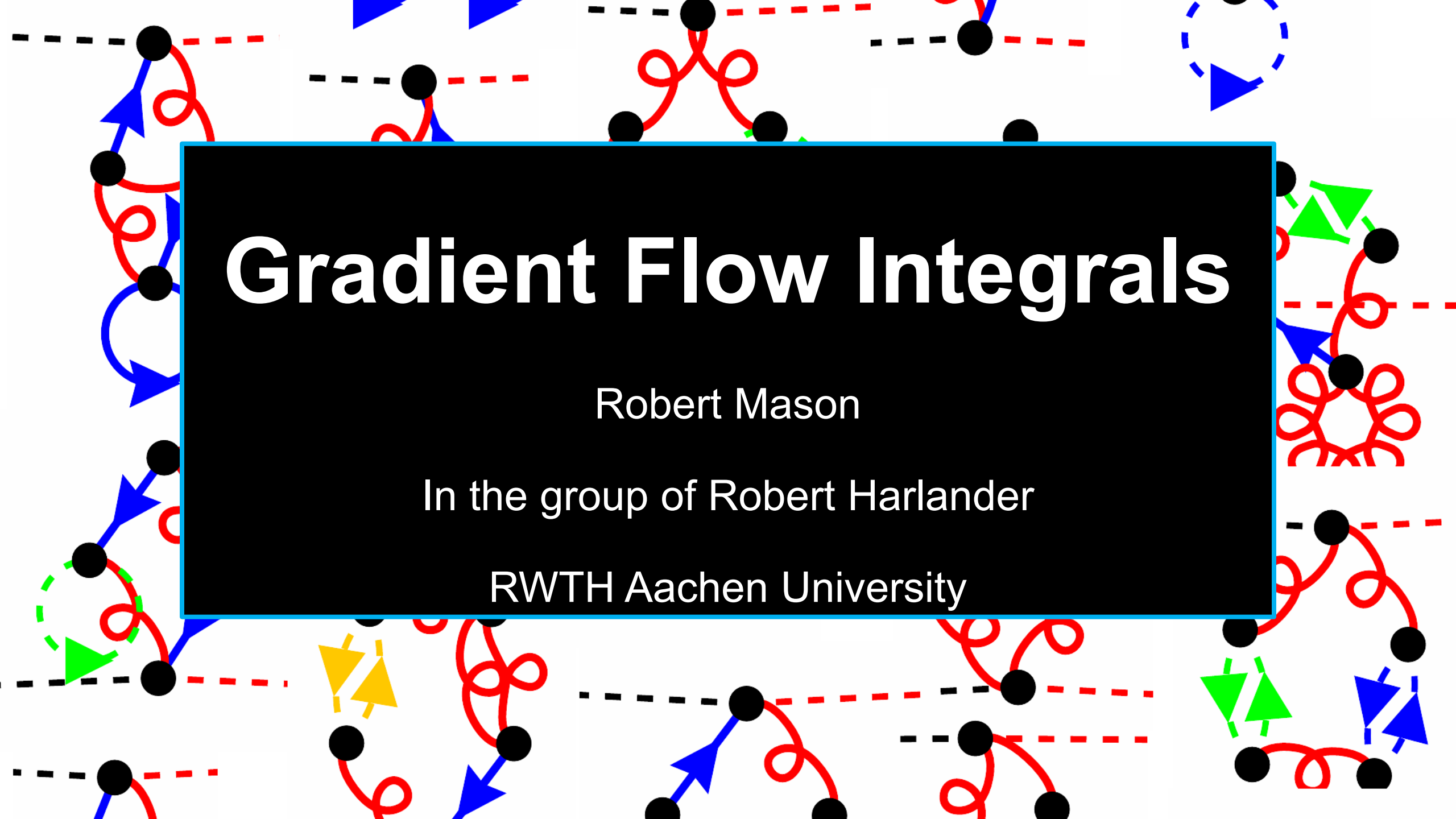


The background of the slide is a complex, abstract pattern. It features a variety of elements: solid red lines forming loops and curves; solid blue lines with arrowheads pointing in different directions; dashed black lines; and dashed red lines. Small black dots are scattered throughout, often serving as nodes or endpoints for the lines. There are also some green and yellow triangular shapes interspersed among the lines. The overall effect is a dynamic, almost chaotic, geometric design.

