CRC Young Scientist Meeting 2026

Tripole contribution to soft gluon emission at two loops

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In collaboration with: Michał Czakon

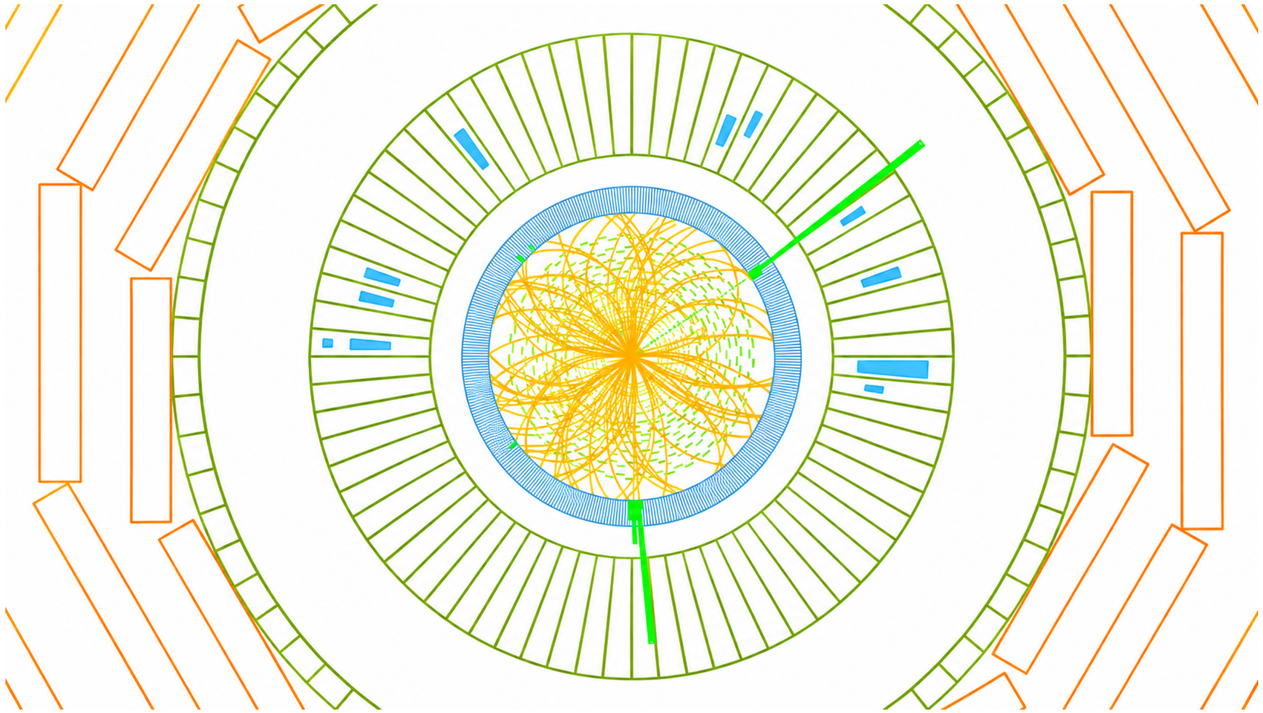
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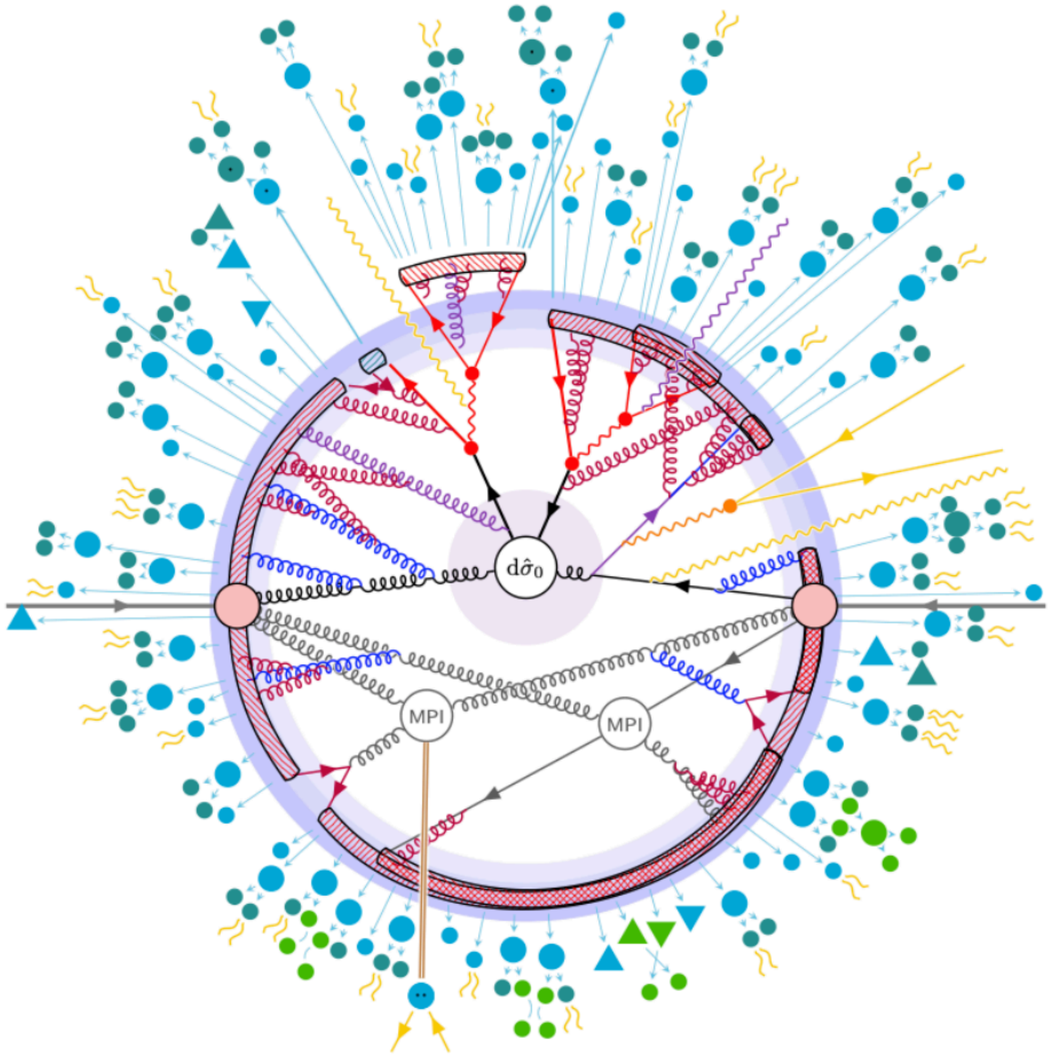
Research Training Group Physics of the Heaviest Particles at the LHC | RWTH AACHEN UNIVERSITY

