Standard Model Prediction of ϵ_K

Joachim Brod

UNIVERSITY OF Cincinnati

Ulifest, KIT
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Ancient history: strangeness

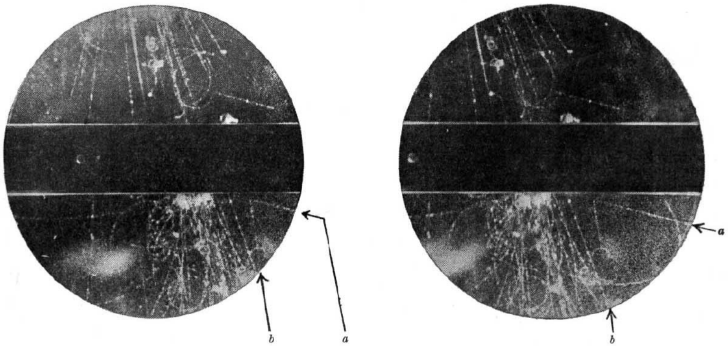


Fig. 1. STEREOSCOPIC PHOTOGRAPHS SHOWING AN UNUSUAL FORK (*a b*) IN THE GAS. THE DIRECTION OF THE MAGNETIC FIELD IS SUCH THAT A POSITIVE PARTICLE COMING DOWNWARDS IS DEVIATED IN AN ANTICLOCKWISE DIRECTION

$$K_S \to \pi^+\pi^-$$

[Rochester & Butler, 1947]

Ancient history: strangeness

856 NATURE December 20, 1947 Vol. 160

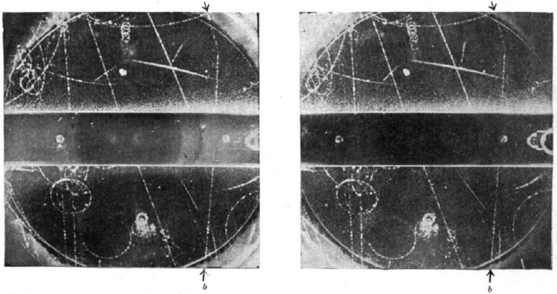
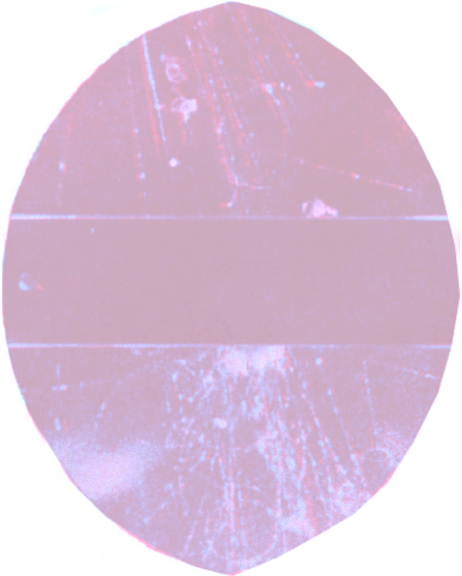


Fig. 2. STEREOSCOPIC PHOTOGRAPHS SHOWING AN UNUSUAL FORK (a b). THE DIRECTION OF THE MAGNETIC FIELD IS SUCH THAT A POSITIVE PARTICLE COMING DOWNWARDS IS DEVIATED IN A CLOCKWISE DIRECTION

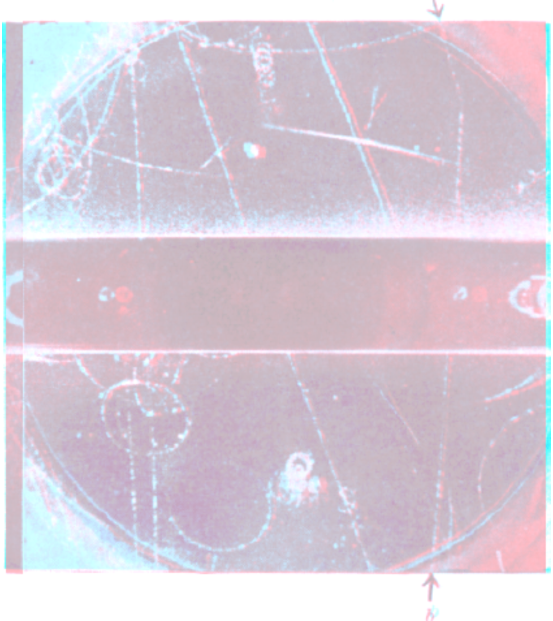
$$K^{\pm} \to \pi^0 e^{\pm} \nu_e$$

[Rochester & Butler, 1947]

