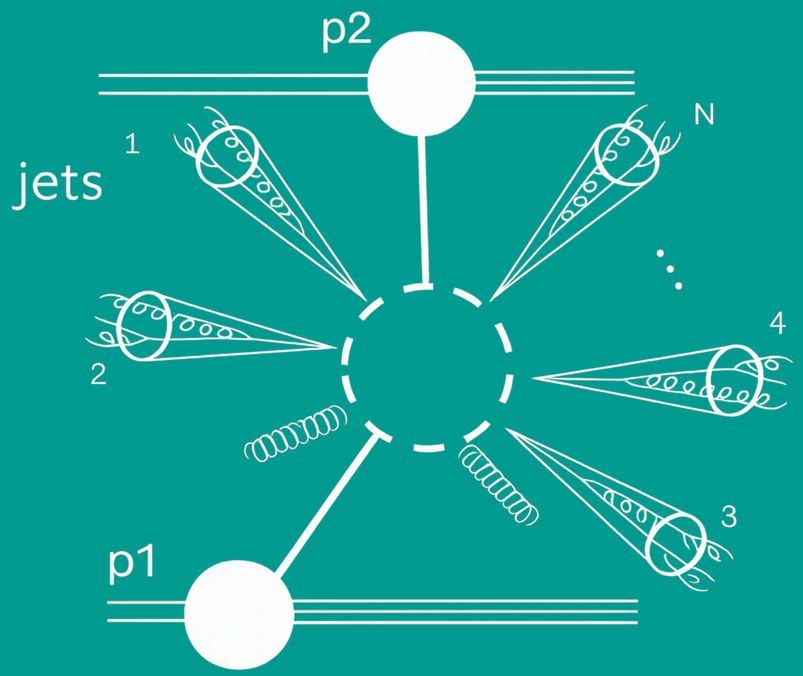


N-jettiness soft function: Towards arbitrary collider processes at NNLO QCD

Prem Agarwal

High Precision for Hard Processes (HP2) 2026

In collaboration with Kirill Melnikov, Ivan Pedron and Andrey Pikelner

Based on **2610.06040** **Appeared today!**

Subtraction and Slicing

- Substantional progress in providing precise predictions of cross sections at NNLO and N3LO QCD in the last few years, especially in handling IR divergences.

Subtraction Schemes

$$\int d^d\Phi |\mathcal{M}|^2 F_M = \int d^4\Phi (|\mathcal{M}|^2 F_M - S) + \int d^d\Phi S$$

Slicing Schemes

$$\int d^d\Phi \, |\mathcal{M}|^2 F_M = \int_0^\delta d^d\Phi \, |\mathcal{M}|^2 F_M + \int_\delta d^4\Phi \, |\mathcal{M}|^2 F_M + \mathcal{O}(\delta)$$

- Many subtractions and slicing schemes at NNLO and beyond: **See Chiara's talk**

Recent developments in NNLOCAL
Speaker
Gabor Somogyi (HUN-REN Wigner Research Centre for Physics)

Local Analytic Sector Subtraction with initial-state hadrons: Infrared kernels and their integration
Speaker
Yashasvee Goel (University of Torino and INFN Torino)

A general subtraction formula at NNLO in QCD with massless partons
Speaker
Matteo Tresoldi (KIT TTP)

Many examples in this conference

Gluon Jet Function for transverse-momentum-like variables in NNLO QCD
Speaker
Giovanbattista Favorito (University of Zurich)

Collinear functions up to NNLO in QCD
Speaker
Prasanna Dhani (University of Zurich)

Precise predictions for Higgsstrahlung at future colliders
Speaker
Aude Gehrmann-De Ridder (ETH Zurich)

N-jettiness Slicing

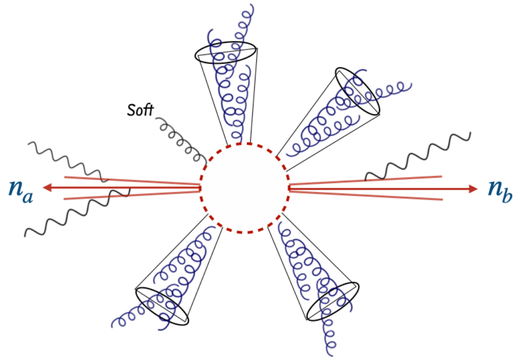
[Implemented in GENEVA framework, MiNNLOPS, MCFM]

[Gaunt et al.'15] [Boughezal et al.'15]

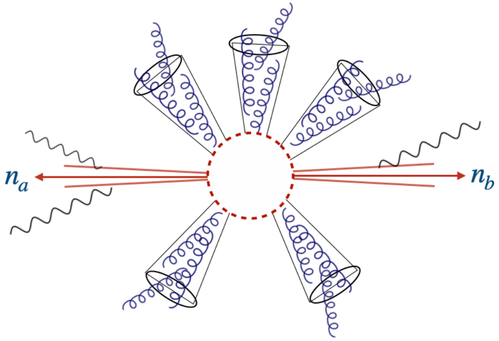
N-jettiness variable: $\mathcal{T}_N = \sum_k \min\left(\frac{2p_1 \cdot p_k}{P_1}, \frac{2p_2 \cdot p_k}{P_2}, \ldots, \frac{2p_N \cdot p_k}{P_N}\right)$ k unresolved momenta N final state jets

[Stewart, Tackmann, Waalewijn '10]

$$\sigma = \int_0^{\tau_{\text{cut}}} d\tau \frac{d\sigma}{d\tau} + \int_{\tau_{\text{cut}}} d\tau \frac{d\sigma}{d\tau}$$



$\mathcal{T}_N \to 0$



$\mathcal{T}_N \gg \tau_{\text{cut}}$

$$\int_0^{\tau_{\text{cut}}} d\tau \frac{d\sigma}{d\tau} = \int B_\tau \otimes B_\tau \otimes S_\tau \otimes H_\tau \otimes \prod^N J_{i,\tau} + \mathcal{O}(\tau_{\text{cut}})$$

- Hard Function H_τ
- Beam function B_τ, Jet function $J_{i,\tau}$
- **Soft Function** S_τ

See talk by Kevin, Giovanbattista, Prasanna

- Factorization violation beyond 4 loops — See talk by Thomas Becher

N-jettiness Slicing at NNLO: What do we know

$$\int_0^{\tau_{\text{cut}}} d\tau \frac{d\sigma}{d\tau} = \int B_\tau \otimes B_\tau \otimes S_\tau \otimes H_\tau \otimes \prod^{N} J_{i,\tau} + \mathcal{O}(\tau_{\text{cut}})$$

- Beam functions B and Jet functions J_i are known up to N3LO. *[Ebert, Mistleberger, Vita‘ 20] [Brüser, Liu, Stahlhofen‘18] [‘Baranowski, Behring, Melnikov, Tancredi, Wever’ 22]*
- Process dependent Hard function H encodes virtual corrections.

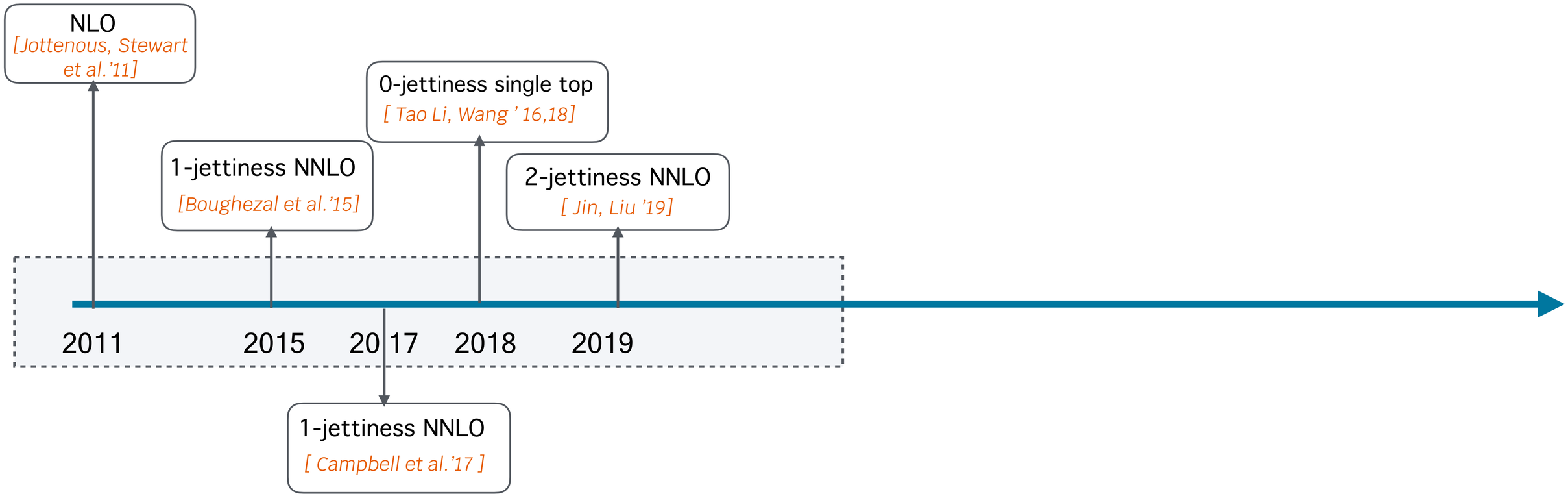
  — See various talks in this conference for progress at NNLO