Collaborative Research Center TRR 257
PPH
Particle Physics Phenomenology after the Higgs Discovery

Hadron collisions



[CMS collaboration]



[Pythia collaboration]

Cross section computation: Perturbative QCD

$$\sigma = \alpha_s^k \sigma_{LO} + \alpha_s^{k+1} \sigma_{NLO} + \alpha_s^{k+2} \sigma_{NNLO} + \alpha_s^{k+3} \sigma_{N^3LO} + \ldots + \mathcal{O}(\alpha_s^n)$$

$\mathcal{O}(20\% - 50\%)$ $\mathcal{O}(10\% - 20\%)$ $\mathcal{O}(1\% - 5\%)$ $\mathcal{O}(< 1\%)$

My work is required for the construction of a subtraction scheme at N^3LO

But what is and why a subtraction scheme is required can be understood taking NLO as an example

$$\sigma_{LO} = \int d\Phi_{N_f} \langle \mathscr{M}^{(0)}_{N_f} \,|\, \mathscr{M}^{(0)}_{N_f} \rangle$$

Beyond leading order regularization is required $\quad d = 4 - 2\epsilon$

$$\sigma_{NLO} = \sigma^R_{NLO} + \sigma^V_{NLO} = \overbrace{\int d\Phi_{N_f+1} \langle \mathscr{M}^{(0)}_{N_f+1} \,|\, \mathscr{M}^{(0)}_{N_f+1} \rangle + \int d\Phi_{N_f} 2Re\left(\langle \mathscr{M}^{(0)}_{N_f} \,|\, \mathscr{M}^{(1)}_{N_f} \rangle \right)}^{\text{Finite}}$$