Ancient history: strangeness



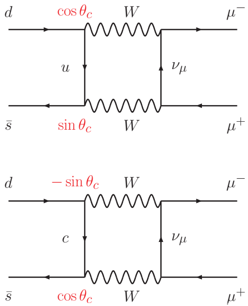
Ancient history: strangeness



History: Parity noninvariance, GIM, CPV

- Discovery of parity violation [Dalitz 1953; Lee & Yang, 1956]

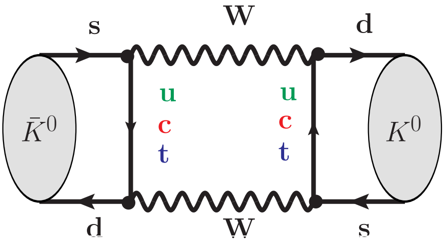
- Why is $Br(K_L \to \mu^+\mu^-) = 6.84(11) \times 10^{-9}$ so small compared to $Br(K_L \to \pi^\pm \mu^\mp \nu_\mu) = 27.04(7)\%$?
  - GIM Mechanism (1970) [Glashow, Iliopoulos, Maiani 1970]
  - $\Rightarrow$ Prediction of the charm quark!



- Observation of the decay $K_L \to \pi\pi$ [Christenson et al., 1964]
  - $\Rightarrow$ violation of CP symmetry; KM mechanism [Kobayashi, Maskawa 1973]

CP Violation

Reminder: neutral kaons



$\bar{K}^0 \sim [s\bar{d}]$ $K^0 \sim [\bar{s}d]$

Neutral Kaon Mixing

$$i\frac{d}{dt}\begin{pmatrix}|K^0(t)\rangle \\ |\bar{K}^0(t)\rangle\end{pmatrix} = \left(M - i\frac{\Gamma}{2}\right)\begin{pmatrix}|K^0(t)\rangle \\ |\bar{K}^0(t)\rangle\end{pmatrix}$$

Neutral Kaon Mixing

$$i\frac{d}{dt}\begin{pmatrix}|K^0(t)\rangle\\|\bar{K}^0(t)\rangle\end{pmatrix} = \left(M - i\frac{\Gamma}{2}\right)\begin{pmatrix}|K^0(t)\rangle\\|\bar{K}^0(t)\rangle\end{pmatrix}.$$

- The Hamiltonian is diagonalized by

$$|K_S(t)\rangle = e^{-(iM_S+\Gamma_S)t}|K_S\rangle \qquad |K_L(t)\rangle = e^{-(iM_L+\Gamma_L)t}|K_L\rangle$$

- where

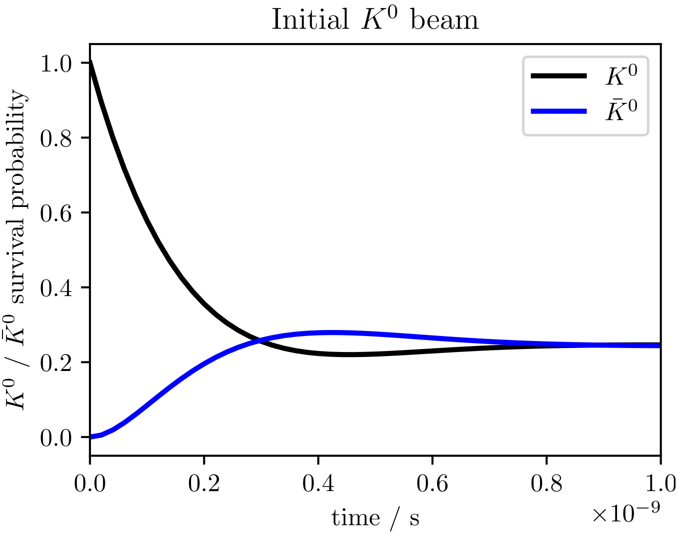
$$|K_S\rangle = p|K^0\rangle + q|\bar{K}^0\rangle \qquad |K_L\rangle = p|K^0\rangle - q|\bar{K}^0\rangle$$