- Soft function S : *[Bell,Dehnadi,Mohnrmann, Rahn ’23] [P.A., Melnikov, Pedron ’24]*
  - — Known for a generic N for massless emitters at NNLO *[Buonocore, Delto, Melnikov, Monni, Pikelner, Vita’26]*
  - — At N3LO, Zero-jettiness soft function is known. *[Baranowski, Delto, Melnikov, Pikelner, Wang ’24]*
  - — For massive emitters at NNLO: Zero-jettiness single top production *[Tao Li, Wang ’ 16,18]*

    Very recent calculation of zero-jettiness heavy quark pair production *[Bell, Broggio, Edelmann, Lim, Rahn ’26]*

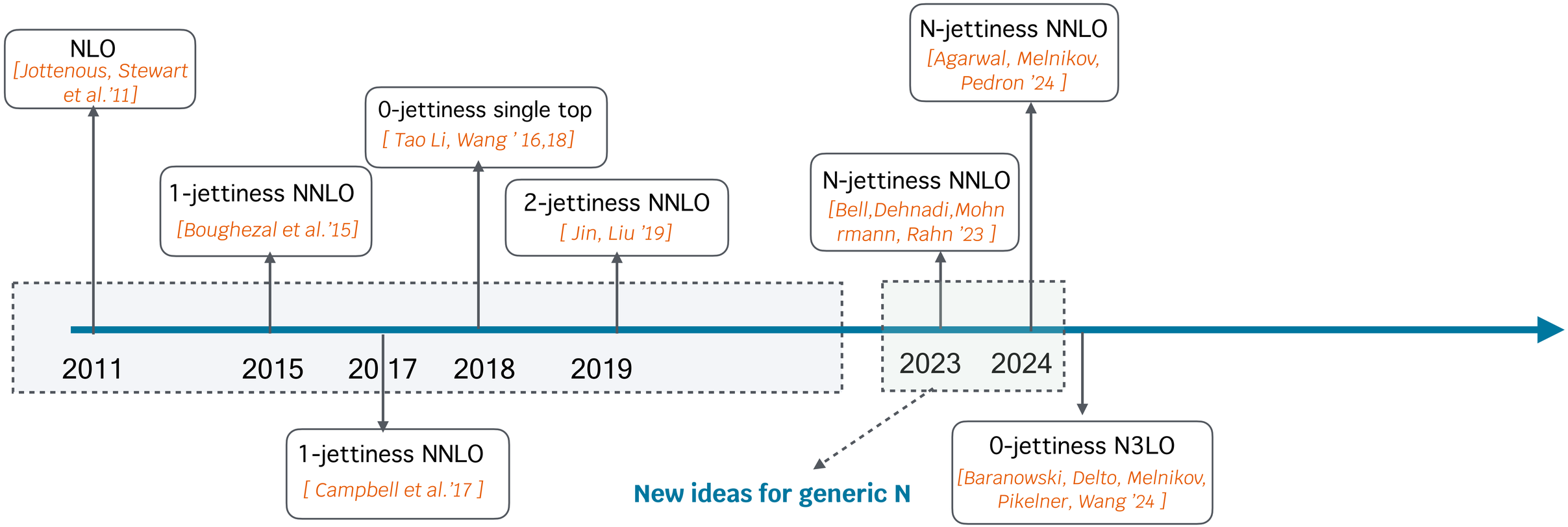
Missing at NNLO: Soft function with massive partons and additional jets

Progress in N-jettiness soft function

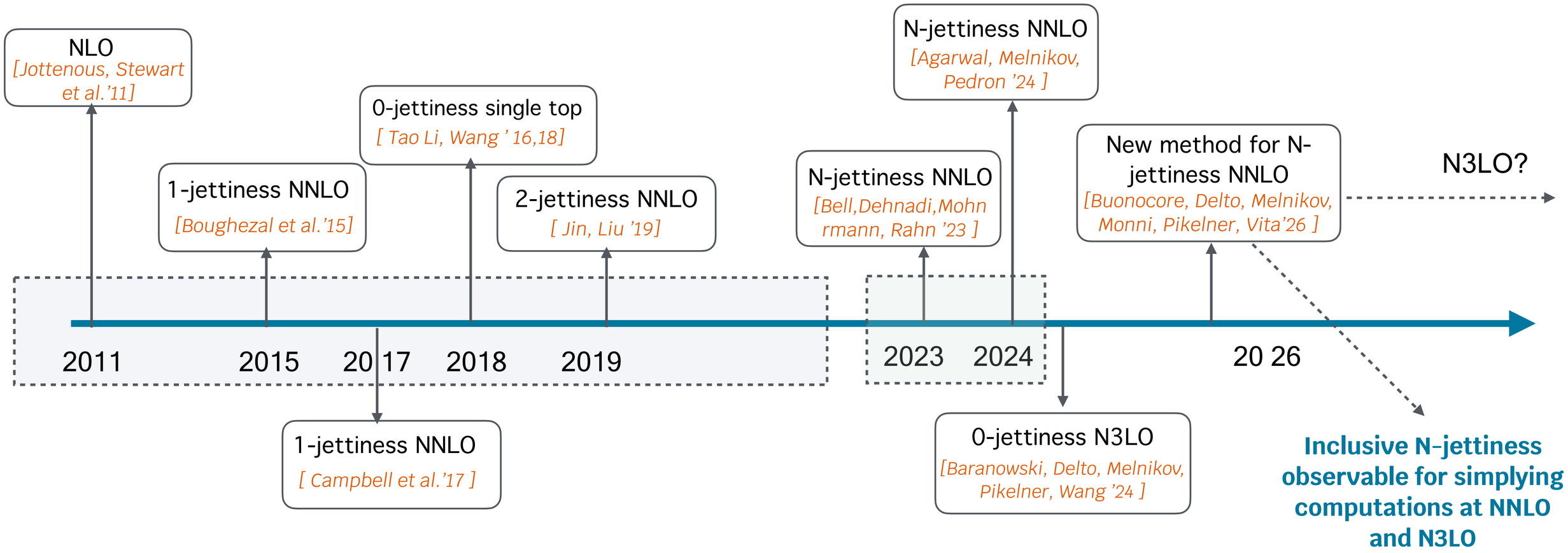


Phase-space partitions using Heaviside functions

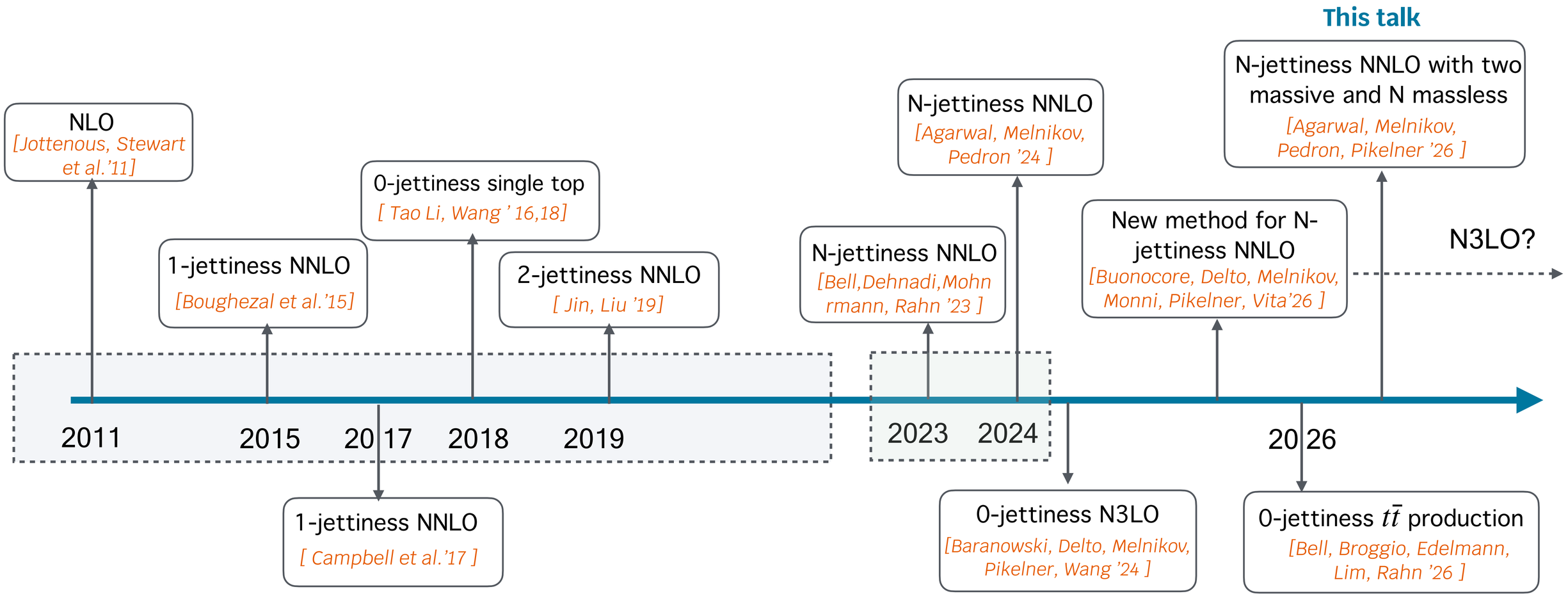
Progress in N-jettiness soft function



Progress in N-jettiness soft function



Progress in N-jettiness soft function



In this work we compute the soft function at NNLO for two massive and N massless emitters ($pp \to Q\bar{Q} + N$ jets) using the new ideas in the last few years

Renormalized soft function at NNLO

- We now begin discussing the calculation of the soft function. Its bare result is of the form:

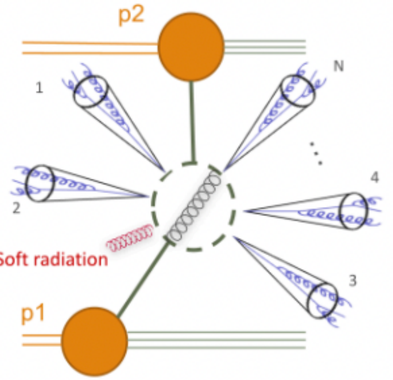
$$S(\tau) = \delta(\tau) + S_1(\tau) + S_2(\tau) + \cdots$$

- The soft function is renormalized multiplicatively in Laplace space: $S = Z\,\tilde{S}\,Z^{+}$

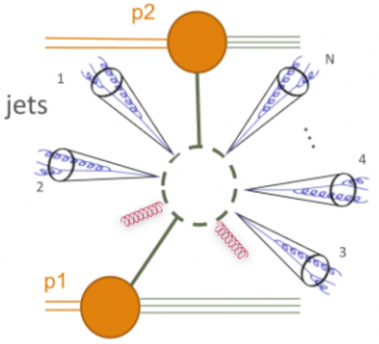
$$\tilde{S} = 1 + \tilde{S}_1 + \frac{1}{2}\tilde{S}_1^2 + S_2^{\text{dip}} + S_2^{\text{tri}} + \mathcal{O}(\alpha_s^3)$$

$$\tilde{S}_1 = S_1 - Z_1 - Z_1^{+}$$

Iterations of NLO contribution



Real-virtual



Double real

Renormalized soft function at NNLO

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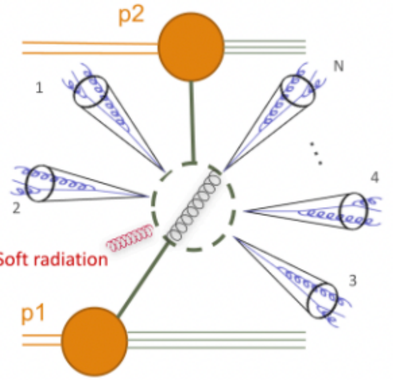
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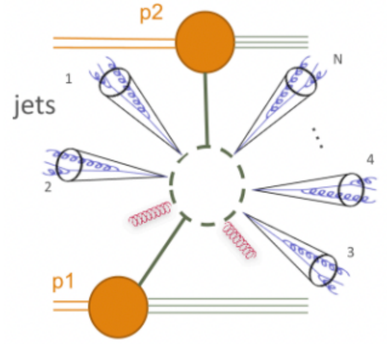
$$\tilde{S}_1 = S_1 - Z_1 - Z_1^{+}$$

Iterations of NLO contribution

$$\tilde{S}_2^{\text{dip}} = S_{RR,T^2} + S_{RV,T^2} - Z_{2,\text{dip}} - Z_{2,\text{dip}}^{+} - \frac{\beta_0\, a_s}{\epsilon} S_1$$



Real-virtual



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- We now begin discussing the calculation of the soft function. Its bare result is of the form:

$$S(\tau) = \delta(\tau) + S_1(\tau) + S_2(\tau) + \cdots$$

- The soft function is renormalized multiplicatively in Laplace space: $S = Z\,\tilde{S}\,Z^{+}$

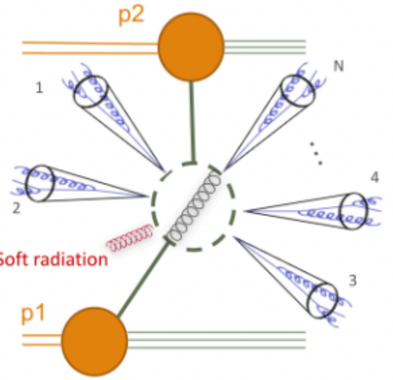
$$\tilde{S} = 1 + \tilde{S}_1 + \frac{1}{2}\tilde{S}_1^2 + S_2^{\text{dip}} + S_2^{\text{tri}} + \mathcal{O}(\alpha_s^3)$$

Iterations of NLO contribution

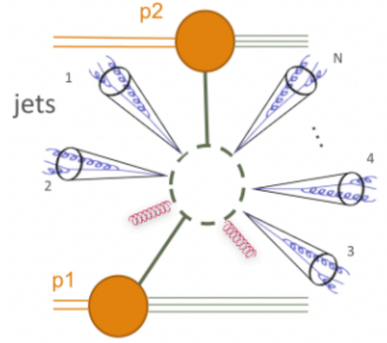
$$\tilde{S}_1 = S_1 - Z_1 - Z_1^{+}$$

$$\tilde{S}_2^{\text{dip}} = S_{RR,T^2} + S_{RV,T^2} - Z_{2,\text{dip}} - Z_{2,\text{dip}}^{+} - \frac{\beta_0\, a_s}{\epsilon} S_1$$

$$\tilde{S}_2^{\text{tri}} = S_{RV,T^3} + \frac{1}{2}[Z_1, Z_1^{+}] + \frac{1}{2}[S_{1,t}, Z_1 - Z_1^{+}] - Z_{2,\text{tri}} - Z_{2,\text{tri}}^{+}$$



Real-virtual



Double real

- We begin by discussing NLO followed by dipoles and tripoles.

Soft function for generic N: Main ideas

The N-jettiness soft function involving massless partons is known for generic N. There are a few main ideas that have made this possible, which can be further extended to massive partons.

- The N-jettiness observable has well-defined limits at the soft and collinear singularities of the eikonal function for a generic number of hard partons.
- This idea can be used to isolate singularities analytically and derive finite numerical representations involving the remenants of N-jettiness function.

Soft function for generic N: Main ideas

The N-jettiness soft function involving massless partons is known for generic N. There are a few main ideas that have made this possible, which can be further extended to massive partons.

- The N-jettiness observable has well-defined limits at the soft and collinear singularities of the eikonal function for a generic number of hard partons.
- This idea can be used to isolate singularities analytically and derive finite numerical representations involving the remenants of N-jettiness function.
- To simplify the calculation, we employ the recently proposed *[Buonocore, Delto, Melnikov, Monni, Pikelner, Vita'26]* Inclusive N-jettiness observable:

$$\mathcal{T}_N = \sum_k \min\left(\frac{p_1\cdot p_k}{Q_1}, \frac{p_2\cdot p_k}{Q_2}, \ldots, \frac{p_N\cdot p_k}{Q_N}\right) \quad \rightarrow \quad T_N = \min\left(\frac{p_1\cdot q}{Q_1}, \frac{p_2\cdot q}{Q_2}, \ldots, \frac{p_N\cdot q}{Q_N}\right) \quad \text{where} \quad q = \sum_{k=1} p_k$$

Regular N-jettiness — Inclusive N-jettiness

— At NNLO, the two observables coincide at the soft and collinear limits when the double-real eikonal function diverges.

— We can replace the regular N-jettiness observable with the inclusive one, since their difference is finite.

Soft function at NLO: Parametrization

- We first illustrate these ideas at NLO. In this case, the regular and inclusive N-jettiness variable are the same.

Bare soft function: $$S_1(\tau) = -\, g_s^2 \sum_{\mathrm{IJ}} \mathbf{T}_\mathrm{I} \cdot \mathbf{T}_\mathrm{J} \int [\mathrm{d}k_1]\, \delta(\tau - T_N)\, \tilde{S}_{\mathrm{IJ}}(k_1)$$

- We select the light cone versions of the momenta as basis vectors and employ the sudakov decomposition

$$p_\mathrm{I} = \tilde{p}_\mathrm{I} + \kappa_{\mathrm{IJ}}\, \tilde{p}_\mathrm{J}, \qquad p_\mathrm{J} = \tilde{p}_\mathrm{J} + \kappa_{\mathrm{JI}}\, \tilde{p}_\mathrm{I} \quad \Rightarrow \quad k_1 = \alpha \tilde{p}_\mathrm{I} + \beta \tilde{p}_\mathrm{J} + k_{1,\perp}$$

- Using this, we obtain the suitable phase-space parametrization for a generic parton in terms of sudakov parameters α , β and x as

$$\mathrm{d}^{\mathrm{d}}\mathrm{k}_1 = \Omega^{(\mathrm{d}-2)} \frac{\tilde{\mathrm{p}}_\mathrm{I}\tilde{\mathrm{p}}_\mathrm{J}}{2} \left(2\tilde{\mathrm{p}}_\mathrm{I}\tilde{\mathrm{p}}_\mathrm{J}\alpha\beta\right)^{1-\epsilon} \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{dx}\, (1-\mathrm{x})^{-\epsilon}\, [\mathrm{d}\Omega_\perp]$$

$$[\mathrm{d}k_1] = \frac{\Omega^{(d-2)}}{2(2\pi)^{d-1}} \left(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J}\right) \left(2\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J}\right)^{-\epsilon} \mathrm{d}\alpha\, \mathrm{d}\beta\, \mathrm{d}x\, (\alpha\beta)^{-\epsilon}\, \delta(x)\, [\mathrm{d}\Omega_\perp]$$

- The N-jettiness function T_N is homogeneous in sudakov parameters α and β. Using this property, we integrate in one of the sudakov variables by introduce the ordering

$$1 = \theta(\alpha - \beta) + \theta(\beta - \alpha) \qquad \beta = \xi\, \alpha, \quad \alpha = \xi\, \beta$$