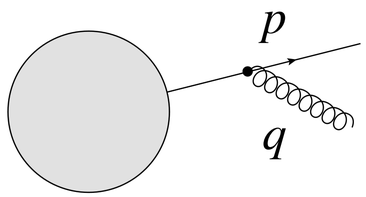
Separately divergent: contains IR-pole

IR-Singularities, Factorization and $d\sigma^a$

Amplitudes **diverge** when **soft** or **collinear** radiations are present

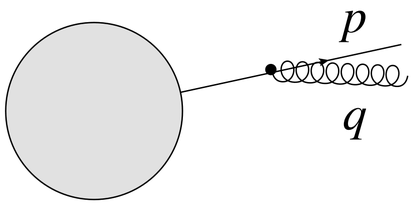
$$\frac{1}{(p+q)^2} \approx \frac{1}{2p \cdot q}$$

soft



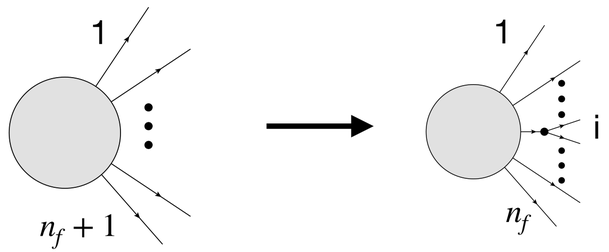
$q^0 \ll \sqrt{s}$

collinear



$p \cdot q \ll s$

$$|M(\{p_i\}_{i=1}^n, q)\rangle \sim \epsilon_{\pm}^{\mu *}(q) J_\mu^a(\{\beta_m\}, q)\,|M(\{p_i\}_{i=1}^n)\rangle \equiv S^{\pm,a}(\{\beta_m\}, q)\,|M(\{p_i\}_{i=1}^n)\rangle$$



Factorization ensures the currents $J_\mu^a(\{\beta_m\}, q)$ and $S^{\pm,a}(\{\beta_m\}, q)$ universality

Cross Section computation: Subtraction scheme

Introduce a *local counter-term* $d\sigma^a$ to cure the divergence of the real part

$$\sigma_{NLO} = \int d\Phi_{N_f+1} \left(d\sigma^R - d\sigma^a \right) + \int d\Phi_{N_f+1}\, d\sigma^a + \int d\Phi_{N_f}\, d\sigma^V$$

The singularity is confined in the last two terms

$$\sigma_{NLO} = \int d\Phi_{N_f+1} \left(d\sigma^R_{\epsilon=0} - d\sigma^a_{\epsilon=0} \right) + \int d\Phi_{N_f} \left(d\sigma^V + \int d\Phi_1\, d\sigma^a \right)$$

Example:

$$\int_0^1 dx \frac{f(x)}{x^{1+a\epsilon}} = \int_0^1 dx \frac{f(x) - f(0) + f(0)}{x^{1+a\epsilon}} = \int_0^1 \frac{f(x) - f(0)}{x} dx - \frac{f(0)}{a\epsilon} + \mathcal{O}(\epsilon)$$

N^3LO subtraction scheme

The construction of an N^3LO subtraction scheme require:

	Soft Current	Splitting Functions
Tree level	Triple Soft	Quadruple collinear
1 loop	Double soft	Triple Collinear
2 loops	Soft	Collinear

My goal ⇒ computation of the *soft* contribution at *2 loops* at $\mathcal{O}(\epsilon)$

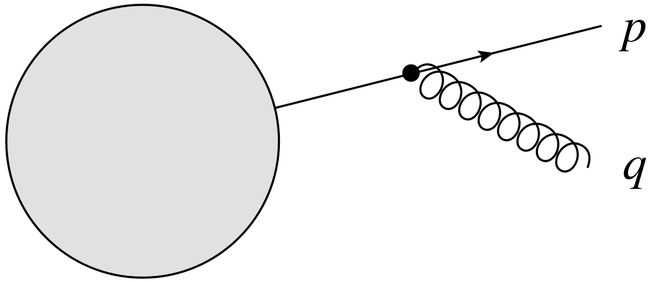
Example:

$$\int_0^1 dx \frac{f(x)}{x^{1+a\epsilon}} = \int_0^1 dx \frac{f(x)-f(0)+f(0)}{x^{1+a\epsilon}} = \int_0^1 \frac{f(x)-f(0)}{x} dx - \frac{f(0)}{a\epsilon} + \mathcal{O}(\epsilon)$$