- Mass difference $\Delta M_K \equiv M_L - M_S = 3.484 \times 10^{-15}\,\mathrm{GeV}$

CP properties

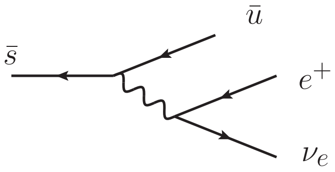
- CP transformation $(|K^0\rangle \sim |\bar{s}d\rangle, \quad |\overline{K^0}\rangle \sim |\bar{d}s\rangle)$
  - $CP|K^0\rangle = -|\overline{K^0}\rangle$
  - $CP|\overline{K^0}\rangle = -|K^0\rangle$
- CP eigenstates
  - $|K_1\rangle \equiv \left(|K^0\rangle - |\overline{K^0}\rangle\right)/\sqrt{2}$ (CP even)
  - $|K_2\rangle \equiv \left(|K^0\rangle + |\overline{K^0}\rangle\right)/\sqrt{2}$ (CP odd)
- $K_2^0 \to \pi\pi$ is forbidden by CP
  - $1/\Gamma_S = \tau_S = 8.954 \times 10^{-11}\,\mathrm{s} = 2.68\,\mathrm{cm}$
  - $1/\Gamma_L = \tau_L = 5.116 \times 10^{-8}\,\mathrm{s} = 15.3\,\mathrm{m}$
  - $\Gamma_{K_2} \ll \Gamma_{K_1}$

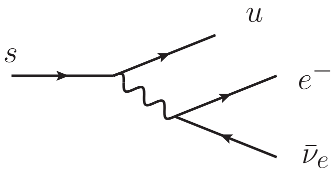
Kaon mixing



Initial K^0 beam
K^0
$\bar{K}^0$
1.0
0.8
0.6
0.4
0.2
0.0
K^0 / $\bar{K}^0$ survival probability
0.0
0.2
0.4
0.6
0.8
1.0
$\times 10^{-9}$
time / s

Kaon mixing

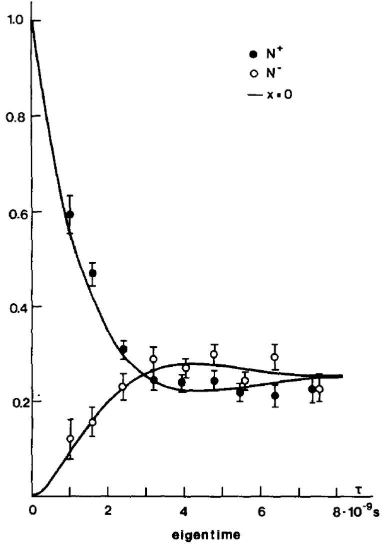




$[\bar{s}d] \sim K^0 \to \pi^- e^+ \nu_e$

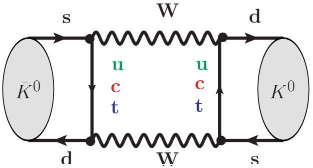
$[sd] \sim \bar{K}^0 \to \pi^+ e^- \bar{\nu}_e$

Kaon mixing



[Niebergall et al., 1974]

Calculating the neutral kaon mass difference

$\Delta M_K \propto \mathrm{Re}\Big\{$  $\Big\}$

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$\lambda_i \equiv V_{is}^* V_{id} \qquad\qquad \lambda_u = -\lambda_c - \lambda_t$$

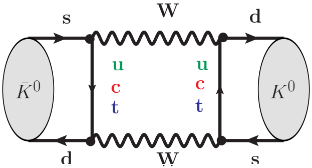
$$\Rightarrow \Delta M_K \propto \mathrm{Re}\Big\{ \lambda_c^2 \; F(m_c^2/M_W^2) \Big\}$$