# Gradient Flow Integrals

Robert Mason

In the group of Robert Harlander

RWTH Aachen University

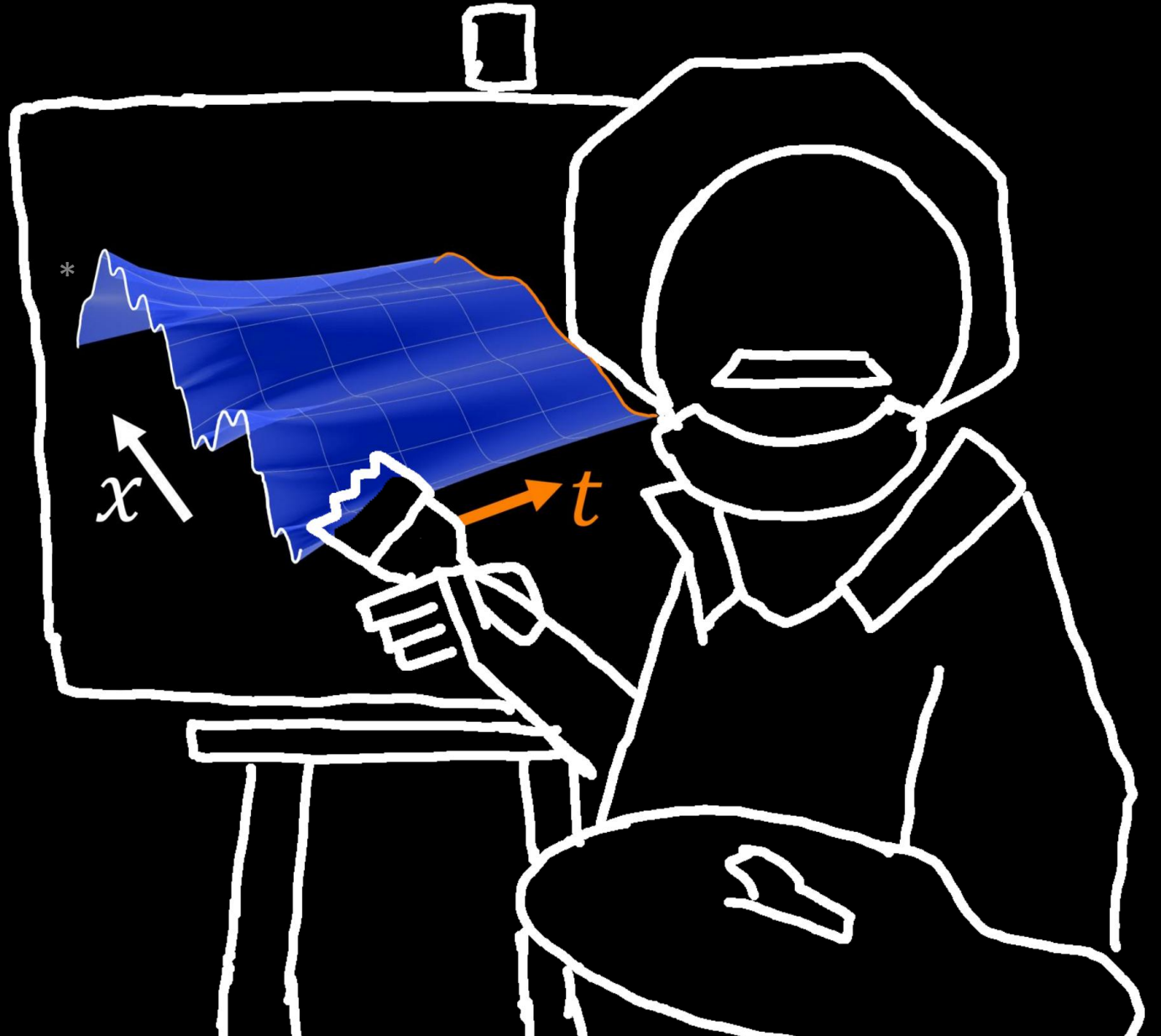
# Aim

Provide a methodology to efficiently reduce and calculate any two-loop gradient flow integral.

## Outline

- State of the art in the perturbative gradient flow
- Computational bottlenecks
- Gradient flow integrals
- Outlook

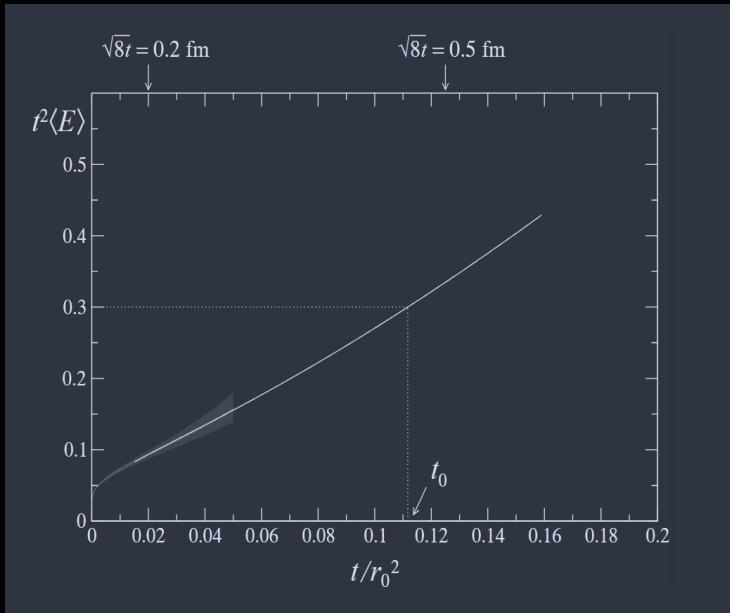
# State of the art



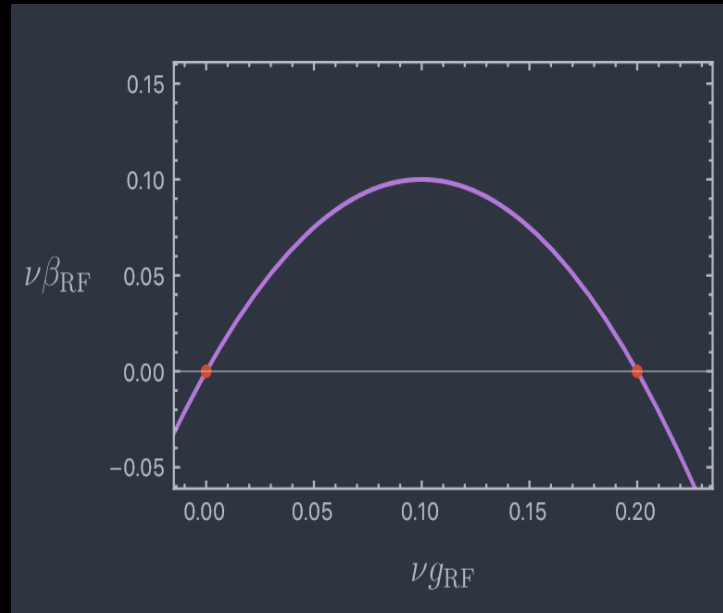
# Gradient flow applications

Flow equations  $\partial_t B_\mu = D_\nu G_{\nu\mu} + \kappa D_\mu \partial_\mu B_\nu$   
such that  $B_\mu(t=0, x) = A_\mu(x)$

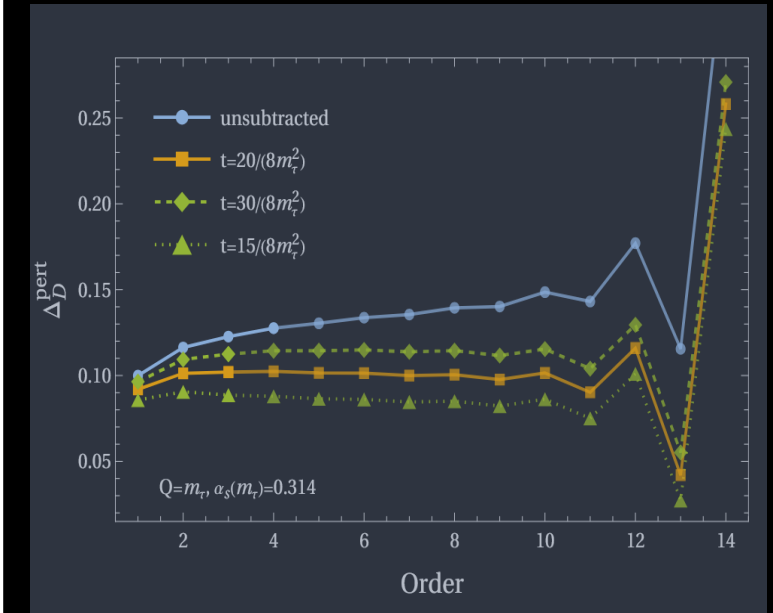
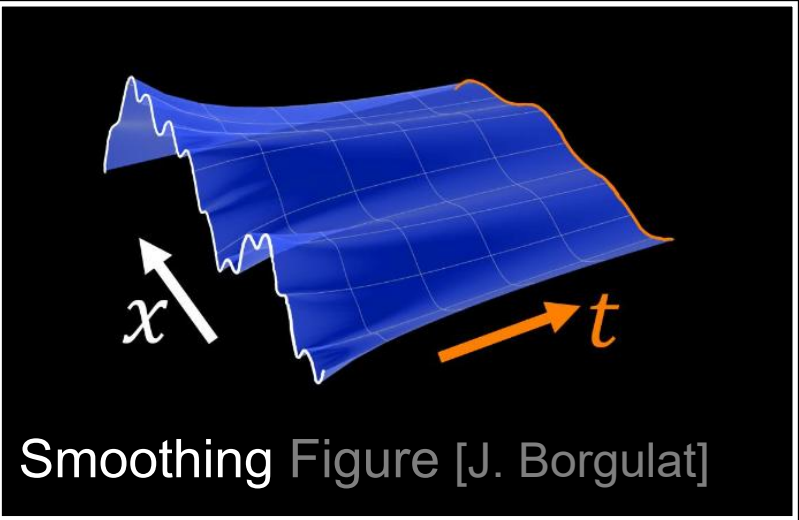
[Narayanan, Neuberger 2006] [Lüscher 2010] [Lüscher, Weisz 2011]



