Particle Physics Phenomenology
after the Higgs Discovery

Dealing with real emissions in NNLO computations

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Hard collisions at the LHC can be described in terms of quarks and gluons thanks to the collinear factorization theorem for hadronic cross sections [Collins, Soper, Sterman]

$$\mathrm{d}\sigma = \int \mathrm{d}x_1 \mathrm{d}x_2 f_i(x_1) f_j(x_2) \mathrm{d}\sigma_{ij} \mathcal{F} \left(1 + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{Q}\right)\right)$$

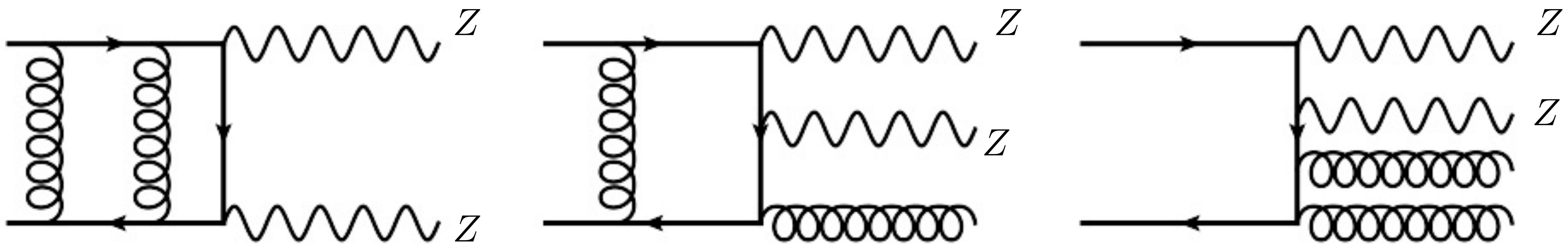
If a partonic process does not involve hierarchies of kinematic scales and an observable that we are interested in is not strongly sensitive to infrared physics, fixed order perturbative calculations provide robust and reliable framework for theoretical predictions.

Non-perturbative corrections limit the usefulness of perturbative calculations: for generic observables at the LHC scale, they can be at the percent level. Since $\alpha_s \sim 0.1$, this means that we can reliably compute up to NNLO and perhaps one order higher. After that, we will have to address the non-perturbative issues.

The LHC experiments are already able to measure some basics processes with a few percent precision (DY, top quarks etc.), and it is expected that, in the future, more complex final states will be studied with a few-percent precision. Interpretation of these results requires theoretical description of these processes in high orders of perturbative QCD.

Fixed order calculations also serve as important ingredients for resummations and parton shower predictions through matching and merging, extraction of boundary conditions for higher-log resummations etc.

To compute NNLO corrections to $pp \to X$, one needs two-loop matrix elements for $ff \to X$, one-loop matrix elements for $ff \to X+f$, and tree-level matrix elements for $ff \to X+ff$

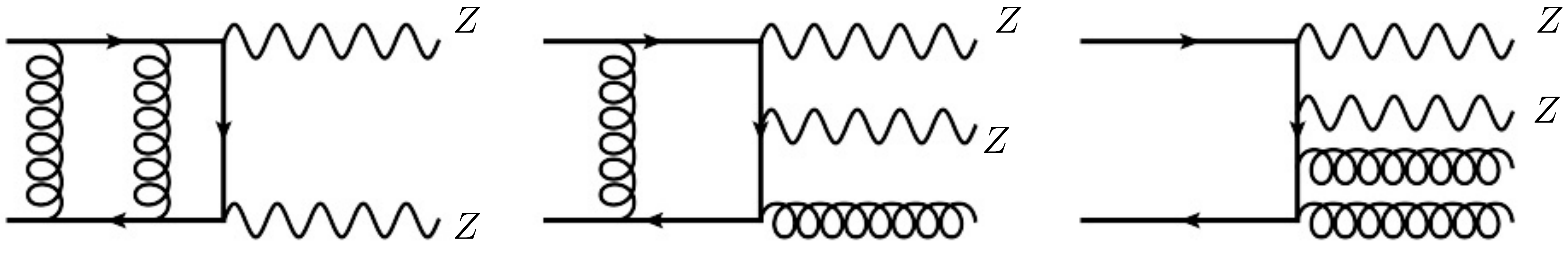


Virtual corrections Real and virtual Real corrections

During the past years, there has been a lot of progress in calculations of two-loop amplitudes. Both analytic and numerical approaches have been developed so that right now, we know almost all relevant scattering amplitudes for $2 \to 2$ processes at two-loops. First results for $2 \to 3$ amplitudes are starting to appear.