Semileptonic CP asymmetry

$$\delta(\ell) = \frac{\Gamma(K_L \to \ell^+ \nu \pi^-) - \Gamma(K_L \to \ell^- \bar{\nu} \pi^+)}{\Gamma(K_L \to \ell^+ \nu \pi^-) + \Gamma(K_L \to \ell^- \bar{\nu} \pi^+)} = \frac{1 - |q/p|^2}{1 + |q/p|^2} = (3.32 \pm 0.06) \times 10^{-3}$$

- $\Rightarrow \quad |q| \neq |p|$
- $\Rightarrow$ CP violation in mixing
- $\delta(\ell) = 2\mathrm{Re}(\epsilon_K)$

Calculating the size of indirect CP violation

$\epsilon_K \propto \mathrm{Im}\Big\{$  $\Big\}$

$$V_{\mathrm{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

$$\lambda_i \equiv V_{is}^* V_{id} \qquad\qquad \lambda_u = -\lambda_c - \lambda_t$$

$$\Rightarrow \epsilon_K \propto \mathrm{Im}\Big\{\lambda_c^2\ F(m_c^2/M_W^2) + \lambda_c\lambda_t\ F(m_c^2/M_W^2, m_t^2/M_W^2) + \lambda_t^2\ F(m_t^2/M_W^2)\Big\}$$

LO prediction of $\mathcal{L}_{|\Delta S|=2}$

- First estimate using a free four-quark model [Vaintshein, Kriplovich, Pisma Zh. Eksp. Teor. Fiz. 18, 1973]
- LL resummation, η_1: [Vaintshein, Zakharov, Novikov, Shifman, Yad. Fiz. 23, 1976]
- LL resummation, η_2 (light top): [Vysotsky, Sov. J. Nucl. Phys 31, 1980]
- Witten explains [Witten, Nucl. Phys. B 122 (1977)] how to use Wilson's operator expansion [Wilson, Phys.Rev. 179 (1969)]
- Following this method, η_1, η_2, and η_3 have been obtained at LL [Gilman, Wise Phys.Rev.D 27 (1983)]
- Finally, the results have been extended for a heavy top quark [Flynn Mod.Phys.Lett.A 5 (1990); Datta, Fröhlich, Paschos, Z.Phys.C 46 (1990)]

The “NLO revolution”

The “NLO revolution”

- NLO top contribution (η_2) [Buras, Jamin, Weisz Nucl.Phys.B 347 (1990)]
- NLO charm contribution (η_1)[Herrlich, Nierste Nucl.Phys.B 419 (1994)]
- NLO charm-top contribution (η_3)[Herrlich, Nierste Nucl.Phys.B 476 (1996)]

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Nuclear Physics B419 (1994) 292–322
NUCLEAR PHYSICS B

Enhancement of the K_L–K_S mass difference by short-distance QCD corrections beyond leading logarithms ⋆

Stefan Herrlich [1], Ulrich Nierste [2]
Physik-Department T31, Technische Universität München, D-85747 Garching, Germany

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Abstract

We calculate the next-to-leading order short-distance QCD corrections to the coefficient η_1 of the effective $\Delta S = 2$ hamiltonian in the standard model. This part dominates the short-distance contribution $(\Delta m_K)^{SD}$ to the K_L–K_S mass difference. The next-to-leading order result enhances η_1 and $(\Delta m_K)^{SD}$ by 20% compared to the leading order estimate.

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Nuclear Physics B 476 (1996) 27–88
NUCLEAR PHYSICS B

The complete $|\Delta S| = 2$ Hamiltonian in the next-to-leading order ⋆

Stefan Herrlich [a,1], Ulrich Nierste [b,2]
[a] *DESY-IfH, Platanenallee 6, D-15738 Zeuthen, Germany*
[b] *Physik-Department, TU München, D-85747 Garching, Germany*

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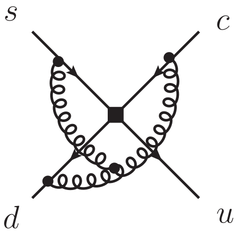
Abstract

