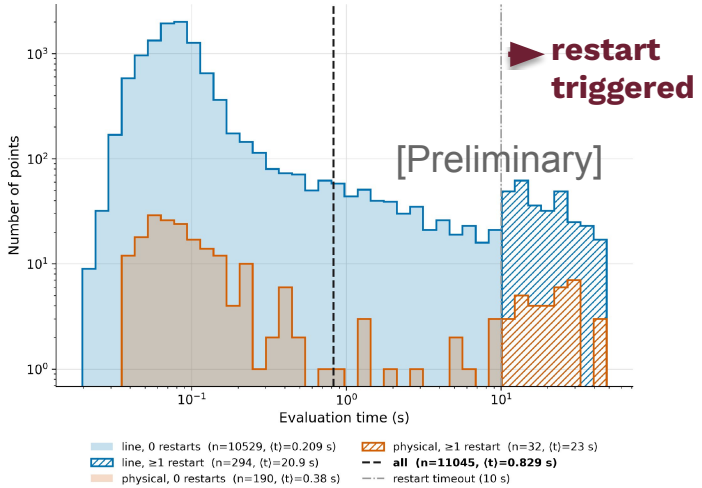
Special functions evaluation benchmarks

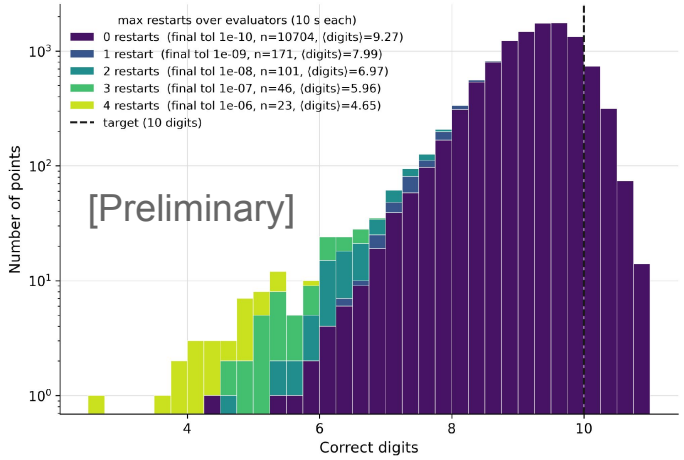
PBD ⟨t⟩ = **0.011 s**



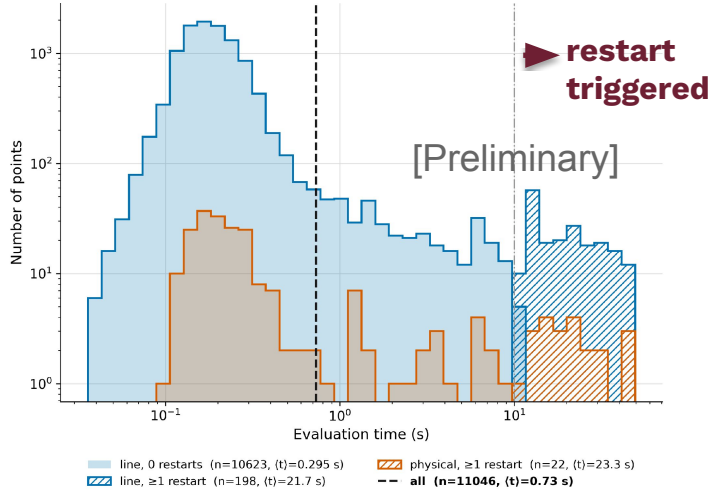


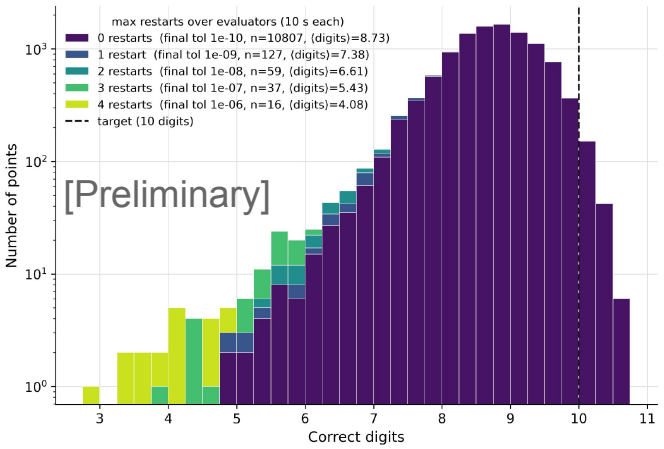
PBE ⟨t⟩ = **0.83 s**





PBF ⟨t⟩ = **0.73 s**





Conclusions

- Computation of the two-loop Feynman integrals for **NNLO QCD corrections to pp → tt̄γ** in the leading-colour approximation

- Construction of a potentially overcomplete set of **special functions** to achieve
  - analytic UV/IR pole subtraction
  - simplification of the finite remainder

- **Efficient numerical evaluation**, hopefully enough for phenomenological applications



Image: © Matt Groening / 20th Television

Outlook

- Two-loop leading-colour **amplitude** computation

Is what we have for the numerical evaluation of the integrals good enough for phenomenological applications?

What is the best way to proceed with the amplitude computation?

Conclusions

- Computation of the two-loop Feynman integrals for **NNLO QCD corrections to pp → tt̄γ** in the leading-colour approximation

- Construction of a potentially overcomplete set of **special functions** to achieve
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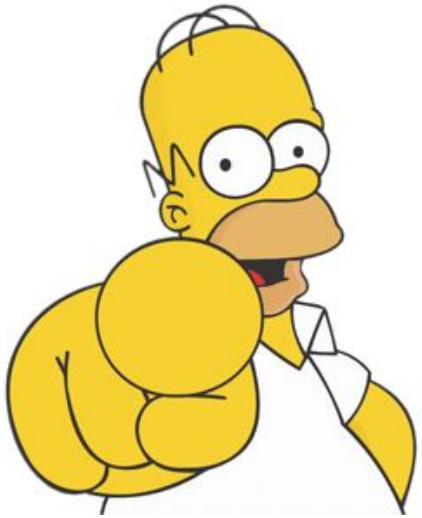


Image: © Matt Groening / 20th Television

Outlook

- Two-loop leading-colour **amplitude** computation

Thank you for your attention!!!