Scale-setting:  $t^2 \langle G_{\mu\nu} G^{\mu\nu} \rangle = 0.3$   
Figure [Lüscher 2010]



Asymptotic safety:  $\beta = 0$   
Figure [Harlander et al 2026]



Renormalon subtraction  
Figure [Beneke, Takaura 2025]

# Gradient Flow Calculations

## VEVs



$$\ni \int_p \frac{e^{-2tp^2}}{p^{2n}}$$

Defines GF scheme:

$$\alpha_{GF} = \langle G_{\mu\nu} G^{\mu\nu}(t) \rangle \Big|_{t=\rho/\mu^2}$$

State of the art: Three-loop QCD

Bubble integrals

## SFTX



$$\ni \int_p \frac{e^{-2tp^2}}{p^{2n} (p - q_1)^{2n_1} (p - q_2)^{2n_2}}$$

OPE Wilson coefficients:

$$Z_\phi \phi^\dagger(t) \phi(t) = \left( \frac{1}{t} \right) c_0(t) \mathbb{I} + c_1(t) Z_{\phi^\dagger \phi} \phi^\dagger \phi$$

State of the art: Two-loop

N-point integrals?

# Gradient Flow Calculations

## VEVs



$$\ni \int_p \frac{e^{-2tp^2}}{p^{2n}}$$

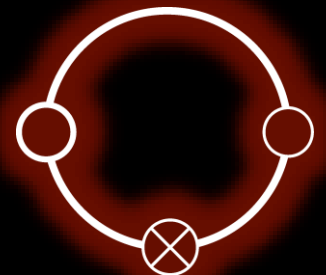
Defines GF scheme:

$$\alpha_{GF} = \langle G_{\mu\nu} G^{\mu\nu}(t) \rangle \Big|_{t=\rho/\mu^2}$$

State of the art: **Five-loop QED**

Bubble integrals

## SFTX



$$\ni \int_p \frac{e^{-2tp^2}}{p^{2n}}$$

OPE Wilson coefficients

$$Z_\phi \phi^\dagger(t) \phi(t) = \left(\frac{1}{t}\right) c_0(t) \mathbb{I} + c_1(t) Z_{\phi^\dagger \phi} \phi^\dagger \phi$$

State of the art: Two-loop

Bubble integrals

# Next steps

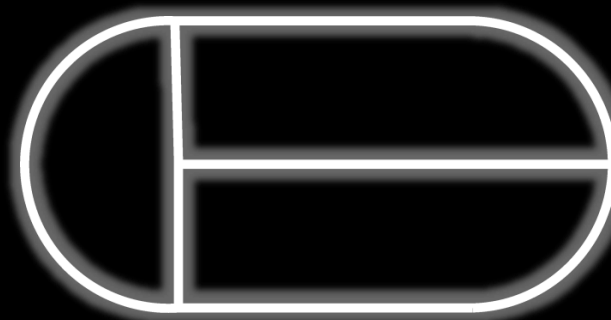
## Two Loop



**Example:** PDF  $n=4$  moments

- Four flow-times
- $\sim 150$  sectors
- $\sim 80,000$  integrals
- Diagrams  $O(\text{weeks})$
- Integrals  $O(\text{months})$
- Six masters (so far)

## Three Loop

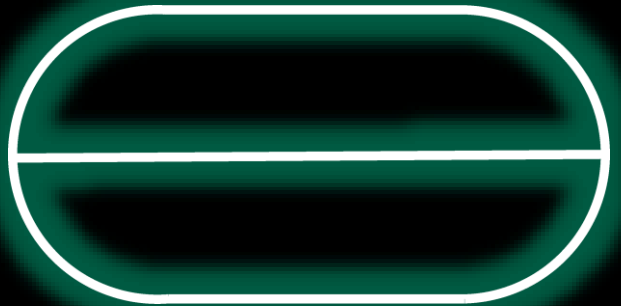


**Example:** SFTX MDM

- Five flow-times
- 12,000 integrals
- Diagrams  $O(\text{weeks})$
- Integral calculation unfinished

# Next steps

## Two Loop

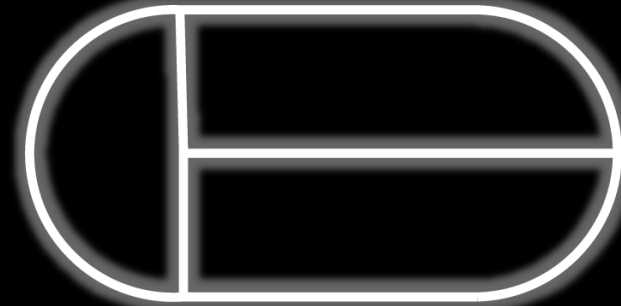


- Complicated operators
- Too many flow-times
- Too many integrals
- Reduction slow

But...