We present the complete next-to-leading order short-distance QCD corrections to the effective $|\Delta S| = 2$ Hamiltonian in the Standard Model. The calculation of the coefficient η_3 is described in great detail. It involves the two-loop mixing of bilocal structures composed of two $|\Delta S| = 1$ operators into $|\Delta S| = 2$ operators. The next-to-leading order corrections enhance η_3 by 27% to $\eta_3 = 0.47^{+0.03}_{-0.04}$ thereby affecting the phenomenology of the *CP* parameter ϵ_K sizeably. η_3 depends on the physical input parameters m_t, m_c and $\Lambda_{\overline{MS}}$ only weakly. The quoted error stems from renormalization scale dependences, which have reduced compared to the old leading log result.

Evanescent operators and all that

- Calculation performed in $d = 4 - 2\epsilon$ spacetime dimensions
  - $E = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3} c_L)(\bar{u}_L \gamma^{\mu_1\mu_2\mu_3} d_L) - 16(\bar{s}_L \gamma_\mu c_L)(\bar{u}_L \gamma^\mu d_L) = \mathcal{O}(\epsilon)$
  - $E = (\bar{s}_L \gamma_{\mu_1\mu_2\mu_3} c_L)(\bar{u}_L \gamma^{\mu_1\mu_2\mu_3} d_L) - 16(\bar{s}_L \gamma_\mu c_L)(\bar{u}_L \gamma^\mu d_L) + \epsilon Q_{\text{phys}} = \mathcal{O}(\epsilon)$
- $\epsilon \times (1/\epsilon^2) = 1/\epsilon$
- Can use this to "engineer" anomalous dimensions starting at two-loop



Evanescent operators and all that

- Many details had to be clarified:
  - Are the evanescent operators uniquely defined? (No)
  - How do Wilson coefficients and ADM transform?
  - Generalization to double insertions (required for ϵ_K)
- Thoroughly discussed in
  [Herrlich, Nierste Nucl.Phys.B 455 (1995)]

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Nuclear Physics B 455 (1995) 39–58
NUCLEAR PHYSICS B

Evanescent operators, scheme dependences and double insertions ★

Stefan Herrlich [a,1], Ulrich Nierste [b,2]
[a] Paul Scherrer Institut, CH-5232 Villigen PSI, Switzerland
[b] Physik Department, TU München, D-85747 Garching, Germany

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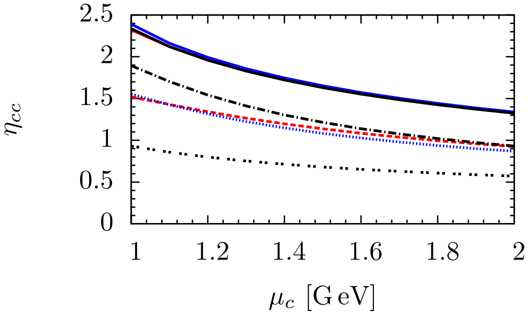
Abstract

The anomalous dimension matrix of dimensionally regularized four-quark operators is known to be affected by evanescent operators, which vanish in $D = 4$ dimensions. Their definition, however, is not unique, as one can always redefine them by adding a term proportional to $(D-4)$ times a physical operator. In the present paper we compare different definitions used in the literature and find that they correspond to different renormalization schemes in the *physical* operator basis. The scheme transformation formulae for the Wilson coefficients and the anomalous dimension matrix

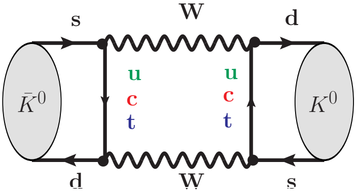
NNLO

- NNLO charm-top (η_3) [Brod, Gorbahn Phys.Rev.D 82 (2010)]
- NNLO charm (η_1) [Brod, Gorbahn Phys.Rev.Lett 108 (2012)]

- At that stage, the calculations hit an impasse:
  the charm contribution did not seem to converge.