Soft contribution at tree level: warm up

Eikonal approx.:

$$\frac{1}{(p_i+q)^2} \approx \frac{1}{2p_i\cdot q}$$



$$J_\mu^{a(0)}(\{\beta_i\},q) = \sum_{i=1}^{n} \mathbf{T}_i^a \frac{p_{i,\mu}}{p_i\cdot q} = \frac{1}{2n}\sum_{i\neq j=1}^{n}\left(\mathbf{T}_i^a - \mathbf{T}_j^a\right)\left(\frac{p_{i,\mu}}{p_i\cdot q} - \frac{p_{j,\mu}}{p_j\cdot q}\right)$$

Dependence only by the hard partons angular direction

$$\beta_i^\mu = \frac{p_i^\mu}{p_i^0}$$

Using:

$$\frac{\epsilon_+^*\cdot p_i}{q\cdot p_i} - \frac{\epsilon_+^*\cdot p_j}{q\cdot p_j} = \sqrt{2}\frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}$$

$$S_+^{a(0)}(\{\beta_i\},q) = \frac{1}{\sqrt{2}n}\sum_{i\neq j=1}^{n}\left(\mathbf{T}_i^a - \mathbf{T}_j^a\right)\frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}$$

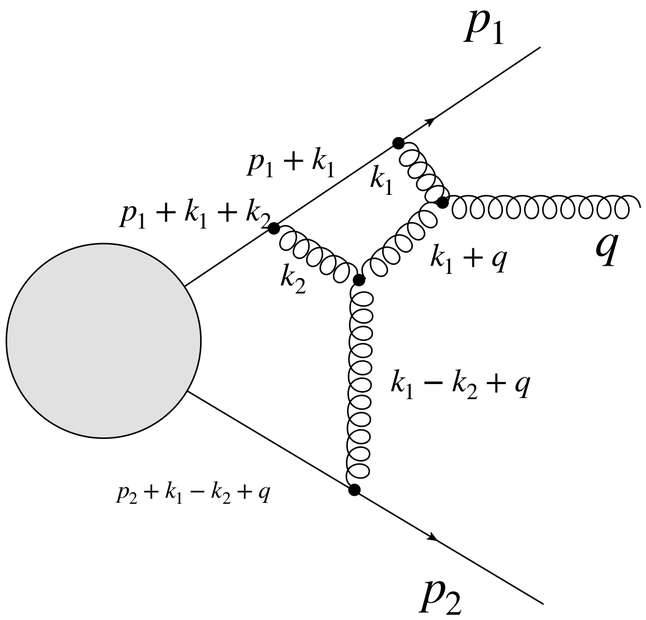
Soft contribution at 2 loops

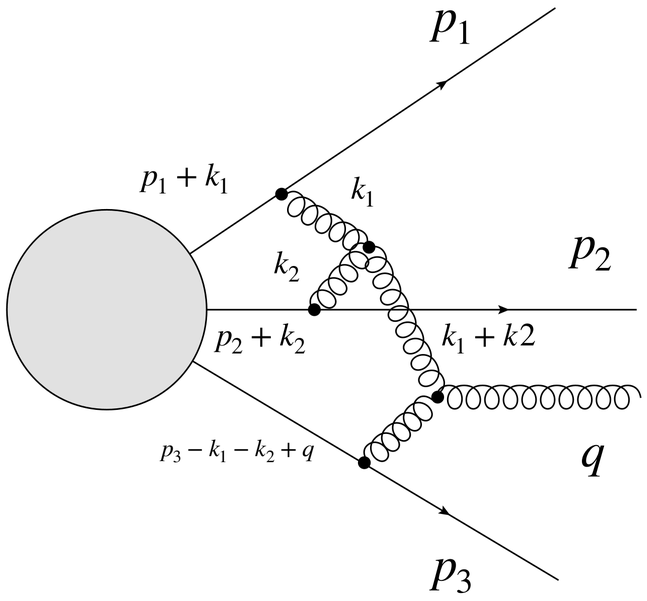
At two loops we can have contribution from diagrams with two or three hard legs

$$S_{+}^{a(2)} = \frac{1}{2}\sum_{i\neq j} S_{+,ij}^{a(2)} - \frac{1}{4}\sum_{i\neq j\neq k} S_{+,ijk}^{a(2)}\,.$$

Dipole

(Already available)





Tripole

(Not yet available)

Toolchain

- *Diagram Generation via private software in Form readable files;* **[DiaGen]**

- *Color algebra and conservation for extracting the correct color factor;*
- *Kinematic simplification;* **[FORM]**
- *Helicity amplitude projection;*