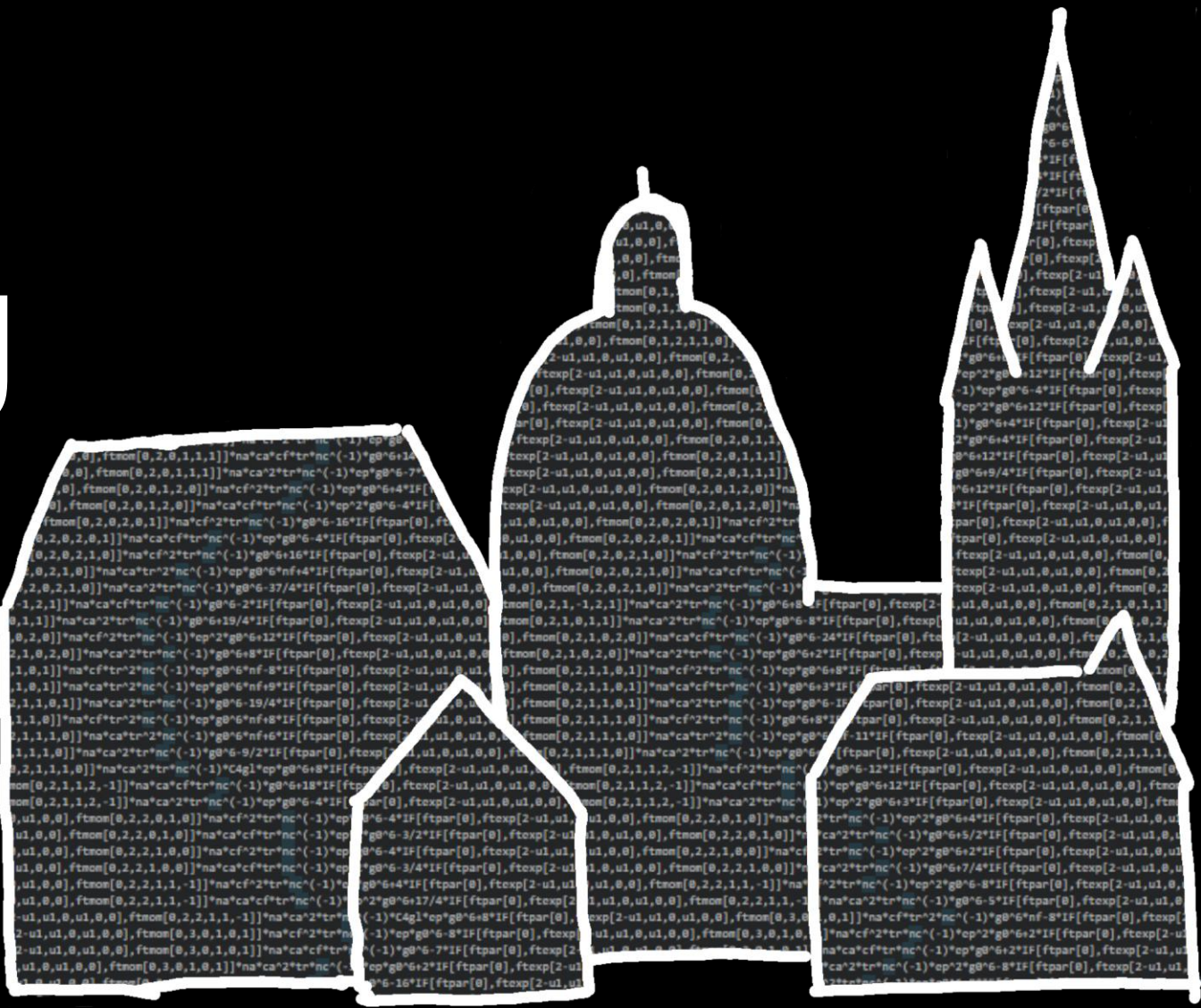
- System (mostly) well understood

## Three Loop



- No analytic results
- $O(100s)$  masters – not known if number bounded
- Too many flow-times
- Too many integrals
- Reduction stalls

# Simplifying gradient flow integrals



# Integrals

Flow-times are in general partially ordered

Note: standard notation projects integrals onto hypercube e.g.

$$\int_0^t ds_1 \int_0^{s_1} ds_2 = t^2 \int_0^1 du_1 u_1 \int_0^1 du_2$$

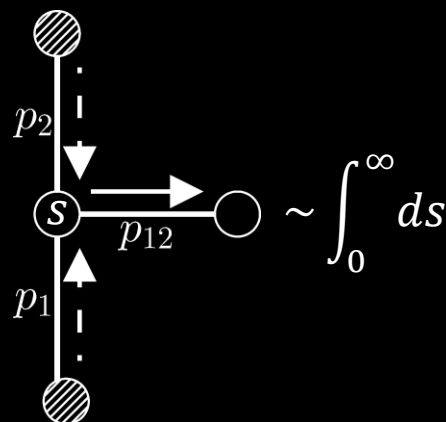
General two loop gradient flow integral:

$$\int_0^t ds_1 s_1^{c_1} \dots \int_0^{s_{n-1}} ds_n s_n^{c_1} \int_{p_1} \int_{p_2} \frac{e^{-ta_1(s)p_1^2 - ta_2(s)p_2^2 - ta_3(s)p_{12}^2}}{p_1^{2b_1} p_2^{2b_2} p_{12}^{2b_3}}$$

Definition:

$$p_{12} = p_1 - p_2$$

## Flow vertex



## Operator



## Unflowed propagators

## Flowed propagators

# Current gradient Flow IBP

General two loop gradient flow integral:

$$\int_0^t ds_1 s_1^{c_1} \dots \int_0^{s_{n-1}} ds_n s_n^{c_1} \int_{p_1} \int_{p_2} \frac{e^{-ta_1(s)p_1^2 - ta_2(s)p_2^2 - ta_3(s)p_{12}^2}}{p_1^{2b_1} p_2^{2b_2} p_{12}^{2b_3}}$$

Exponential sector

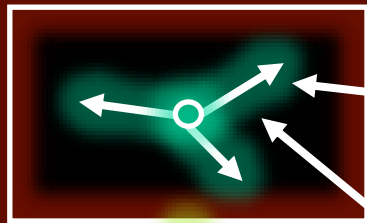
**Example:** PDF n=4 moments

- Four flow-times
- ~150 sectors\*
- Integrals O(months)

**Example:** QED

- One simple sector  $e^{-2tp^2}$

$s_1, s_2, \dots, s_n$



Standard IBP relations

$$\int_p \partial_p^\mu (k_\mu f(p)) = 0$$

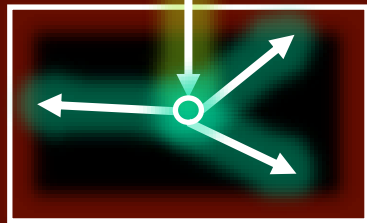
Only moves within a sector

Flow-time IBP relations:

$$c \int_0^{s_0} ds s^{c-1} f(s) - \int_0^{s_0} ds s^c \partial_s f(s) = [s^c f(s)]_{s=0}^{s=s_0}$$

Also moves to two sectors with one fewer flow times

$s_1, s_2, \dots, s_{n-1}$



# Aim

Provide a methodology to efficiently reduce and calculate any two-loop gradient flow integral analytically.