Computation of NNLO corrections also requires 1-loop amplitudes; in variance with ``regular'' NLO computations, these amplitudes have to be evaluated very close to degenerate kinematic limits. Nevertheless, it seems that existing one-loop providers are able to cope with this situation. For example, OpenLoops has been used for calculations of NNLO corrections to di-boson production at the LHC etc.

Apart from loop amplitudes, at NNLO we also need a framework to deal with emissions of two additional on-shell particles (gluons, quark pairs etc.)



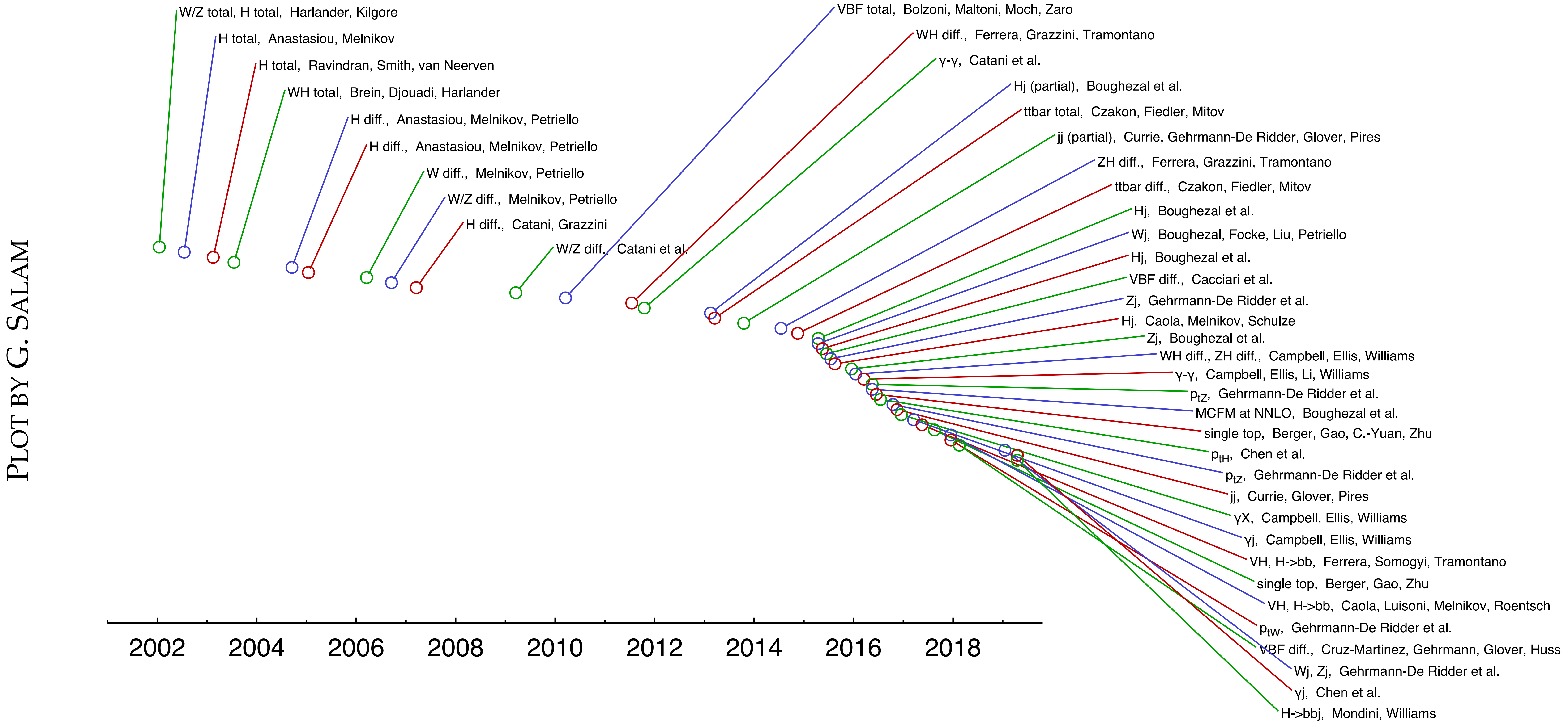
Virtual corrections | Real and virtual | Real corrections

These real-emission contributions are finite in the bulk of the allowed phase-space, but (infra-red and collinear) divergencies appear upon integration of matrix elements squared over energies and angles of the emitted partons.