c-t vs. u-t Unitarity



$$\lambda_u = V_{us} V_{ud}^* , \quad \lambda_c = V_{cs} V_{cd}^* , \quad \lambda_t = V_{ts} V_{td}^* , \quad \lambda_u + \lambda_c + \lambda_t = 0$$

c-t unitarity

	Im	Re
λ_t^2	$\sim \lambda^{10}$	$\sim \lambda^{10}$
$\lambda_c \lambda_t$	$\sim \lambda^6$	$\sim \lambda^6$
λ_c^2	$\sim \lambda^6$	$\sim \lambda^2$

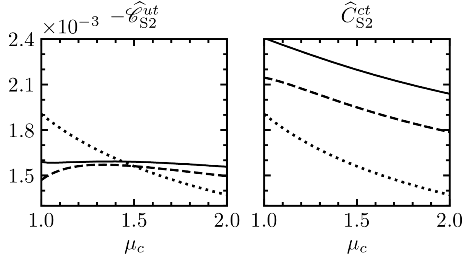
u-t unitarity

	Im	Re
λ_t^2	$\sim \lambda^{10}$	$\sim \lambda^{10}$
$\lambda_u \lambda_t$	$\sim \lambda^6$	$\sim \lambda^6$
λ_u^2	0	$\sim \lambda^2$

$$\mathrm{Im}(M_{12}) \to \epsilon_K \qquad \mathrm{Re}(M_{12}) \to \Delta M_K$$

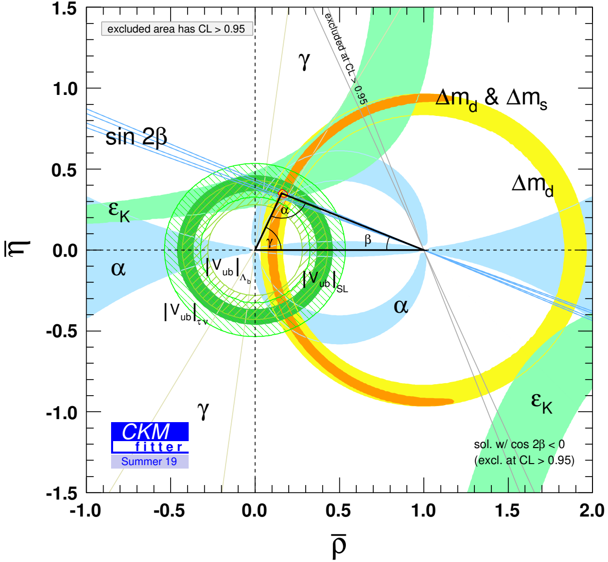
How to calculate ϵ_K efficiently

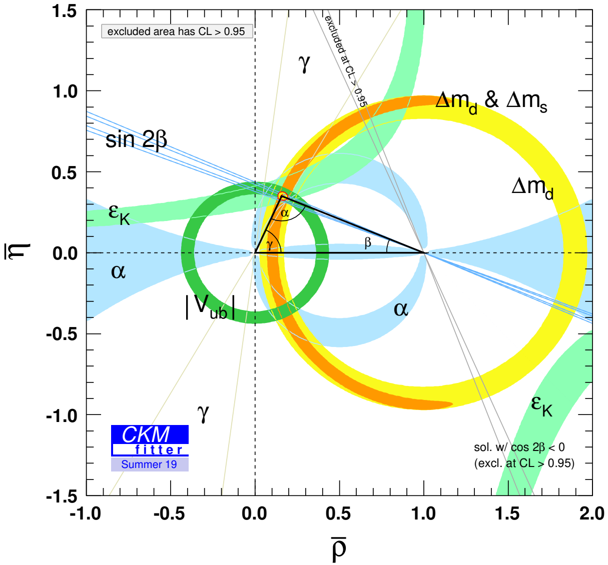
- Application to ϵ_K short-distance contribution [Brod, Gorbahn, Stamou 1911.06822]
  - Can "recycle" NNLO anomalous dimensions and matching conditions
- "Simple" rearrangement of effective Hamiltonian has reduced perturbative uncertainty (@ NNLO!) from $\sim 30\%$ to $\sim 1\%$