GFRReduce: FORM program aimed at projecting any massless two loop gradient flow integral to minimal number of **exponential sectors** and using predefined **IBP routines** reduce these down to known masters

————→ show there are only six master integrals.

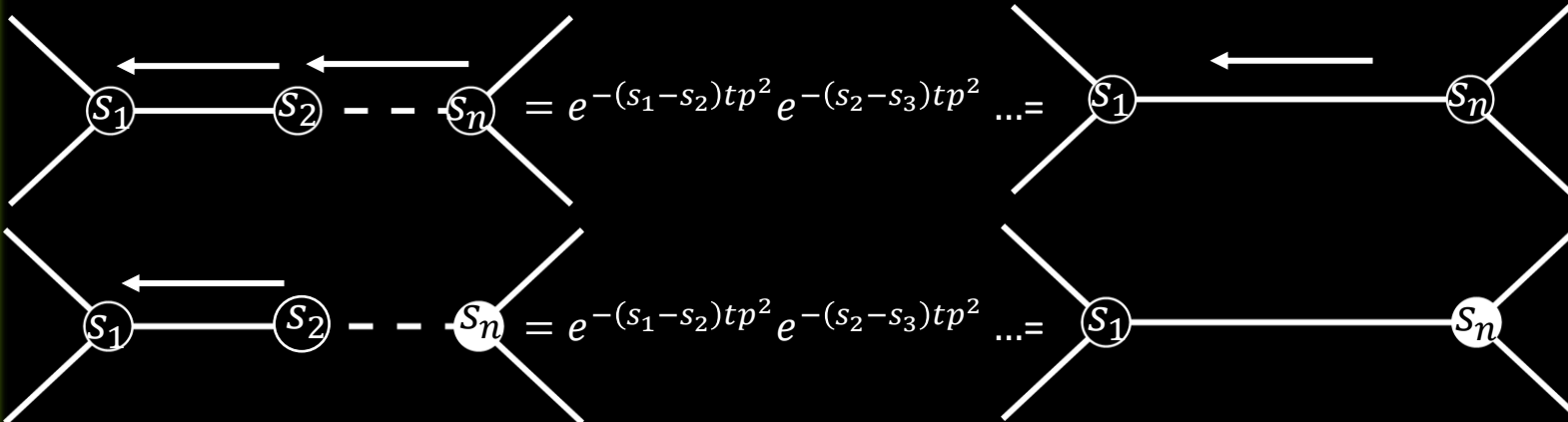
# Exponential structure

Two loop gradient flow integral:

$$f(c, a, b) = \int_0^t ds_1 s_1^{c_1} \dots \int_0^{s_{n-1}} ds_n s_n^{c_1} \int_{p_1} \int_{p_2} \frac{e^{-ta_1(s)p_1^2 - ta_2(s)p_2^2 - ta_3(s)p_{12}^2}}{p_1^{2b_1} p_2^{2b_2} p_{12}^{2b_3}}$$

Interested in integrals with many external legs

Consider the chains

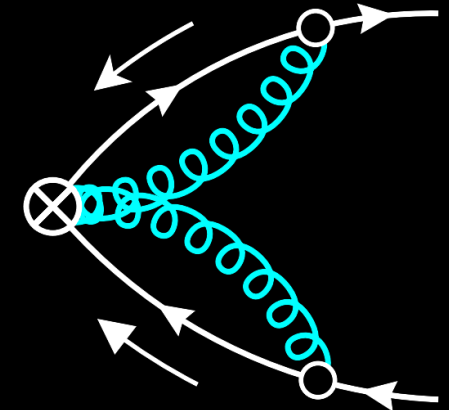


difference appears only in limits of integration.

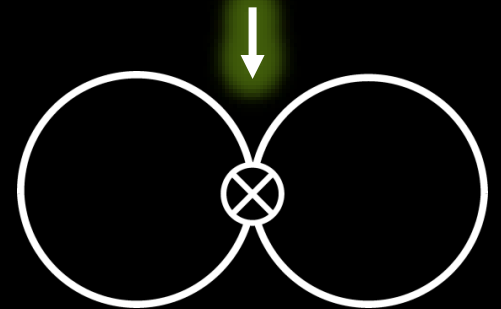
Any flow-time not in exponential can be integrated e.g.

$$\int_0^{s_0} ds_1 s_1^{c_1} \int_0^{s_1} ds_2 i(s_2) = \int_0^{s_0} ds_2 i(s_2) \int_{s_2}^{s_0} ds_1 s_1^{c_1}$$

Example diagram

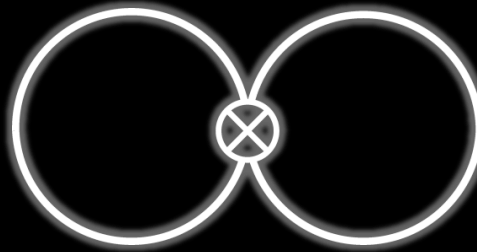


NS Twist-Two SFTX  
[Harlander et al 2025]

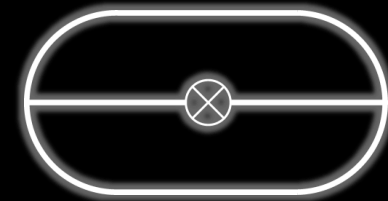


# Sectors

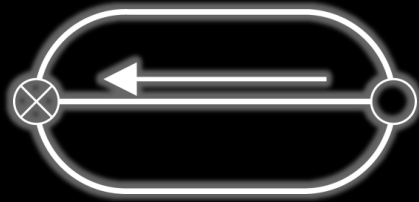
Any flow-time not in exponential can be integrated (+other relations)



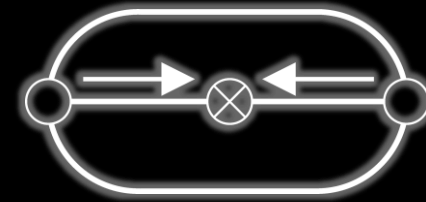
$$= i_0 = e^{-2tp_1^2 - 2tp_2^2}$$



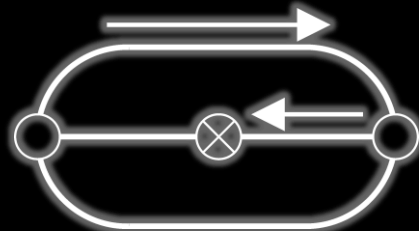
$$= i_1 = e^{-2tp_1^2}$$



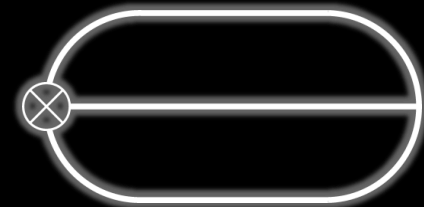
$$= i_2 = e^{-(2t-s_1)p_1^2 - s_1p_2^2 - s_1p_{12}^2}$$