- *Integration By Parts Identities for Master Integrals reduction;* **[KIRA]**

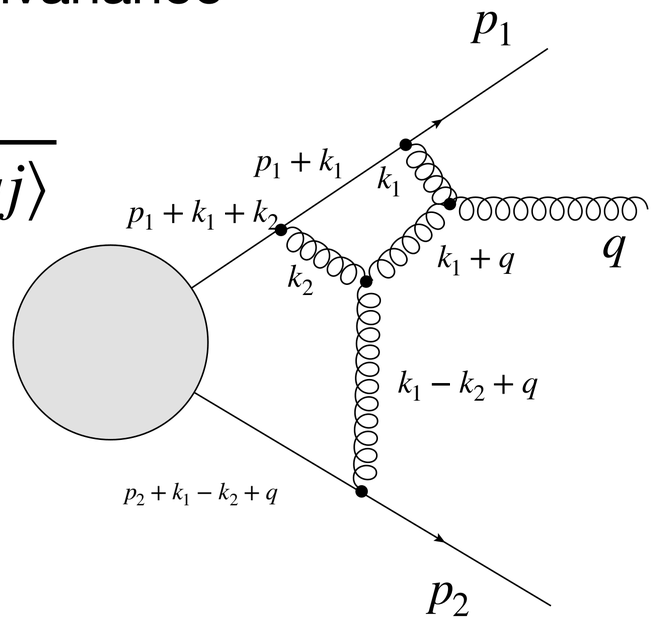
- *Numerical MIs evaluation;* **[AMFlow], [PySecDec]**

- *Analytical MIs evaluation with differential equations method.* **[LiteRed2], [Canonica], [PolyLogTools]**

Dipole contributions $S^{a(2)}_{+,\,ij}$

For the dipole factor we can completely fix the kinematic dependence on parton momenta by dimension analysis and scale invariance

$$S_+^{a(2)} = i\sqrt{2}\; C_2(\epsilon) \quad f^{aa_ia_j}\mathbf{T}_i^{a_i}\mathbf{T}_j^{a_j} \quad (V_{ij}^q)^2 \quad \frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}$$



Where:

$$V_{ij}^q = \left[\frac{\mu^2(-s_{ij})}{(-s_{iq})(-s_{jq})}\right]^{\epsilon}$$

$$s_{ab} = \langle ab\rangle[ba] = -\,2\;|p_a\cdot p_b|\;e^{-i\pi\lambda_{ab}} \qquad \log(-s_{ab}) \to \log|s_{ab}| - i\pi\Theta(s_{ab})$$

where $\lambda_{ab} = 1$ if both a and b are incoming or outgoing , and 0 otherwise

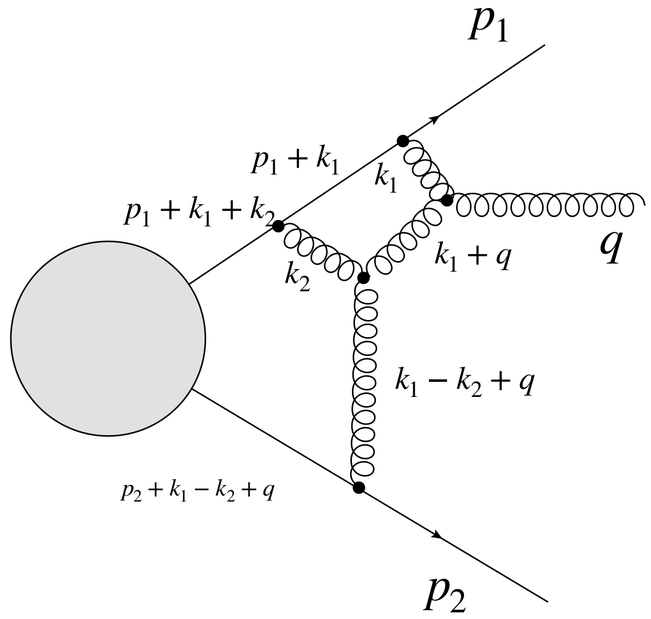
The analytical continuation is completely specified by these expressions

Dipole contributions $S^{a(2)}_{+,\,ij}$

$$S^{a(2)}_{+,ij} = i\sqrt{2}\; C_2(\epsilon) \quad f^{aa_i a_j}\mathbf{T}^{a_i}_i\mathbf{T}^{a_j}_j \quad (V^q_{ij})^2 \quad \frac{\langle ij\rangle}{\langle iq\rangle\langle qj\rangle}$$