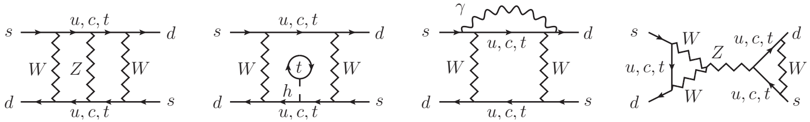
Interpretation

- $|\Delta S| = 2$ Hamiltonian has one real ($\mathcal{C}_1$) and two imaginary ($\mathcal{C}_2$, $\mathcal{C}_3$) parameters.
  - $\mathcal{C}_1$ is long-distance dominated ($\Rightarrow \Delta M_K$)
  - $\mathcal{C}_2$, $\mathcal{C}_3$ are short-distance dominated ($\Rightarrow \epsilon_K$)
- New choice:
  - $\mathscr{C}_{S2}^{uu} \equiv \mathcal{C}_1$, $\mathscr{C}_{S2}^{tt} \equiv \mathcal{C}_2$, $\mathscr{C}_{S2}^{ut} \equiv \mathcal{C}_3$
- Traditional choice:
  - $C_{S2}^{cc} \equiv \mathcal{C}_1$, $C_{S2}^{ct} \equiv 2\mathcal{C}_1 - \mathcal{C}_3$, $2C_{S2}^{tt} \equiv \mathcal{C}_1 + \mathcal{C}_2 - \mathcal{C}_3$


excluded area has CL > 0.95
excluded at CL > 0.95
γ
Δm_d & Δm_s
sin 2β
Δm_d
ε_K
α
β
γ
α
$|V_{ub}|_{\Lambda_b}$
$|V_{ub}|_{SL}$
α
$|V_{ub}|_{\tau\nu}$
γ
ε_K
CKM fitter
Summer 19
sol. w/ cos 2β < 0
(excl. at CL > 0.95)
$\bar{\eta}$
$\bar{\rho}$
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0

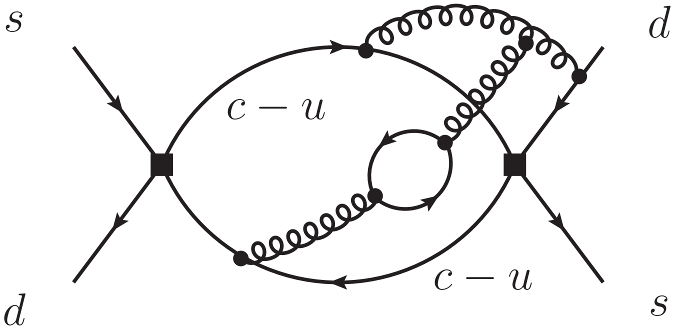

excluded area has CL > 0.95
excluded at CL > 0.95
γ
Δm_d & Δm_s
sin 2β
Δm_d
ε_K
α
β
γ
α
|V_ub|
α
γ
ε_K
CKM fitter
Summer 19
sol. w/ cos 2β < 0
(excl. at CL > 0.95)
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
η̄
ρ̄

SD contribution to ϵ_K – EW corrections



- Electroweak effects are large because of
  - Large couplings (e.g. top Yukawa) [Brod, Kvedaraite, Polonsky, Youssef 2207.07669]
  - Large logarithms [Brod, Kvedaraite, Polonsky, Youssef 2207.07669]
- Fix scheme of electroweak input parameters
  - Removes $\sim 5\%$ ambiguity
- Caveat: Scheme dependence due to missing QED contribution to B_K
  - Effect is expected to be tiny, $\mathcal{O}(\alpha/4\pi)$

Double insertions at NNNLO



SD contribution to ϵ_K – Outlook

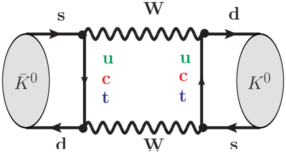
- NNLO QCD corrections to the top contribution
  [Gorbahn, Sondermann, Stamou, W.I.P.]
  - Residual error expected below percent level

- N^{3}LO QCD corrections to the charm contribution
  [Brod, Stamou, Steudtner, W.I.P.]
  - ✓ 4-loop operator mixing (current-current operators)
    [Brod, Stamou, Steudtner, 2604.16691]
  - $\Rightarrow$ 4-loop operator mixing (double insertions)
  - 3-loop weak-scale initial conditions
  - Full RG analysis
  - Residual error expected below percent level