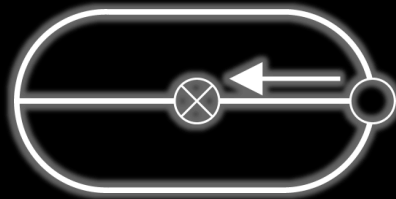
$$= i_3 = e^{-(2t-s_1-s_2)p_1^2 - (s_1+s_2)p_2^2} \cdot e^{-(s_1+s_2)p_{12}^2}$$



$$= i_4 = e^{-(2t-s_1+s_2)p_1^2 - (s_1-s_2)p_2^2} \cdot e^{-(s_1+s_2)p_{12}^2}$$



$$= i_5 = e^{-tp_1^2 - tp_2^2 - tp_{12}^2}$$



$$= i_6 = e^{-(t-s_1)p_1^2 - (t+s_1)p_2^2 - (t+s_1)p_{12}^2}$$

# Symmetric sectors

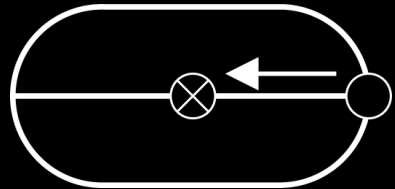
Note: standard notation projects integrals onto hypercube e.g.

$$\int_0^t ds_1 \int_0^{s_1} ds_2 = t^2 \int_0^1 du_1 u_1 \int_0^1 du_2$$

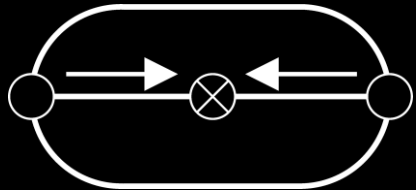
Flow-time IBP relations:

$$c \int_0^1 du u^{c-1} f(u) \quad Q$$

$$= \text{Limits} - \int_0^1 du u^c \partial_u f(u)$$



$$= \int_0^1 du_1 i_2 = \int_0^1 du_1 e^{-(2-u_1)tp_1^2 - u_1 tp_2^2 - u_1 tp_{12}^2}$$



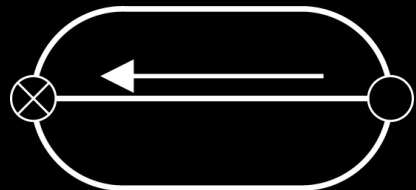
$$= \int_0^1 du_1 \int_0^1 du_2 i_3$$

$$= \int_0^1 du_1 \int_0^1 du_2 e^{-(2-u_1-u_2)tp_1^2} \cdot e^{-(u_1+u_2)tp_2^2 - (u_1+u_2)tp_{12}^2}$$

$$u_1 \rightarrow u_1 - u_2$$

$$u_2 \rightarrow u_2$$

$$= \int_1^2 du_1 i_2 \int_{1-u_1}^1 du_2 + \int_0^1 du_1 i_2 \int_0^{u_1} du_2$$



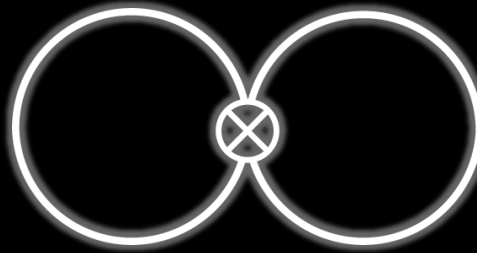
$$= \int_0^1 du_1 i_6 = \int_0^1 du_1 e^{-(1-u_1)tp_1^2 - (1+u_1)tp_2^2 - (1+u_1)tp_{12}^2}$$

$$u_1 \rightarrow u_1 - 1$$

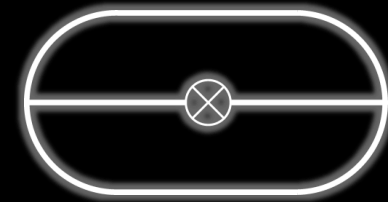
$$= \int_1^2 du_1 i_2$$

# Sectors

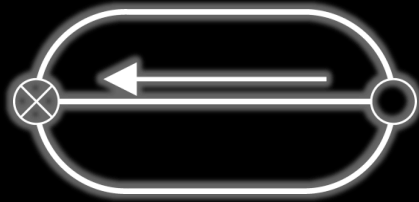
Any flow-time not in exponential can be integrated (+other relations)



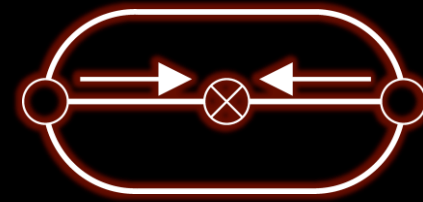
$$= i_0 = e^{-2tp_1^2 - 2tp_2^2}$$



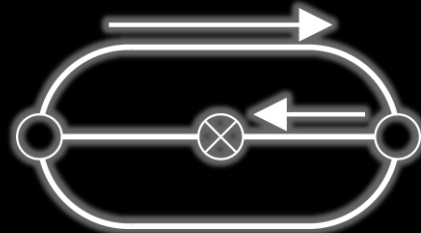
$$= i_1 = e^{-2tp_1^2}$$



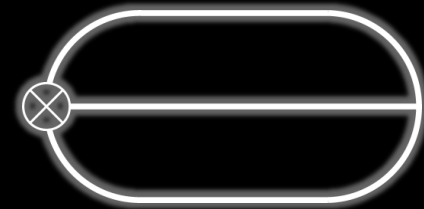
$$= i_2 = e^{-(2t-s_1)p_1^2 - s_1p_2^2 - s_1p_{12}^2}$$