Soft function at NLO

- Upon integration and performing Laplace transform, the NLO contribution is of the form

$$S_{1,\mathrm{IJ}} = \sum_{\mathrm{IJ}} \mathbf{T}_\mathrm{I}\cdot\mathbf{T}_\mathrm{J}\frac{N_1(u)}{\epsilon}\left(\frac{\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J}}{2}\right)^{\epsilon}\int_0^1 d\xi\ \xi^{-1-\epsilon}\,[d\Omega^{(d-2)}]\left[\left(\frac{T_{N,\mathrm{IJ}}(1,\xi,0)}{(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})}\right)^{2\epsilon} R_{\mathrm{IJ}}(1,\xi) + \left(\frac{T_{N,\mathrm{IJ}}(\xi,1,0)}{(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})}\right)^{2\epsilon} R_{\mathrm{IJ}}(\xi,1)\right]$$

Collinear : $\xi \to 0$

- To extract this divergence, we normalize the N-jettiness function with its limit in $\xi \to 0$ such that

$$\left(\frac{T_{N,\mathrm{IJ}}(1,\xi,0)}{(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})}\right)^{2\epsilon} = (C_\mathrm{I} f_\mathrm{I}(\xi))^{2\epsilon}\left(\frac{T_{N,\mathrm{IJ}}(1,\xi,0)}{C_\mathrm{I} f_\mathrm{I}(\xi)(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})}\right)^{2\epsilon} \quad\Rightarrow\quad \left(\frac{T_{N,\mathrm{IJ}}(1,\xi,0)}{C_\mathrm{I} f_\mathrm{I}(\xi)(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})}\right)^{2\epsilon} = 1 + 2\epsilon\ g(\xi)$$

- This allows us to isolate the singualities and construct a finite numerical respresentation at NLO

Master formula for generic emitters:

$$S_{1,\mathrm{IJ}} = \sum_{\mathrm{IJ}} \mathbf{T}_\mathrm{I}\cdot\mathbf{T}_\mathrm{J}\left\{\frac{N_1(u)}{\epsilon}\left(\frac{\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J}}{2}\right)^{\epsilon}\int_0^1 \mathrm{d}\xi\ \xi^{-1-\epsilon}[d\Omega_\perp]\left(C_\mathrm{I}^{2\epsilon} f_\mathrm{I}^{2\epsilon} R_{\mathrm{IJ}}(1,\xi) + C_\mathrm{J}^{2\epsilon} f_\mathrm{J}^{2\epsilon} R_{\mathrm{IJ}}(\xi,1)\right)\right.$$

$$\left. +2\int_0^1 \frac{d\xi}{\xi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\left[\ln\left(\frac{T_{N,\mathrm{IJ}}(1,\xi,0)}{(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})C_\mathrm{I} f_\mathrm{I}(\xi)}\right) R_{\mathrm{IJ}}(1,\xi) + \ln\left(\frac{T_{N,\mathrm{IJ}}(\xi,1,0)}{(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})C_\mathrm{J} f_\mathrm{J}(\xi)}\right) R_{\mathrm{IJ}}(\xi,1)\right]\right\}$$

Finite numerical representation

Soft function at NLO: Results

Master formula for generic emitters:

$$S_{1,\mathrm{IJ}} = \sum_{\mathrm{IJ}} \mathbf{T}_\mathrm{I}\cdot\mathbf{T}_\mathrm{J}\left\{\frac{N_1(u)}{\epsilon}\left(\frac{\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J}}{2}\right)^{\epsilon}\int_0^1 \mathrm{d}\xi\,\xi^{-1-\epsilon}[d\Omega_\perp]\left(C_\mathrm{I}^{2\epsilon}f_\mathrm{I}^{2\epsilon}R_{\mathrm{IJ}}(1,\xi)+C_\mathrm{J}^{2\epsilon}f_\mathrm{J}^{2\epsilon}R_{\mathrm{IJ}}(\xi,1)\right)\right.$$
$$\left.+2\int_0^1\frac{d\xi}{\xi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\left[\ln\left(\frac{T_{N,\mathrm{IJ}}(1,\xi,0)}{(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})C_\mathrm{I}f_\mathrm{I}(\xi)}\right)R_{\mathrm{IJ}}(1,\xi)+\ln\left(\frac{T_{N,\mathrm{IJ}}(\xi,1,0)}{(\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J})C_\mathrm{J}f_\mathrm{J}(\xi)}\right)R_{\mathrm{IJ}}(\xi,1)\right]\right\}$$

Finite numerical representation

- Using this formula, we obtain the renormalized soft function at NLO for all cases of mass assignments

Two massless:

$$\tilde{S}_{1,ij} = a_s\left[\frac{1}{2}L_{ij}^2+\frac{1}{2}\ln^2\frac{Q_i}{Q_j}+\frac{\pi^2}{4}+\int_0^1\frac{d\xi}{\xi}\int_0^{2\pi}\frac{d\varphi}{2\pi}\ln\left(\frac{T_{N,ij}(1,\xi,0)T_{N,ij}(\xi,1,0)Q_iQ_j}{\xi^2(p_ip_j)^2}\right)\right]$$

One massive one massless:

$$\tilde{S}_{1,Ij} = a_s\left[\frac{1}{4}L_{Ij}^2-\frac{L_{Ij}}{2}+\frac{5\pi^2}{24}+\int_0^1\frac{\mathrm{d}\xi}{\xi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\left[\mathscr{R}_{Ij}(1,\xi)\ln\left(\frac{T_{N,Ij}(1,\xi,0)Q_j}{(p_Ip_j)}\right)+\mathscr{R}_{Ij}(\xi,1)\ln\left(\frac{T_{N,Ij}(\xi,1,0)Q_j}{\xi(p_Ip_j)}\right)\right]\right]$$

Two massive:

$$\tilde{S}_{1,\mathrm{IJ}} = \left[\frac{1}{v_{IJ}}\left\{-\left(\frac{1}{2}L_{v_{IJ}}+v_{IJ}\right)L_{IJ}+\frac{1}{4}L_{v_{IJ}}^2+\frac{v_{IJ}}{2}L_{v_{IJ}}+\frac{\pi^2}{6}-v_{IJ}\ln[(1+\kappa_{IJ})(1+\kappa_{JI})]+\mathrm{Li}_2(-\kappa_{IJ})+\mathrm{Li}_2(-\kappa_{JI})-\frac{1}{2}\ln\kappa_{IJ}\ln\kappa_{JI}\right.\right.$$
$$\left.\left.+v_{IJ}\int_0^1\frac{\mathrm{d}\xi}{\xi}\int_0^{2\pi}\frac{\mathrm{d}\varphi}{2\pi}\left[\mathscr{R}_{IJ}(1,\xi)\ln\left(\frac{T_{N,IJ}(1,\xi,0)Q}{\tilde{p}_I\tilde{p}_J}\right)+\mathscr{R}_{IJ}(\xi,1)\ln\left(\frac{T_{N,IJ}(\xi,1,0)Q}{\tilde{p}_I\tilde{p}_J}\right)\right]\right\}\right]$$

Dipole contributions at NNLO

$$\tilde{S}_2^{\text{dip}} = S_{RR,T^2} + S_{RV,T^2} - \frac{\beta_0\, a_s}{\epsilon} S_1 - Z_{2,\text{dip}} - Z_{2,\text{dip}}^{+}$$

The renormalization constants are known even for massive cases

[Neubert, Becher ' 09]

$$S_{RV,T^2} = [\alpha_s]\, 2^{-\epsilon} g_s^2 \sum_{\text{IJ}} \mathbf{T}_\text{I} \cdot \mathbf{T}_\text{J} \int [\mathrm{d}k_1]\, F_{RV,\text{IJ}}^{\text{dip}}\, \delta\,(\tau - T_N)$$

Related to the one-loop soft current

[Bierenbaum, Czakon, Mitov.'11,18]

Dipole contributions at NNLO

$$\tilde{S}_2^{\text{dip}} = S_{RR,T^2} + S_{RV,T^2} - \frac{\beta_0\, a_s}{\epsilon} S_1 - Z_{2,\text{dip}} - Z_{2,\text{dip}}^{+}$$

The renormalization constants are known even for massive partons

[Neubert, Becher ' 09]

$$S_{RV,T^2} = [\alpha_s]\, 2^{-\epsilon} g_s^2 \sum_{\text{IJ}} \mathbf{T}_\text{I} \cdot \mathbf{T}_\text{J} \int [\mathrm{d}k_1]\, F^{\text{dip}}_{RV,\text{IJ}}\, \delta\,(\tau - T_N)$$

Related to the one-loop soft current

[Bierenbaum, Czakon, Mitov.'11,18]

$$S_{RR,T^2} = g_s^4 \sum_{\text{IJ}} \mathbf{T}_\text{I} \cdot \mathbf{T}_\text{J} \int [\mathrm{d}k_1][\mathrm{d}k_2]\, \tilde{S}_{\text{IJ}}(k_1, k_2)\, \delta(\tau - \mathcal{T}_N)$$

- We use the Inclusive N-jettiness approach: $S_{RR,T^2} = S^{\text{inc}}_{RR,T^2} + \Delta S_{RR}$

The difference term is finite at NNLO and can be computed without an infrared regulator.