(What is really the) Definition of ϵ_K

$$\epsilon_K \equiv \frac{\eta_{00} + 2\eta_{+-}}{3} \approx \frac{\langle (\pi\pi)_{I=0} | K_L \rangle}{\langle (\pi\pi)_{I=0} | K_S \rangle} \approx e^{i\phi_\epsilon} \sin\phi_\epsilon \left(\frac{\mathrm{Im} M_{12}}{2\mathrm{Re} M_{12}} - \frac{\mathrm{Im}\Gamma_{12}}{2\mathrm{Re}\Gamma_{12}} \right)$$

- $\phi_\epsilon \equiv \arctan \frac{\Delta M_K}{\Delta\Gamma_K/2} = (43.5 \pm 0.5)^\circ$
- M_{12} – dispersive part of $|\Delta S| = 2$ amplitude
- Γ_{12} – absorptive part of $|\Delta S| = 2$ amplitude
- Approximations good up to
  $|A_2|^2/|A_0|^2 \sim 0.002, \qquad \Gamma_L/\Gamma_S \sim 0.002.$



LD contribution to ϵ_K – hadronic matrix element

- Traditionally parameterized as deviations from V.I.A.

$$B_K(\mu) \equiv \frac{\langle \bar{K}^0 | Q^{|\Delta S|=2} | K^0 \rangle}{\frac{8}{3} f_K^2 M_K^2}$$

- $\hat{B}_K = 0.7533(91)$ ($N_f = 2 + 1$, NLO conversion)[FLAG Review 2024]
- RI-(S)MOM – $\overline{\text{MS}}$ conversion has recently been calculated at NNLO [Gorbahn, Jäger, Kvedaraite 2411.19861]
  - They quote $\hat{B}_K = 0.7627(60)$
  - N^3LO perturbative QCD correction would require three-loop conversion

LD contribution to ϵ_K – power corrections

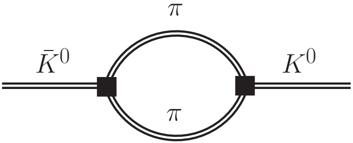
- What is the contribution of dimension-eight operators?
  - Naive estimate gives $m_K^2/m_c^2 \sim 15\%$ correction to charm contribution
- Leading term ($SU(3)$ chiral limit, large N_c): [Cata, Peris 0303162]
  - $O_4^{(8)} = g_s(\bar{s}_L\gamma^\mu \widetilde{G}_{\mu\nu} d_L)(\bar{s}_L\gamma^\nu d_L)$
  - gives $\sim 1\%$ correction to ϵ_K
- Contribution of all eight operators gives $+(1 \pm 0.3)\%$ shift in ϵ_K [Ciuchini et al. 2111.05153]
  - Matrix elements calculated in V.I.A. approximation
  - Can be improved by lattice QCD

Bilocal LD contribution to ϵ_K

$$\epsilon_K = e^{i\phi_\epsilon} \sin\phi_\epsilon \left(\frac{\mathrm{Im} M_{12}}{2\mathrm{Re} M_{12}} - \frac{\mathrm{Im}\Gamma_{12}}{2\mathrm{Re}\Gamma_{12}} \right)$$

Dispersive (real) and

absorptive (imaginary) part of



$$\int d^4x \langle \bar{K}^0 | H^{|\Delta S|=1}(x) H^{|\Delta S|=1}(0) | K^0 \rangle$$

- Estimate $\mathrm{Im}\Gamma_{12}/2\mathrm{Re}\Gamma_{12} = -\mathrm{Im}A_0/\mathrm{Re}A_0$ from ϵ'/ϵ: $-(6 \pm 2)\%$
  [Nierste, 0201071; Buras et al., 0805.3887]
- Estimate dispersive part in ChPT: $+(2.4 \pm 1.2)\%$ [Buras et al., 1002.3612]
- A lattice calculation with dynamical charms seems to be the way forward
  [Laiho et al., 0910.2928; Blum et al., 1502.00263; Bai et al. 1505.07863, 2309.01193]