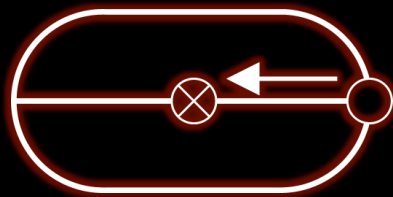
$$= i_3 = e^{-(2t-s_1-s_2)p_1^2 - (s_1+s_2)p_2^2} \cdot e^{-(s_1+s_2)p_{12}^2}$$



$$= i_4 = e^{-(2t-s_1+s_2)p_1^2 - (s_1-s_2)p_2^2} \cdot e^{-(s_1+s_2)p_{12}^2}$$



$$= i_5 = e^{-tp_1^2 - tp_2^2 - tp_{12}^2}$$



$$= i_6 = e^{-(t-s_1)p_1^2 - (t+s_1)p_2^2 - (t+s_1)p_{12}^2}$$

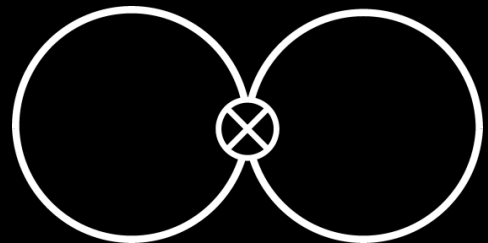
# Example sector reduction

Definition:

$$i_0 = e^{-2tp_1^2 - 2tp_2^2}$$

$$i_1 = e^{-2tp_1^2}$$

In general, for sector  $i_0$

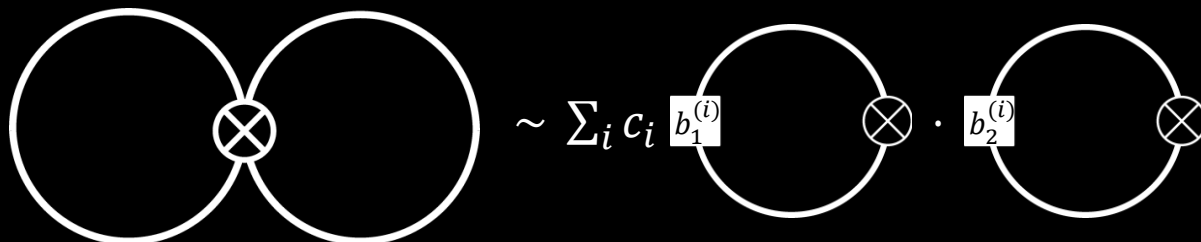


$$= I_0(b_1, b_2, b_3) = \int_{p_1} \int_{p_2} \left( \frac{i_0}{p_1^{2b_1} p_2^{2b_2}} (p_1 - p_2)^{-b_3} \right), \quad b_3 \leq 0$$

This is directly factorizable, e.g.

$$I_0(b_1, b_2, -1) = \int_{p_1} \frac{e^{-2tp_1^2}}{p_1^{2b_1}} \int_{p_2} \frac{e^{-2tp_2^2}}{p_2^{2b_2}} (p_1^2 + p_2^2 - 2p_1 \cdot p_2) = I_0(b_1 + 1, b_2, 0) + I_0(b_1, b_2 + 1, 0)$$

Or graphically



$$\sim \sum_i c_i \boxed{b_1^{(i)}} \cdot \boxed{b_2^{(i)}} \longrightarrow I_1(b_1, b_2, b_3) = 0 \quad \text{if } b_1 \text{ or } b_2 \leq 0$$

# Results

**Example:** PDF  $n=4$  moments

- Diagrams  $O(\text{weeks})$
- Integrals  $O(\text{months})$



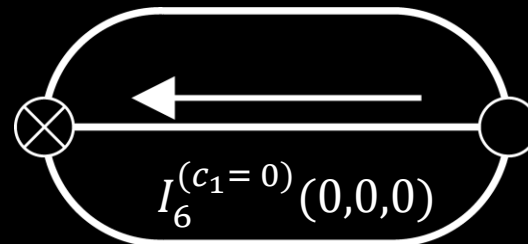
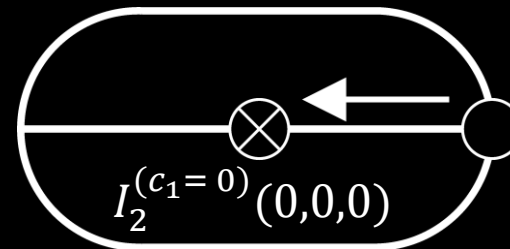
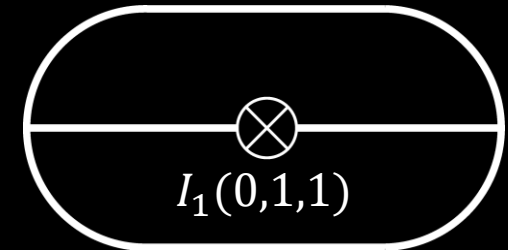
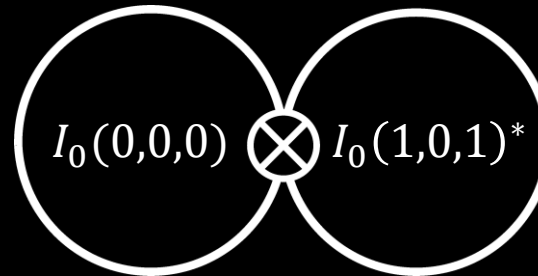
**Example:** PDF  $n=4$  moments

- Diagrams  $O(\text{weeks})$
- Integrals  $O(\text{day})$

**Example:** PDF  $n=6$  moments

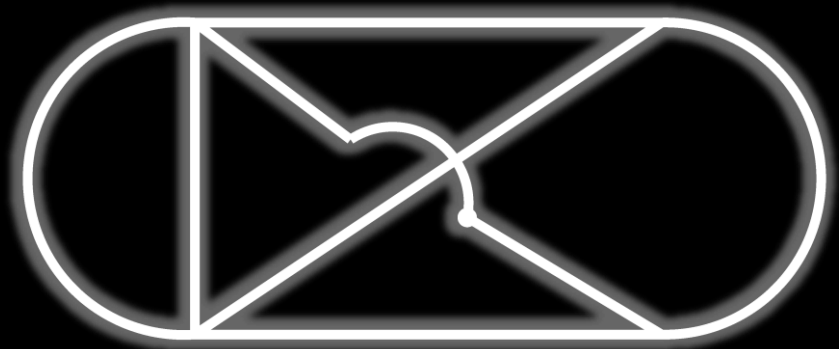
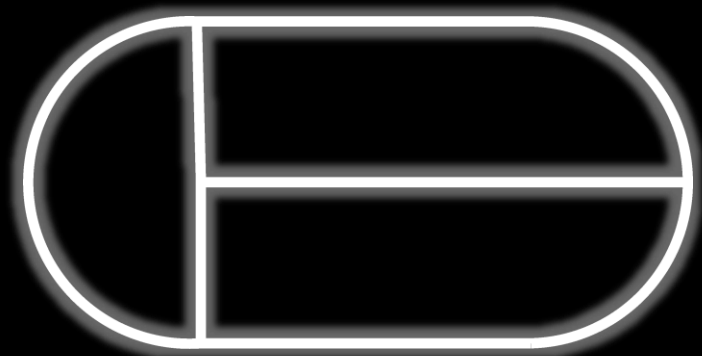
- Diagrams  $O(\text{weeks})$
- Integrals  $O(\text{days})^\dagger$