And

$$\begin{aligned}
C_2(\epsilon) =& N_f\left[\frac{1}{6\epsilon^3}+\frac{5}{18\epsilon^2}+\left(\frac{19}{54}+\frac{\zeta_2}{6}\right)\frac{1}{\epsilon}+\left(\frac{65}{162}+\frac{5\zeta_2}{18}-\frac{31\zeta_3}{9}\right)\right.\\
&\left.+\left(\frac{211}{486}-\frac{35\zeta_2}{54}-\frac{155\zeta_3}{27}-\frac{185\zeta_4}{24}\right)\epsilon\right]\\
&+C_A\left[\frac{1}{2\epsilon^4}-\frac{11}{12\epsilon^3}+\left(-\frac{67}{36}+\zeta_2\right)\frac{1}{\epsilon^2}+\left(-\frac{193}{54}-\frac{11\zeta_2}{12}-\frac{11\zeta_3}{6}\right)\frac{1}{\epsilon}\right.\\
&+\left(-\frac{571}{81}-\frac{67\zeta_2}{36}+\frac{341\zeta_3}{18}+\frac{7\zeta_4}{8}\right)\\
&\left.+\left(-\frac{3410}{243}-\frac{139\zeta_2}{54}+\frac{2077\zeta_3}{54}+\frac{2035\zeta_4}{48}-\frac{247\zeta_5}{10}-\frac{7\zeta_2\zeta_3}{6}\right)\epsilon\right]+\mathcal{O}(\epsilon^2)
\end{aligned}$$



→ No explicit subleading N_c terms

These are generated by the color operators acting on an amplitude with more than two colored hard partons

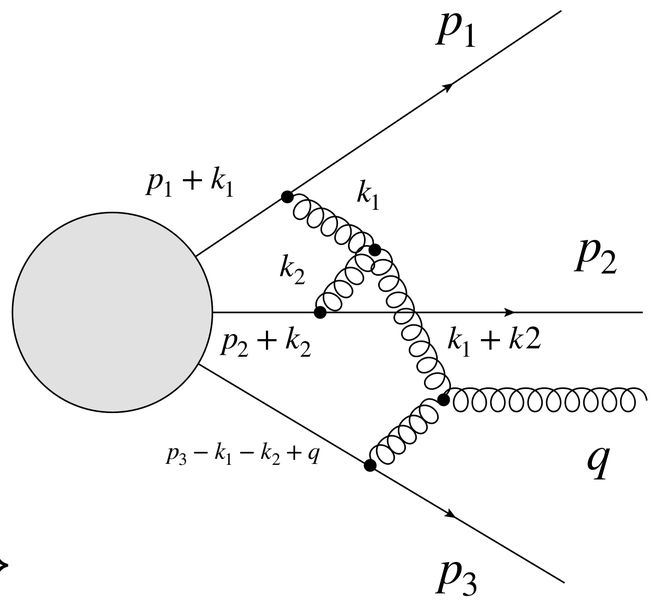
Tripole contributions $S^{a(2)}_{+,\,ijk}$

This computation is more involved, we can now built two dimensionless rescaling invariant variables

$$u_k^{ij} \equiv \frac{s_{ik}s_{jq}}{s_{ij}s_{kq}} \qquad\qquad v_k^{ij} \equiv \frac{s_{jk}s_{iq}}{s_{ij}s_{kq}}$$

Using the Schouten and Jacobi identity, we know it will have the form:

$$\sum_{\{ijk\}} S^{a(2)}_{+,\{ijk\}} = \mathbf{T}_i^{a_i}\mathbf{T}_j^{a_j}\mathbf{T}_k^{a_k} \left\{ \xi_\epsilon (V^q_{ik})^2 \left[f^{aa_jb} f^{ba_ia_k} D_1(u,v) + f^{aa_ib} f^{ba_ja_k} D_2(u,v) \right] \frac{\sqrt{2}\langle ik\rangle}{\langle iq\rangle\langle qk\rangle} \right.$$
$$\left. + \xi_\epsilon (V^q_{jk})^2 \left[f^{aa_ib} f^{ba_ja_k} D'_1(u,v) + f^{aa_jb} f^{ba_ia_k} D'_2(u,v) \right] \frac{\sqrt{2}\langle jk\rangle}{\langle jq\rangle\langle qk\rangle} \right\}$$