To obtain fully differential results, we need to extract these singularities *without* integrating over resolved phase space. It is clear that the larger the number of extra emissions that we have to consider, the more complicated this problem is.

For a long time, understanding how to ``integrate without integrating'' was the main issue preventing us for performing NNLO QCD computations. For example, di-jet amplitudes at two loops have been known for almost 20 years, but NNLO predictions for di-jet production cross sections and kinematic distirbutions became available only a few years ago.

Over the past few (O(5)) years, several methods have been developed to deal with the real-emission problem. This fact, combined with the availability of 2-loop amplitudes, led to a large number of NNLO predictions for many important 2→2 LHC processes.



Currently, going beyond 2 → 2 processes is not possible because relevant 2-loop amplitudes are unknown. However, even when they will become available, it will require significant effort to obtain NNLO predictions using existing methods to describe real emission contributions.

What have we learned from existing NNLO computations?

1. They work! In general, NNLO QCD results improve agreement between theory and data (WW, WZ, Z+j, di-photons, top pairs);

2. It is important to have NNLO QCD computations for fiducial cross sections measured in experiments; corrections to inclusive cross sections and to fiducial cross sections may be quite different (Higgs production in WBF, single top production with decay, WW pairs);

3. NNLO QCD computations work in ``hard kinematic regions''. For an object with an invariant mass O(100) GeV, ``hard'' seems to mean ``down to transverse momenta of about 30 GeV''. However, to go that low in p_t requires NNLO. Resummations are important but, as NNLO results become widely available, resummations become relevant at low(er) transverse momenta (Z/H p_t studies) which is their natural domain of applicability;

4. Sometimes, thanks to NNLO computations, it becomes possible to get access to physics that otherwise would have been inaccessible (e.g. Higgs width constraints from off-shell ZZ production).

Further progress with fixed order calculations for the LHC will require improvements in computing two-loop (and higher-loop) amplitudes and devising more efficient and transparent subtraction schemes.

In what follows, I will discuss the various approaches to dealing with the double real emission contributions, eventually focusing on the so-called nested soft-collinear subtraction scheme.

Large number of methods have been developed to deal with real emissions at NNLO; they fall into two general categories, slicing and subtraction.

"q_t" slicing [Catani, Grazzini]

"Antenna" subtraction [Gehrmann-de Ridder, Gehrmann, Glover et al.

"Jettiness" slicing [Boughezal et al., Gaunt et al.]

"Sector decomposition & FKS" subtraction [Czakon, Heyman, Caola, Röntsch, K.M.]

"Projection-to-Born" [Cacciari et al.]

"Colourful NNLO" [Del Duca, Troscanyi et al.]

"Local Analytic Sector Subtraction" [Magnea, Maina et al.]

"Geometric IR subtraction" [Herzog]

PHASE-SPACE SLICING: $$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int\limits_0^\delta \left[|\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d \right]_{\rm s.c.} + \int\limits_\delta^1 |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_4 + \mathcal{O}(\delta)$$

SUBTRACTION $$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int \left[|\mathcal{M}|^2 \mathcal{F}_J - \mathcal{S} \right] \mathrm{d}\phi_4 + \int \mathcal{S} \mathrm{d}\phi_d$$

Slicing and subtraction approaches have very different pros and cons, already known from NLO computations.

PHASE-SPACE SLICING

$$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int_0^\delta \left[|\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d\right]_{\mathrm{s.c.}} + \int_\delta^1 |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_4 + \mathcal{O}(\delta)$$

- conceptually simple, straightforward to implement since known NLO results can be re-used;
- requires care with the residual δ-dependence (especially in kinematic distributions);
- non-local → severe numerical cancellations.