- Only need to focus on the Inclusive contribution: 'simpler' to compute

$$S^{\text{inc}}_{RR,T^2} = g_s^4 \sum_{(\text{IJ})} \mathbf{T}_\text{I} \cdot \mathbf{T}_\text{J} \int [\mathrm{d}p_m][\mathrm{d}p_n]\, \tilde{S}_{\text{IJ}}(m,n)\, \delta\big(\tau - T_N\,(p_m + p_n)\big) \qquad T_{N,\text{IJ}} = \min\left(\frac{p_1\cdot q}{Q_1}, \frac{p_2\cdot q}{Q_2}, \ldots, \frac{p_N\cdot q}{Q_N}\right)$$

Dipole RR contribution: Inclusive approach

- **Inclusive RR contribution:** $$S^{\text{inc}}_{RR,T^2} = g_s^4 \sum_{(\text{IJ})} \mathbf{T}_\text{I} \cdot \mathbf{T}_\text{J} \int [\mathrm{d}p_m][\mathrm{d}p_n]\, \tilde{S}_\text{IJ}(m,n)\, \delta\big(\tau - T_N(p_m + p_n)\big)$$

$$\int [\mathrm{d}p_m][\mathrm{d}p_n]\, \tilde{S}_\text{IJ}(m,n)\, \delta\big(\tau - T_N(p_m + p_n)\big) = \int \mathrm{d}^d q\, F^{\text{inc}}_\text{IJ}(q)\, \delta\big(\tau - T_N(q)\big)$$

$$F^{\text{inc}}_\text{IJ}(q) = \int [\mathrm{d}p_m][\mathrm{d}p_n]\, \delta^{(d)}\big(q - p_m - p_n\big)\, \tilde{S}_\text{IJ}(p_m, p_n)\,.$$

Simpler to compute in the COM frame of q

- The calculation reduces to a one phase-space integral

$$\int \mathrm{d}^d q\, F^{\text{inc}}_\text{IJ}(q)\, \delta\big(\tau - T_N(q)\big) \qquad q \propto \alpha, \beta, x$$

Proceed as NLO

Introduce $\beta = \xi\,\alpha$ → Integrate over α → $$\left(\frac{T_{N,\text{IJ}}(1,\xi,x)}{\tilde{p}_\text{I}\tilde{p}_\text{J}\, f_\text{I}(\xi)}\right)^{4\epsilon}$$

- The soft and collinear singularities appear as divergences in the introduced parameters x and ξ

Dipole RR contribution: Inclusive approach

- To extract the divergences in x and ξ, we use ideas from subtractions to express the remenant of N-jettiness function in terms of three contributions, regulating one divergence at a time.

$$\left(\frac{T_{N,\mathrm{IJ}}(1,\xi,x)}{\tilde{p}_\mathrm{I}\tilde{p}_\mathrm{J}f_\mathrm{I}(\xi)}\right)^{4\epsilon} = 1 + 4\epsilon\, g^{(4)}_{\mathrm{IJ}}(\xi,0) + 4\epsilon\, \Delta g^{(4)}_{\mathrm{IJ}}(\xi,x)$$

Analytical term. Divergences after integrating in x and ξ cancel upon combining with RV and renormalization terms.

ξ regulated. Divergences in x cancel upon combining with RV and S_1 contributions.

This term is fully regulated in x and ξ

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- This allows us to achieve analytic pole cancellation and express the renormalized dipole contribution into three finite terms.

$$S^{\mathrm{inc}}_{2,\mathrm{dip}} = S^{\mathrm{inc}}_{2,\mathrm{dip}}\big|_{g=0} + S^{\mathrm{inc}}_{2,\mathrm{dip}}\big|_{g(\xi,0)} + S^{\mathrm{inc}}_{2,\mathrm{dip}}\big|_{\Delta g(\xi,x)}$$

Dipole contributions: Two massless emitters

- Fow two massless emitters, we reproduce the known result.

  We present a more general result as a function for arbitrary normalization Q_i of the N-jettiness function.

- The implementation involves a simple analytic term along with terms requiring a numerical integration over 2 or 3 variables.

$$\tilde{W}_{ij}^{(2),\mathrm{inc}} = \left(\frac{11}{36}C_A - \frac{T_R\, n_f}{9}\right) L_{ij}^3 + \left[\left(\frac{67}{36} - \frac{\pi^2}{12}\right) C_A - \frac{5}{9} T_R\, n_f\right] \left(L_{ij}^2 + \ln^2\left(\frac{Q_i}{Q_j}\right)\right)$$
$$+ \left[\left(\frac{101}{27} + \frac{11\pi^2}{36} + \frac{11}{12}\ln^2\left(\frac{Q_i}{Q_j}\right) - \frac{7}{2}\zeta_3\right) C_A - \left(\frac{28}{27} + \frac{\pi^2}{9} - \frac{1}{3}\ln^2\left(\frac{Q_i}{Q_j}\right)\right) T_R n_f\right] L_{ij}$$
$$- \frac{5\pi^4}{72} C_A + \left(\frac{335}{432} C_A - \frac{25}{108} T_R n_f\right) \pi^2 + \left(\frac{121}{36} C_A - \frac{11}{9} T_R n_f\right) \zeta_3 + \left(\frac{607}{162} C_A - \frac{82}{81} T_R n_f\right)$$
$$+ \int_0^1 \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \left\{ - \frac{\tilde{F}_{ij}^{(0)}(x)}{x} \ln\left(\frac{t_i(\xi,x)\, t_j(\xi,x)}{t_i(\xi,0)\, t_j(\xi,0)}\right) + \left[\frac{11}{12} C_A - \frac{1}{3} T_R n_f\right] \right.$$
$$\times \left(\ln^2\left(t_i(\xi,0)\, t_j(\xi,0)\right) + \ln^2\left(\frac{t_i(\xi,0)}{t_j(\xi,0)}\right)\right) + \ln\left(t_i(\xi,0)\, t_j(\xi,0)\right)$$
$$\times \left[\left(\frac{67}{18} - \frac{\pi^2}{6} + \frac{11}{6} L_{ij} + \frac{11}{6}\ln\xi\right) C_A - \left(\frac{10}{9} + \frac{2}{3} L_{ij} + \frac{2}{3}\ln\xi\right) T_R n_f\right]$$
$$\left. + \ln\left(\frac{t_i(\xi,0)}{t_j(\xi,0)}\right) \left(\frac{11}{6} C_A - \frac{2}{3} T_R n_f\right) \ln\left(\frac{Q_j}{Q_i}\right) \right\},$$

$$t_i(\xi,x) = \frac{Q_i T_{N,ij}(1,\xi,x)}{(p_i p_j)\xi} \qquad t_j(\xi,x) = \frac{Q_j T_{N,ij}(\xi,1,x)}{(p_i p_j)\xi} \qquad L_{ij} = \ln\left(\frac{\bar{u}^2 \mu^2\, p_i p_j}{2 Q_i Q_j}\right)$$

Dipole contributions: One massive one massless emitters

- The case of one massive and one massless emitter is more complex since it involves integrations over complex hypergeometric functions.

Dipole contributions: One massive one massless emitters

- The case of one massive and one massless emitter is more complex since it involves integrations over complex hypergeometric functions.
- The analytic part simplifies drastically after combining all contributions!
- We obtain a very simple result for the case involving one massive one massless emitter as well.

$$
\begin{aligned}
\tilde{W}_{Ij}^{(2),\mathrm{inc}} = {} & \frac{1}{2}\Bigg[\left(\frac{11}{36}C_A - \frac{1}{9}n_f T_R\right)L_{Ij}^3 + \left(\left(\frac{17}{18} - \frac{\pi^2}{12}\right)C_A - \frac{2}{9}n_f T_R\right)L_{Ij}^2 \\
& + \left(C_A\left(\frac{55}{54} + \frac{7\pi^2}{9} - \frac{9}{2}\zeta_3\right) - n_f T_R\left(\frac{2\pi^2}{9} - \frac{2}{27}\right)\right)L_{Ij} \\
& + C_A\left(\frac{89}{144}\pi^2 + \frac{163}{162} + \frac{283}{36}\zeta_3 - \frac{107}{720}\pi^4\right) + n_f T_R\left(-\frac{7\pi^2}{36} + \frac{2}{81} - \frac{11}{9}\zeta_3\right)\Bigg] \\
& + \int_0^1 \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \Bigg\{ \frac{1}{x}\left(\frac{C_A}{4}\tilde{F}_{Ij}^{gg,(0)}(1,\xi,x) - \frac{T_R n_f}{2}F_{Ij}^{q\bar{q},(0)}(1,\xi,x)\right)\ln\left(\frac{T_{N,Ij}(1,\xi,x)}{T_{N,Ij}(1,\xi,0)}\right) \\
& + \left(\frac{11}{6}C_A - \frac{2}{3}T_R n_f\right)R_{Ij}(1,\xi)\ln^2\left(\frac{T_{N,Ij}(1,\xi,0)Q_j}{p_I p_j}\right) \\
& + \Bigg[\left[\frac{67}{9} + \frac{11}{3}\left(L_{Ij} + \ln u\right) + 2u\,\mathrm{Li}_2(-u) - u\ln^2 u - 2(1+u)\ln(1+u)\right. \\
& \left. - 2(1-u)\left(\ln(1+u)\ln u + \frac{u\ln u}{1+u} + \frac{\pi^2}{6}\right)\right]\frac{C_A}{2} - \left(\frac{10}{9} + \frac{2}{3}(L_{Ij} + \ln u)\right)T_R n_f\Bigg] \\
& \times R_{Ij}(1,\xi)\ln\left(\frac{T_{N,Ij}(\xi,1,0)Q_j}{\xi\,(p_I p_j)}\right) + (\xi \to \xi^{-1})\Bigg\}
\end{aligned}
$$

where $u = \dfrac{\kappa_{Ij}}{\xi}$, $\qquad L_{Ij} = \ln\left(\dfrac{\bar{u}^2\,\mu^2\,(p_I p_j)^2}{m_I^2\,Q_j^2}\right)$