Where: $\xi_\epsilon = 1 + \epsilon^2\zeta_2$

Tripole contributions $S^{a(2)}_{+,\,ijk}$: D_i

Generation: Diagrams for the configuration: tripole, and dipole + non interacting leg

The color generators are specified for: $\left(\mathbf{T}_i^{a_i}\right)_{\alpha\beta} \to \left(T_i^{a_i}\right)_{\alpha\beta}$, $\left(\mathbf{T}_j^{a_j}\right)_{\alpha\beta} \to \left(-T_j^{a_j}\right)_{\beta\alpha} \left(\mathbf{T}_k^{a_k}\right)_{\alpha\beta} \to i f^{a_k \alpha\beta}$

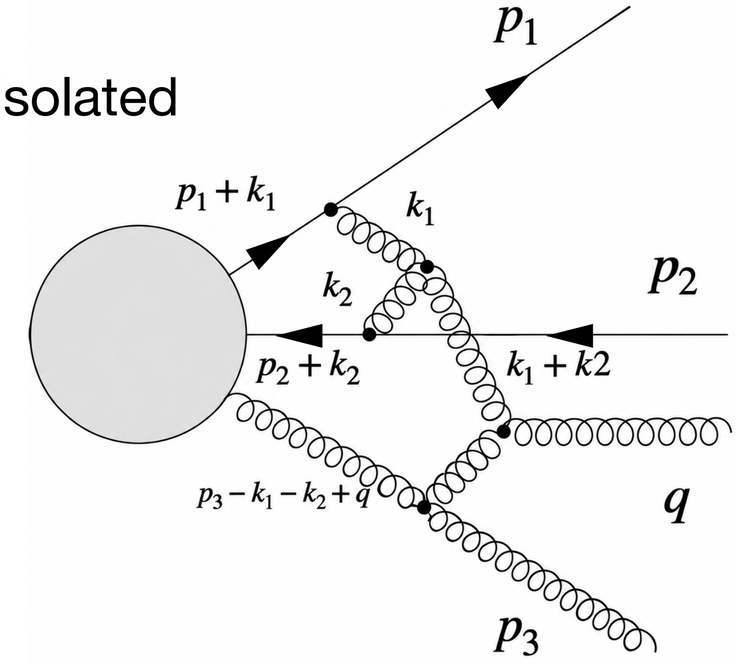
The dipole result, for every couple of hard momenta is subtracted

$\Rightarrow$ The pure tripole contribution has been isolated

$\Rightarrow$ This family contains 8 MIs, computed via the method of differential equations. And then mapped in the 3 possible kinematic families

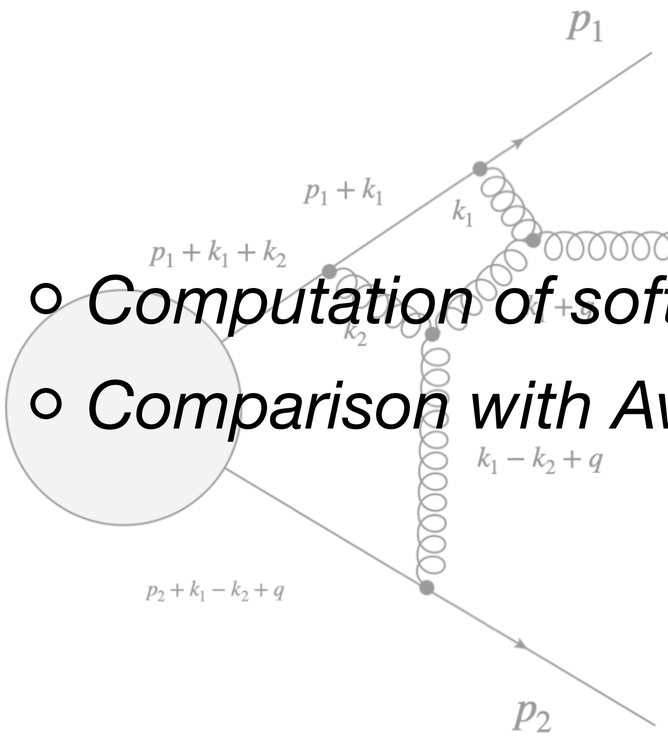
$p_1 \leftrightarrow p_3$ $\qquad$ $p_2 \leftrightarrow p_3$