SUBTRACTION

$$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int \left[|\mathcal{M}|^2 \mathcal{F}_J - \mathcal{S}\right] \mathrm{d}\phi_4 + \int \mathcal{S} \mathrm{d}\phi_d$$

- in principle fully local → better efficiency and reliability;
- requires knowledge of subtraction terms and ability to integrate them.

It appears that we have many options to deal with real emissions at NNLO , but do we have a really good one? Probably not… Slicing techniques are limited in what they can offer; different subtraction procedures have different shortcomings.

A good subtraction scheme should be

1. physically transparent
2. general (scaleable)
3. local
4. analytic
5. efficient

None of the existing subtraction schemes satisfy all these criteria.

Is all of this really necessary for phenomenology? Yes and no. On the one hand, large number of relevant computations can be and has been done. Yet, on the other hand, even relatively simple 2→2 calculations involving coloured initial *and* final states require very significant computing power. This implies that even with large computing farms, CPU requirements are likely to severely limit the breadth of high-precision phenomenological studies for complex processes. This issue may become especially relevant for emerging 2→3 computations.

CPU HOURS FOR FULLY DIFFERENTIAL RESULTS

	2→1	2→2	2→3
NNLO	**100-500**	**10^5-10^7**	**-**
N^3LO	**-**	**-**	**-**

O(10) centuries ….

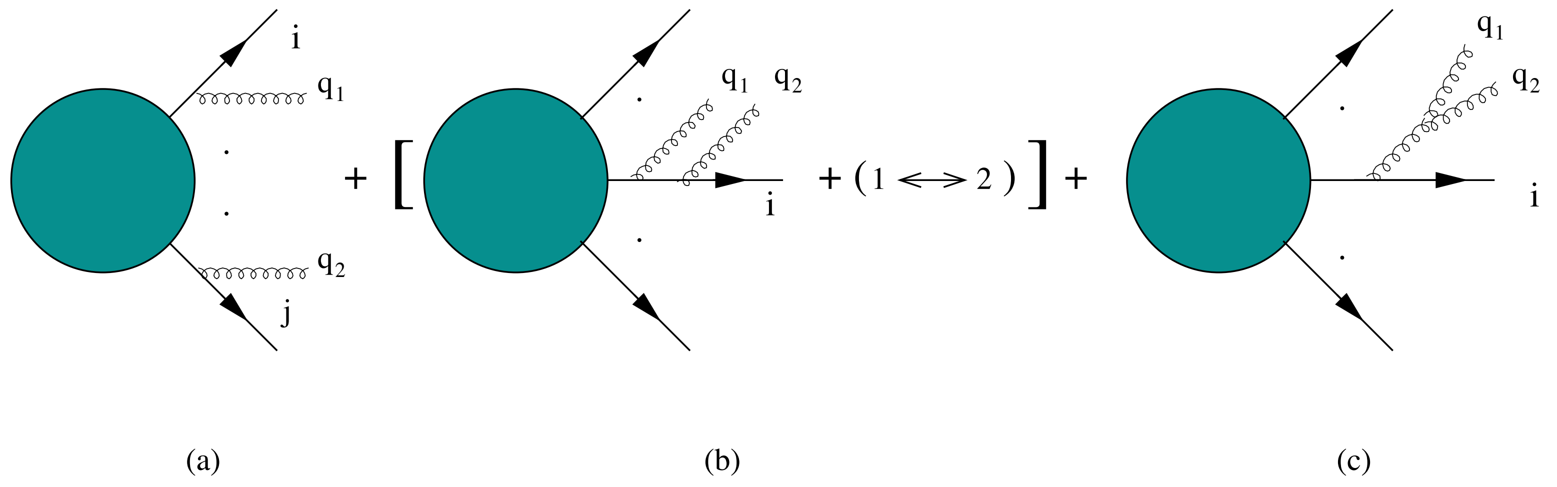
An optimal subtraction method, able to efficiently deal with complex processes has yet to emerge and we are looking for it. In what follows I will discuss an example of a scheme that satisfies the five criteria.