Dipole contributions: Two massive emitters

- While the $q\bar{q}$ case is somewhat simple, the case of two massive partons involving the emission of gg is more involved.
- The complexity of the finite remainder $A_{IJ}^{3,gg}$ suggests that computing its two dimensional integral numerically is a preferable strategy.

$$
\begin{aligned}
\tilde{W}_{IJ}^{(2),\mathrm{inc}} &= C_A\, A_{IJ}^{(3),gg} + \frac{n_f T_R}{6 v_{IJ}} \Bigg\{ \left(\ln\kappa_{IJ} + v_{IJ}\right) L_{IJ}^2 - L_{IJ} \Bigg[4\mathrm{Li}_2(-\kappa_{IJ}) + \ln^2\kappa_{IJ} \\
&+ \left(2v_{IJ} - \frac{10}{3}\right)\ln\kappa_{IJ} - 4v_{IJ}\ln(1+\kappa_{IJ}) - \frac{10}{3}v_{IJ} + \frac{\pi^2}{3} \Bigg] + \frac{28}{9}v_{IJ} \\
&+ \left(\frac{28}{9} + \frac{2}{3}\pi^2 - \frac{10}{3}v_{IJ}\right)\ln\kappa_{IJ} + \left(\frac{2}{3}v_{IJ} - \frac{5}{9}\right)\pi^2 + \frac{1}{3}\ln^3\kappa_{IJ} \\
&+ \left(v_{IJ} - \frac{5}{3}\right)\ln^2\kappa_{IJ} - \frac{20}{3}\left(\mathrm{Li}_2(-\kappa_{IJ}) - v_{IJ}\ln(1+\kappa_{IJ})\right) \Bigg\} + (I \leftrightarrow J) \\
&+ \int_0^1 \mathrm{d}x \frac{\mathrm{d}\xi}{\xi} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \Bigg\{ \frac{1}{x}\tilde{F}_{IJ}^{(0)}(1,\xi,x)\ln\left(\frac{T_{N,IJ}(1,\xi,x)}{T_{N,IJ}(1,\xi,0)}\right) \\
&+ \left(\frac{11}{6}C_A - \frac{2}{3}T_R n_f\right) R_{IJ}(1,\xi)\ln^2\left(\frac{T_{N,IJ}(1,\xi,0)Q}{p_I p_J v_{IJ}^+}\right) + \Big[K_{IJ}(1,\xi) \\
&+ \left(\frac{11}{6}C_A - \frac{2}{3}n_f T_R\right)(L_{IJ} - \log\xi) \Big] R_{IJ}(1,\xi)\ln\left(\frac{T_{N,IJ}(1,\xi,0)Q}{p_I p_J v_{IJ}^+}\right) + (I \leftrightarrow J) \Bigg\},
\end{aligned}
$$

We have achieved pole cancellation and derived finite numerical representaions for all the dipole configurations!

Tripole contributions at NNLO

- In addition to the real-virtual, tripole contributions arise from commutators of dipole Z_1 and S_1 contributions and genuine tripole contributions to Z

$$\tilde{S}_2^{\text{tri}} = S_{RV,T^3} + \frac{1}{2}[Z_1, Z_1^+] + \frac{1}{2}[S_{1,t}, Z_1 - Z_1^+] - Z_{2,\text{tri}} - Z_{2,\text{tri}}^+$$

Only present in case of two or more massive partons

[Neubert, Becher ' 09]
[Ferroglia, Neubert et al ' 09]

$$S_{RV,T^3} = \frac{2^{1-\epsilon}\pi}{\epsilon} \sum_{\text{IJK}} H_{\text{IJK}}\, \sigma_{\text{JK}}\, I_{\text{IJK}}$$

Imaginary part of the one-loop soft current

[Bierenbaum, Czakon, Mitov.'11,18]

Fully antisymmetrix color tensor

$$H_{\text{IJK}} = \frac{1}{2} f_{abc} \left(T_{\text{I}}^a T_{\text{J}}^b T_{\text{K}}^c + T_{\text{K}}^c T_{\text{J}}^b T_{\text{I}}^a \right)$$

- The commutators $\frac{1}{2}[Z_1, Z_1^+]$ and $\frac{1}{2}[S_{1,t}, Z_1 - Z_1^+]$ can be expressed in a similar way

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- The calculation can be organized such that the pole cancellations are obtained independently for the three massless, one massive two massless, and two massive one massless emitters.

$$\tilde{S}_2^{\text{tri}} = a_s^2 \left\{ \sum_{\text{I,J,K}} H_{\text{IJK}}\, U_{\text{I,JK}} \right\} = \tilde{S}_{2,0m}^{\text{tri}} + \tilde{S}_{2,1m}^{\text{tri}} + \tilde{S}_{2,2m}^{\text{tri}}$$

Tripole contributions at NNLO

- To simplify the computation, it is useful to separate the N-jettiness dependent and independent parts

$$\tilde{S}_2^{\text{tri}} = \tilde{S}_2^{\text{tri},(0)} + \tilde{S}_2^{\text{tri},(g)}$$

The N-jettiness dependent contribution is finite upon combining the RV term with the commutator involving S_1

$$\tilde{S}_2^{\text{tri},(g)} = \frac{2a_s^2\pi}{\epsilon}\sum H_{\text{IJK}} S_{\text{IJ}}^t \bar{\sigma}_{\text{JK}} + \frac{2^{1-\epsilon}\pi}{\epsilon}\sum H_{\text{IJK}}\sigma_{\text{JK}} I_{\text{IJK}}^{(g)}\,.$$

The N-jettiness independent term for massive emitters is much more involved since the integrations involve non-trivial dependence on the azimuthal angle.

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The N-jettiness independent term for massive emitters is much more involved since the integrations involve non-trivial dependence on the azimuthal angle.

- Antisymmetrizing I_{IJK} in the real virtual contribution simplifies the calculation, since it typically removes the highest pole.

E.g. for the massless case $$\sum_{ijk} H_{ijk}\, I^{(0)}_{ijk}\sigma_{jk} = \sum_{ijk} H_{ijk}\frac{1}{2}\left(I^{(0)}_{ijk} - I^{(0)}_{ikj}\right)\sigma_{jk}\,. \qquad j \leftrightarrow k$$

- The remaining pole cancellations also becomes explicit if the renormalization coefficients arising from the commutators are also antisymmetrized.

Tripole contributions: Results

- With these simplifications, we achieve explicit pole cancellation and obtain very compact finite expressions for the N-jettiness independnt parts of the cases involving three massless emitters and one massive two massless emitters.

Three massless emitters

$$\tilde{S}^{\mathrm{tri},(0)}_{2,0m} = \frac{\pi a_s^2}{2} \sum H_{ijk}\, \sigma_{jk} \left[-2 \left(S^{0,(1)}_{ij} - S^{0,(1)}_{ik} \right) - \frac{\pi^2}{6} \left(L_{ij} - L_{ik} \right) + \frac{1}{2} \left(\tilde{F}_{0,ijk} - \tilde{F}_{0,ikj} \right) \right]$$

where $S^{0,(1)}_{ij} = \frac{1}{6} L^3_{ij} + \frac{\pi^2}{4} L_{ij} + \frac{1}{2} L_{ij} \ln^2 \left(\frac{C_j}{C_i} \right) + \frac{7}{3} \zeta_3$

and $\tilde{F}_{0,ijk} = \theta(p_k p_i - p_k p_j) \ln^2 \left(\frac{p_k p_j}{p_k p_i} \right) \left(\frac{2}{3} \ln \frac{p_k p_j}{p_k p_i} - 2 \ln \frac{C_i}{C_j} \right)$

One massive two massless emitters

$$\begin{aligned}\tilde{S}^{\mathrm{tri},(0)}_{2,1m} = \alpha_s^2 \pi \sum_{I,j,k} H_{Ijk} \Bigg\{ \sigma_{jk} \Bigg[& \frac{1}{12} \left(L_{Ik} - L_{Ij} \right)^3 + \frac{1}{4} \left(L_{Ik} - L_{Ij} \right) L_{Ij} L_{Ik} \\ & + \frac{(L_{Ik} - L_{Ij})}{4} \left[\pi^2 - \ln^2(y_{Ijk}) - 2\mathrm{Li}_2(1 - y_{Ijk}) \right] \Bigg] + \frac{1}{24} (L_{Ik} - L_{Ij})^3 \\ & + \frac{2}{3} \ln \left(\frac{C_j}{C_k} \right)^3 + \frac{1}{4} L_{Ij} L_{Ik} (L_{Ik} - L_{Ij}) - \frac{1}{4} \left(F^{(A)}_{kj,I} - F^{(A)}_{jk,I} \right) \\ & + \frac{(L_{Ik} - L_{Ij})}{2} \left(\frac{\pi^2}{2} + \ln^2 \frac{C_j}{C_k} - \frac{y_{Ijk} \ln^2(y_{Ijk}) + 2\mathrm{Li}_2(1 - y_{Ijk})}{1 - y_{Ijk}} \right) \Bigg\}.\end{aligned}$$

where $F^{(A)}_{ijK} = 4 \int_0^1 \frac{\mathrm{d}\xi}{\xi} \int_0^{2\pi} \frac{\mathrm{d}\varphi}{2\pi} \ln \left(\frac{C_i}{C_j} \xi \right) \left\{ \ln \left[\frac{\beta_K}{\Delta_K(\xi, \varphi)} \right] + \mathcal{I}^{(1)}_{jK}(q) - \mathcal{I}^{(1)}_{jK}(p_i) \right\}$

Tripole contributions: Results

- The case involving two massive and one massless emitters gets considerably more complex since it requires integrations over Polylogs with complex arguments.