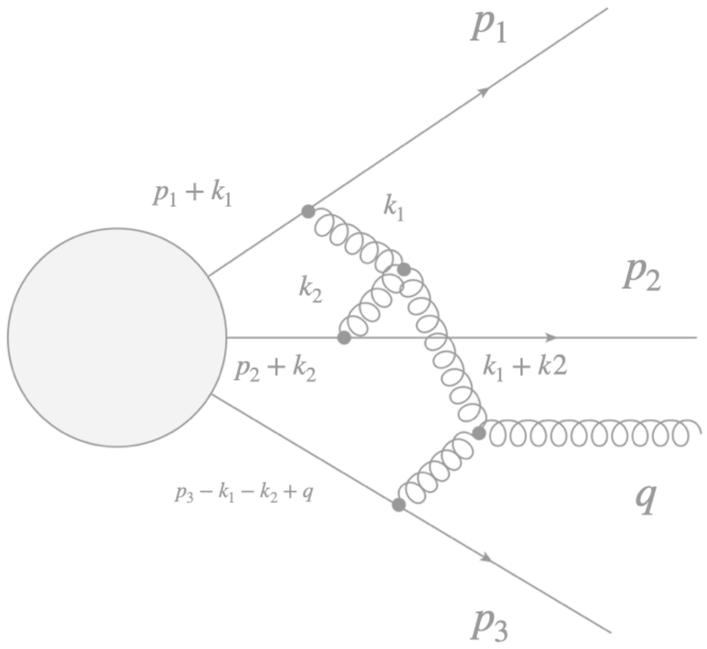
$\Rightarrow$ Solved in terms of GPLs



Outlook

- *Computation of soft contribution at two loops;*
- *Comparison with Available partial results;*





Thank you for your attention!!!

Backup

Tripole contributions $S^{a(2)}_{+,\,ijk}$: D_i

The differential eq. in u_k^{ij} and v_k^{ij} variables how a dependence from

Källén function $$\Delta_k^{ij} = \sqrt{1 - 2u_k^{ij} - 2v_k^{ij} + \left(u_k^{ij} - v_k^{ij}\right)^2}$$

The following change of variables is considered: $$u_k^{ij} \equiv (1 - z_k^{ij})(1 - \bar{z}_k^{ij}), \qquad v_k^{ij} \equiv z_k^{ij}\bar{z}_k^{ij}$$

The system of differential equation is generated and solved via:

`LiteRed2, Canonica, PolyLogTools, AMFlow`

⇒ 8 possible kinematical regions, analytical continuation of u_k^{ij} and v_k^{ij} is necessary

$$u_k^{ij} e^{-i\pi\left(\theta(s_{ik}) + \theta(s_{jq}) - \theta(s_{ij}) - \theta(s_{kq})\right)} \qquad v_k^{ij} e^{-i\pi\left(\theta(s_{jk}) + \theta(s_{iq}) - \theta(s_{ij}) - \theta(s_{kq})\right)}$$

A_0: All outgoing ⇒ $u_k^{ij} = |u_k^{ij}|, v_k^{ij} = |v_k^{ij}|$

A_1: j, k incoming, i, q outgoing ⇒ $u_k^{ij} \to |u_k^{ij}|, v_k^{ij} \to |v_k^{ij}| e^{-i2\pi}$

A_2: i, k incoming, j, q outgoing ⇒ $u_k^{ij} \to |u_k^{ij}| e^{-i2\pi}, v_k^{ij} \to |v_k^{ij}|$

A_3: i, j incoming, k, q outgoing ⇒ $u_k^{ij} \to |u_k^{ij}| e^{i2\pi}, v_k^{ij} \to |v_k^{ij}| e^{i2\pi}$

A_4: i incoming, j, k, q outgoing ⇒ $u_k^{ij} \to |u_k^{ij}|, v_k^{ij} \to |v_k^{ij}|$

A_5: j incoming, i, k, q outgoing ⇒ $u_k^{ij} \to |u_k^{ij}|, v_k^{ij} \to |v_k^{ij}|$

A_6: k incoming, i, j, q outgoing ⇒ $u_k^{ij} \to |u_k^{ij}|, v_k^{ij} \to |v_k^{ij}|$

A_7: i, j, k incoming, q outgoing ⇒ $u_k^{ij} \to |u_k^{ij}|, v_k^{ij} \to |v_k^{ij}|$