Evidence for six physical master integrals  
(one unphysical\*)



Except \* calculated in  
[Harlander et al. 2018]

$^\dagger$  used simplify step  
then separate KIRA  
reduction



# Outlook



# Next steps

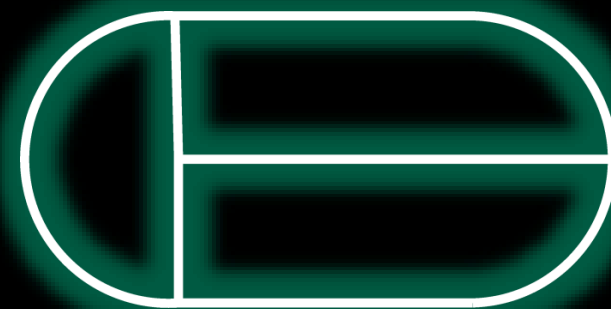
## Two Loop



**Example:** PDF  $n=4$  moments

- Four flow-times
- $\sim 150$  sectors
- $\sim 80,000$  integrals
- Diagrams  $O(\text{weeks})$
- Integrals  $O(\text{months})$
- Six masters (so far)

## Three Loop



**Example:** SFTX MDM

- Five flow-times
- 12,000 integrals
- Diagrams  $O(\text{weeks})$
- Integral calculation unfinished

# Next steps

## Two Loop

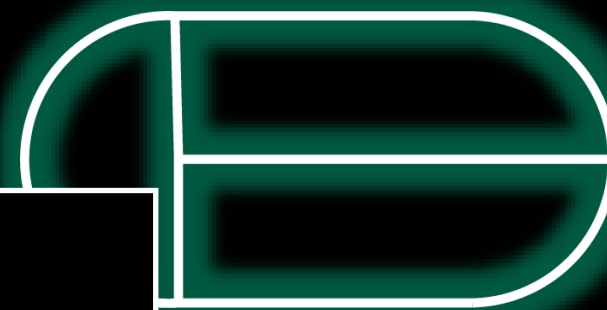


**Example:** PDF n=

- Four flow-times
- ~150 sectors
- ~80,000 integrals
- Diagrams O(weeks)
- Integrals O(months)
- Six masters (so far)

Can we do a similar  
analysis at the three loop  
level?

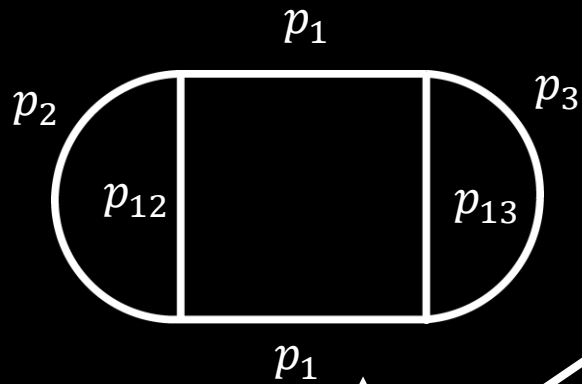
## Three Loop



**Example:** SFTX MDM

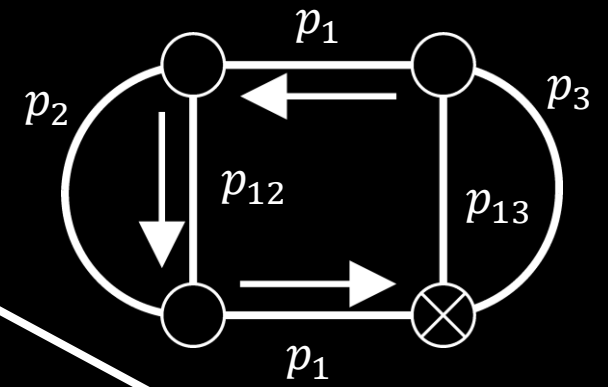
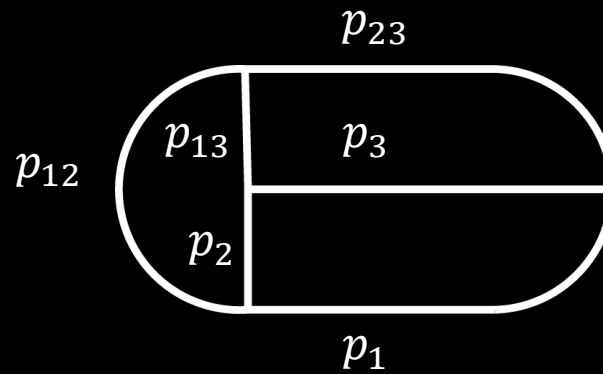
- Four flow-times
- ~150 sectors
- ~80,000 integrals
- Diagrams O(weeks)
- Integral calculation unfinished

# Higher loops



Same propagator topology

General three-loop bubble topology



Different exponentials

Expected total structures 96  
Maximum four flow-times:

29 Factorizable

24



?

19



24



?

Note: we find  
100s three-loop  
masters...

# Thank you for listening

## Aim

Provide a methodology to efficiently reduce and calculate any two-loop gradient flow integral.

## Outline

- State of the art in the perturbative gradient flow
- Computational bottlenecks
- Gradient flow integrals
- Outlook

# Sector reduction

Definition:

$$i_4 = e^{-(2-u_1+u_1u_2)tp_1^2-(u_1-u_1u_2)tp_2^2} \cdot e^{-(u_1+u_1u_2)tp_{12}^2}$$

Consider our remaining sector with two flow-time parameters

$$I_4(\mathbf{c}, \mathbf{b}) = \int_{p_1} \int_{p_2} \int_0^1 du_1 \int_0^1 du_2 \frac{u_1^{c_1} u_2^{c_2} i_4}{p_1^{2b_1} p_2^{2b_2} p_{12}^{2b_3}}, \quad c_1 > c_2$$

Combining flow-time IBP relations we can find

$$2I_4(\{c_1, c_2\}, \{b_1, b_2, b_3\}) + \text{Limits} =$$

$$c_2 I_4(\{c_1-1, c_2-1\}, \{b_1, b_2, b_3+1\}) + (c_1 - c_2 - 1)I_4(\{c_1-1, c_2\}, \{b_1, b_2, b_3+1\})$$