- After considerable efforts, the poles can be cancelled analytically and result largely simplifies.

$$\tilde{S}^{\mathrm{tri},(0)}_{1,2m} = \frac{a_s^2\pi}{2}\sum_{I,J,k} H_{IJk}\left\{\frac{1}{2}D^{(2)}_{IJk} + \ln\left(\frac{\mu^2\bar{u}^2(p_Ip_J)}{2}\right)D^{(1)}_{IJk} + \left[\ln^2\left(\frac{\mu^2\bar{u}^2(p_Ip_J)}{2}\right) + \frac{7\pi^2}{12}\right]D^{(0)}_{IJk} - \left(1-\frac{1}{v_{IJ}}\right)\left[\frac{1}{6}\left(L^3_{Ik}-L^3_{Jk}\right)+\frac{5\pi^2}{12}\left(L_{Ik}-L_{Jk}\right)\right]\right\}.$$

with $D^{(0)}_{IJk} = -\ln\frac{y_{JIk}}{y_{IJk}} + \left(2-\frac{1}{v_{IJ}}\right)\ln\frac{y_{JIk}}{y_{IJk}} = \left(1-\frac{1}{v_{IJ}}\right)\ln\frac{y_{JIk}}{y_{IJk}}$

$$D^{(1)}_{IJk} = 2\ln\left(\frac{y_{JIk}}{y_{IJk}}\right)\left(\ln(1+v_{IJ}) - \frac{(1-v_{IJ})}{v_{IJ}}\ln\frac{2C_k^2(p_Ip_k)(p_Jp_k)}{m_I^2m_J^2} - \frac{1}{v_{IJ}}\ln(2v_{IJ})\right)$$

$D^{(2)}_{IJk}$ can be computed via a one-dimensional numerical integration.

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with $D^{(0)}_{IJk} = -\ln\frac{y_{JIk}}{y_{IJk}} + \left(2-\frac{1}{v_{IJ}}\right)\ln\frac{y_{JIk}}{y_{IJk}} = \left(1-\frac{1}{v_{IJ}}\right)\ln\frac{y_{JIk}}{y_{IJk}}$

$$D^{(1)}_{IJk} = 2\ln\left(\frac{y_{JIk}}{y_{IJk}}\right)\left(\ln(1+v_{IJ}) - \frac{(1-v_{IJ})}{v_{IJ}}\ln\frac{2C_k^2(p_Ip_k)(p_Jp_k)}{m_I^2m_J^2} - \frac{1}{v_{IJ}}\ln(2v_{IJ})\right)$$

$D^{(2)}_{IJk}$ can be computed via a one-dimensional numerical integration.

With this, we have computed all the tripole configurations for processes with up to two massive emiiters. All contributions are finite and suitable for fast numerical implementations!

Results and Checks

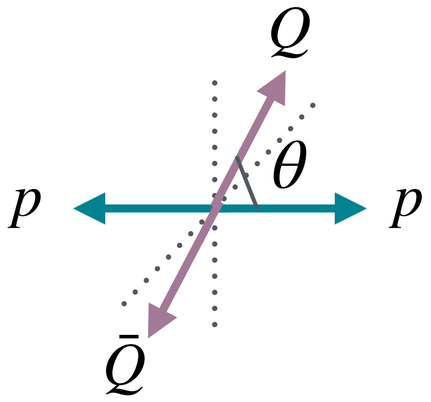
- We obtain the total dipole and tripole contributions to the NNLO soft function for processes with two heavy quarks and N jets by combining the coefficients for various mass assignments.

$$\tilde{S}_2^{\text{dip}} = a_s^2 \left\{ \sum_{i \neq j} \mathbf{T}_i \cdot \mathbf{T}_j \, \tilde{W}_{ij}^{(2)} + 2 \sum_{I,j} \mathbf{T}_I \cdot \mathbf{T}_j \, \tilde{W}_{Ij}^{(2)} + \sum_{I \neq J} \mathbf{T}_I \cdot \mathbf{T}_J \, \tilde{W}_{IJ}^{(2)} \right\}$$

$$\tilde{S}_2^{\text{tri}} = a_s^2 \left\{ \sum_{i,j,k} H_{ijk} \, U_{i,jk} + \sum_{I,j,k} H_{Ijk} \, U_{I,jk} + \sum_{I,J,k} H_{IJk} \, U_{IJ,k} \right\}$$

- To validate our results, we check against the limiting case: 0-jettiness soft function for $pp \to Q\bar{Q}$ in back-to-back kinematics. *[Bell, Broggio, Edelmann, Lim, Rahn '26]*

- We compare both the dipole and tripole coefficients for the grid of 576 points in the $\beta_Q - \theta$ plane provided in *[Bell, Broggio, Edelmann, Lim, Rahn '26]*



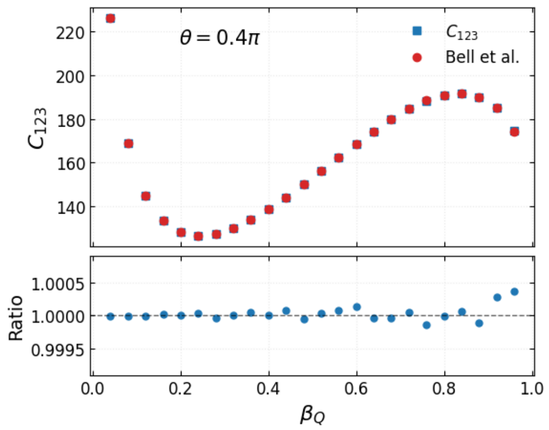
$$p_{1,2} = (1,\, 0,\, 0,\, \pm 1)$$

$$p_{3,4} = \frac{m_Q}{\sqrt{1-\beta_Q^2}} (1,\, \pm \beta_Q \sin\theta,\, 0,\, \pm \beta_Q \cos\theta)$$

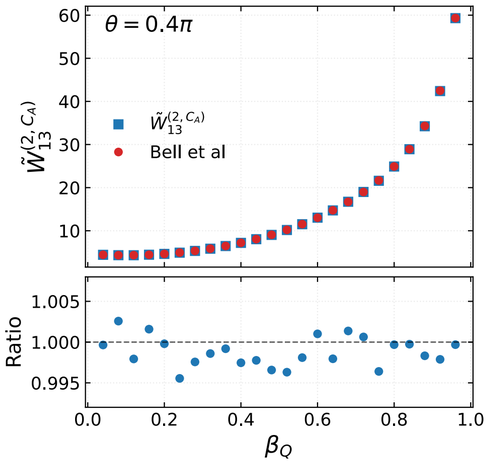
Numerical Checks

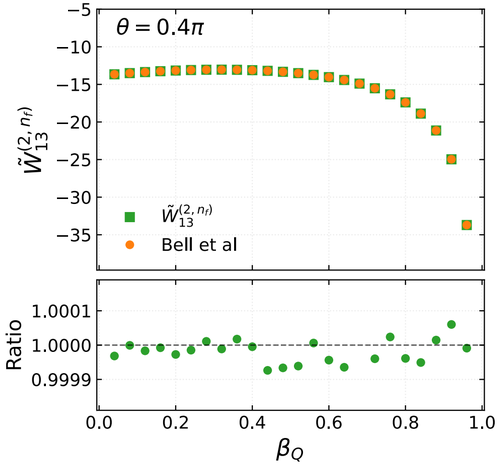
- We obtain agreement at $\sim 0.1\,\%$ level for both the dipole and tripole contributions in all mass cases.

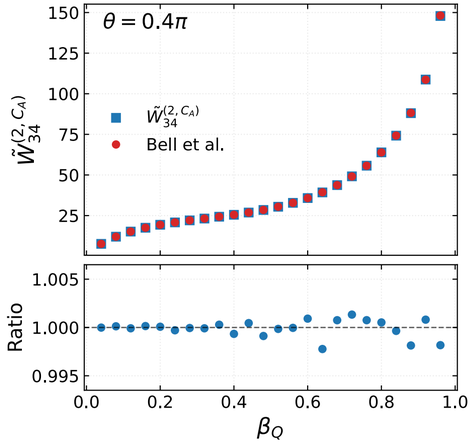
Tripole sum

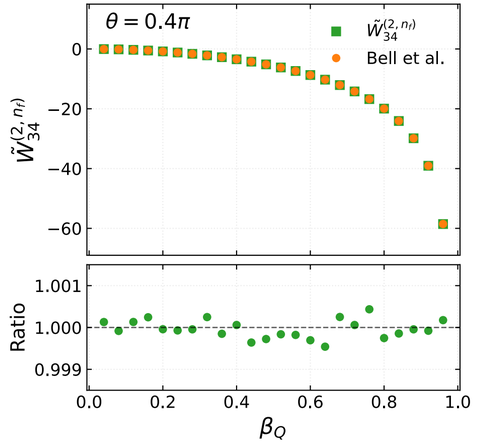


Dipole Contributions: C_A and n_f





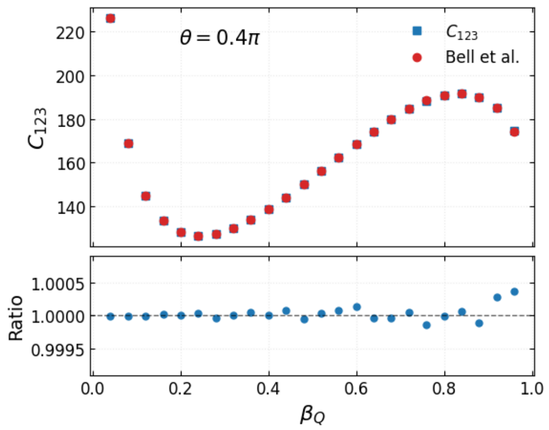




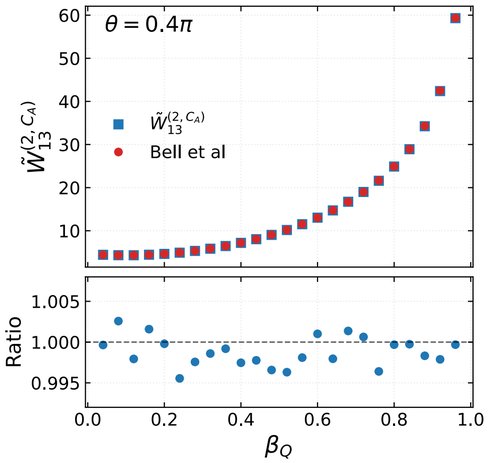
Numerical Checks

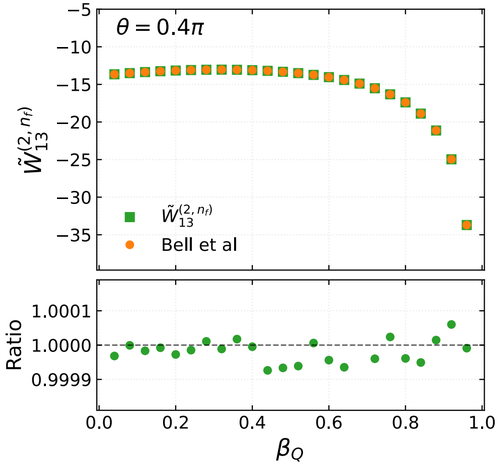
- We obtain agreement at $\sim 0.1\,\%$ level for both the dipole and tripole contributions in all mass cases.

Tripole sum



Dipole Contributions: C_A and n_f



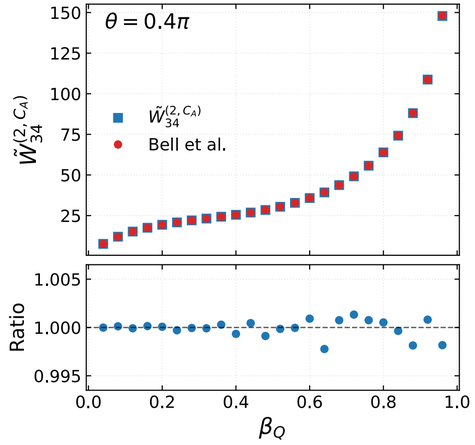


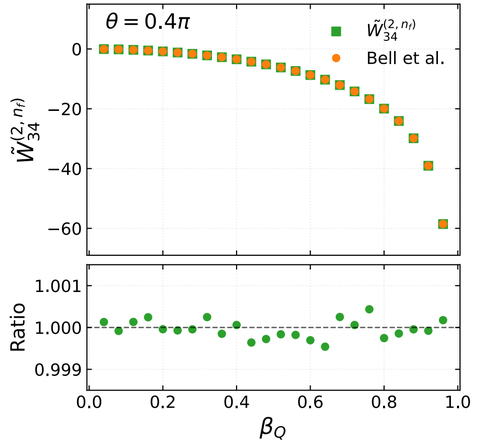
- Since our results are finite simplified expressions, they allow fast numerical evaluations

E.g. to compute the grid of 576 points at subpercent level:

Dipoles	~ 30 mins
Tripoles	$5 - 10$ mins

Further optimizations could allow on-the-fly evaluations





Benchmark numbers: $pp \to Q\bar{Q} + \text{jets}$

- The numerical implementation is general and can be applied to processes involving an arbitrary number of jets.
- For future computations and implementaions of our results, we provide benchmark results for $pp \to Q\bar{Q} + 2j$:

$$p_I = \frac{m}{\sqrt{1-\beta_I^2}}\left(1, \beta_I \sin\theta_I \cos\phi_I, \beta_I \sin\theta_I \sin\phi_I, \beta_I \cos\theta_I\right), \qquad I = 3,4. \qquad n_i = \left(1, \sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i\right), \qquad i = 5,6,\ldots$$

Dipoles	Gluons(C_A part)			Quarks ($T_R n_f$ part)		
(M0)	$\tilde{W}_{Ij}^{(2),\text{inc}}$	$\Delta W_{Ij}^{\text{dip}}$	$\tilde{W}_{Ij}^{(2)}$	$\tilde{W}_{Ij}^{(2),\text{inc}}$	$\Delta W_{Ij}^{\text{dip}}$	$\tilde{W}_{Ij}^{(2)}$
31	36.875(6)	-8.076(9)	28.80(1)	-7.138(1)	-4.6782(8)	-11.817(2)
32	15.439(2)	-6.605(3)	8.835(5)	-3.9967(5)	-2.2474(4)	-6.2441(6)
35	37.335(6)	-8.332(8)	29.003(9)	-6.693(1)	-4.7980(8)	-11.491(2)
36	29.383(4)	-8.745(9)	20.638(9)	-4.9898(9)	-3.7338(6)	-8.723(1)
41	40.965(7)	-8.403(9)	32.56(1)	-8.041(1)	-5.2895(9)	-13.331(2)
42	12.617(2)	-6.505(4)	6.112(5)	-3.3357(4)	-1.9966(3)	-5.3323(5)
45	43.973(9)	-8.06(1)	36.41(1)	-7.880(1)	-5.748(1)	-13.628(2)
46	19.964(3)	-8.198(6)	11.766(7)	-3.5850(6)	-2.7683(4)	-6.3533(6)
(MM)	$\tilde{W}_{IJ}^{(2),\text{inc}}$	$\Delta W_{IJ}^{\text{dip}}$	$\tilde{W}_{IJ}^{(2)}$	$\tilde{W}_{IJ}^{(2),\text{inc}}$	$\Delta W_{IJ}^{\text{dip}}$	$\tilde{W}_{IJ}^{(2)}$
34	11.228(1)	-0.4323(1)	10.796(1)	-0.16252(8)	-0.14659(2)	-0.30912(8)

U_{Ijk}	jk					
I	12	15	16	25	26	56
3	-30.228(8)	0.103(2)	0.625(3)	1.217(3)	0.620(1)	0.746(2)
4	-42.051(7)	0.016(2)	0.378(4)	1.643(4)	0.357(1)	0.299(3)

U_{IJk}	k			
IJ	1	2	5	6
34	15.743(1)	-6.397(1)	23.475(1)	-22.533(1)

Conclusions and Outlook

- We have presented the results of the NNLO N-jettiness soft function for the production of a heavy quark pair and an arbitrary number of jets in hadron collisions.
- We demonstrate analytic cancellation of poles and derive finite numerical representations for dipole and tripole contributions for massless and up to two massive emitters, allowing for fast numerical evaluations.
- For dipole contributions, the use of the inclusive N-jettiness observable largely simplifies the computation.

Conclusions and Outlook

- We have presented the results of the NNLO N-jettiness soft function for the production of a heavy quark pair and an arbitrary number of jets in hadron collisions.

- We demonstrate analytic cancellation of poles and derive finite numerical representations for dipole and tripole contributions for massless and up to two massive emitters, allowing for fast numerical evaluations.

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What's next

- For arbitrary collider processes at NNLO QCD, the only missing piece is the tripole configuration involving three massive emitters.

- A public code performing fast numerical evalauation for the N-jettiness soft function at NNLO.

Thank You for your attention!



Backup Slides

ΔS is finite at NNLO

- The difference between the inclusive and the regular soft function is given by:

$$\Delta S_{RR} = g_s^4 \sum_{\mathsf{IJ}} \mathbf{T}_\mathsf{I} \cdot \mathbf{T}_\mathsf{J} \int [\mathrm{d}p_m][\mathrm{d}p_n]\; \tilde{S}_{\mathsf{IJ}}(m,n)\; \left[\delta(\tau - \mathcal{T}_N(p_m,p_n)) - \delta(\tau - T_N(q))\right]$$

- To simplify this expression, we introduce the ordering of the two soft partons

$$[\mathrm{d}p_m][\mathrm{d}p_n] = E_m^{1-2\epsilon} E_n^{1-2\epsilon} \mathrm{d}E_m \mathrm{d}E_n [\mathrm{d}\Omega_m][\mathrm{d}\Omega_n] \quad \longrightarrow \quad E_m < E_n, \quad E_m = \omega E_n$$

- The homogenity of the N-jettiness function allows us to integrate in the energy of the soft parton. Performing the Laplace transform:

$$\Delta S_{RR} \propto \frac{a_s^2}{2} \sum_{\mathsf{IJ}} \mathbf{T}_\mathsf{I} \cdot \mathbf{T}_\mathsf{J} \int_0^1 \mathrm{d}\omega\; \omega\; [\mathrm{d}\Omega_m][\mathrm{d}\Omega_n]\; \tilde{S}_{\mathsf{IJ}}(m,n)\; \log\left(\frac{T_N(n_m + \omega n_n)}{\mathcal{T}_N(n_m, \omega n_n)}\right)$$

- It follows from this expression that the logarithm of the ratio of N-jettiness observables vanishes in the singular limits of the eikonal function

$$\omega \to 0, \quad n_m || n_n || i, \quad n_m || n_n || j, \quad n_m || n_n$$