Any subtraction or slicing scheme requires us to understand singular limits of scattering amplitudes. These limits have been known for O(30) years at NLO and for O(20) years at NNLO. Hard matrix elements decouple in these limits — it is this feature that allows construction of universal subtraction schemes.

$$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int \left[|\mathcal{M}|^2 \mathcal{F}_J - \mathcal{S} \right] \mathrm{d}\phi_4 + \int \mathcal{S} \mathrm{d}\phi_d \qquad \int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int_0^\delta \left[|\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d \right]_{\mathrm{s.c.}} + \int_\delta^1 |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_4$$

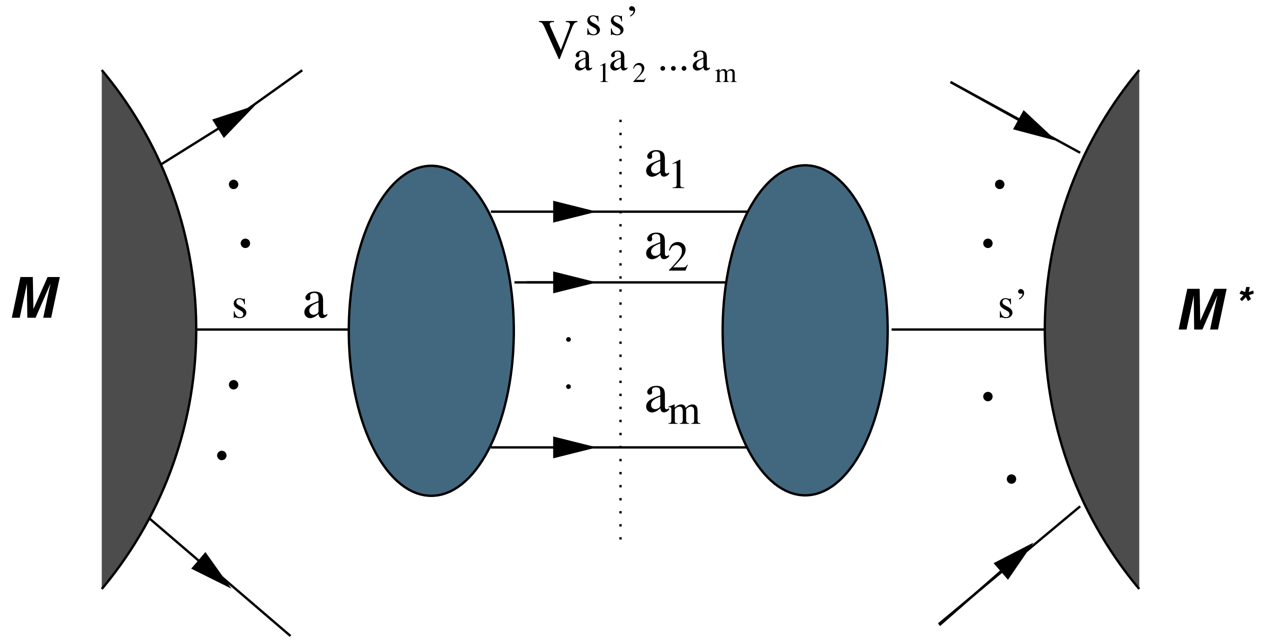
Soft limits, single and double:

$$\lim_{k_{1,2}\to 0} |\mathcal{M}|^2_{n+2}(\{p\}, k_1, k_2) \approx \mathrm{Eik}(\{p\}, k_1, k_2) |M_n(\{p\})|^2$$

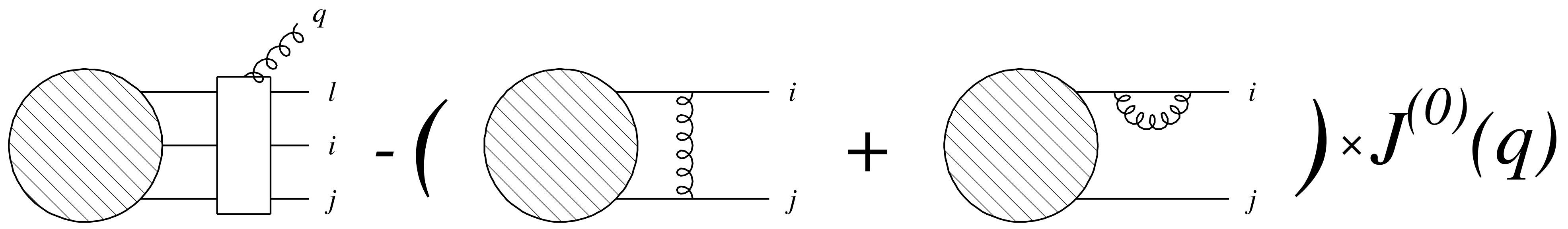


Collinear, double and triple:

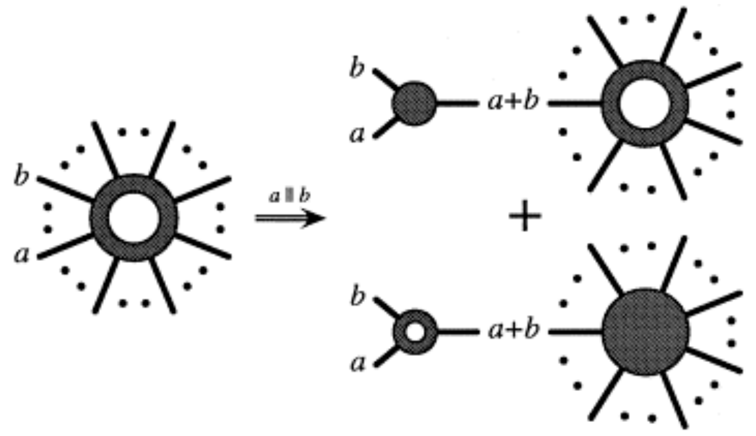
$$\lim_{k_1||k_2||p_j} |\mathcal{M}_{n+2}|^2(\{p\}, k_1, k_2) \approx \frac{1}{s^2_{jk_1k_2}} P(z_1, z_2, k_\perp) \otimes |\mathcal{M}_n(p_{jk_1k_2}, \ldots)|^2$$



Soft limits of one-loop

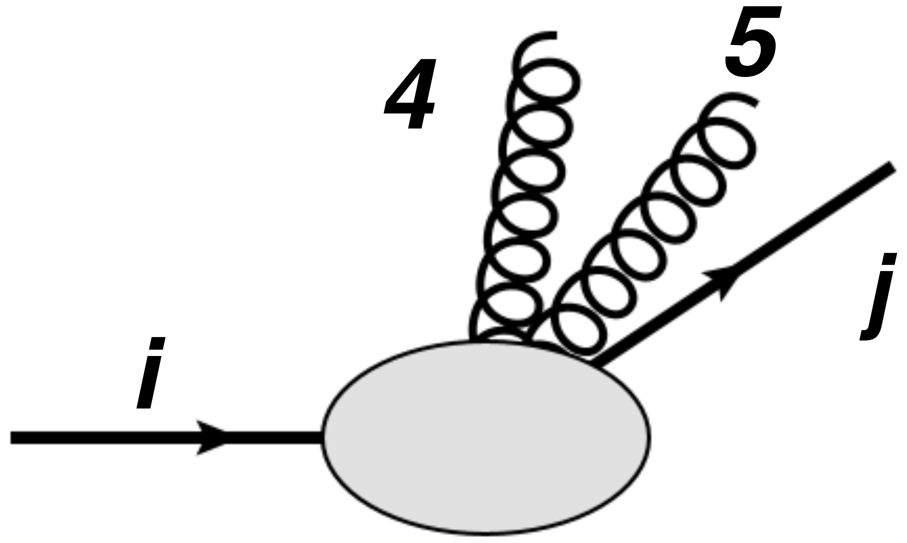


Collinear limits, one-loop



Catani, Grazzini, Glover, Campbell, Kosower, Uwer, Czakon, Mitov

Factorization for soft gluons



$$|\mathcal{M}(g_4, g_5; \{p\})|^2 \approx \left[g_{s,b}^2 \mu^{2\epsilon}\right]^2 \left[\frac{1}{2} \sum_{ij,kl}^{n} S_{ij}(k_4) S_{kl}(k_5) |\mathcal{M}^{(i,j)(k,l)}(\{p\})|^2 - C_A \sum_{i<j}^{n} \widetilde{S_{ij}}(k_4, k_5) |\mathcal{M}^{(ij)}(\{p\})|^2\right],$$

$$\widetilde{S_{ij}}(k_4, k_5) = 2S_{ij}(k_4, k_5) - S_{ii}(k_4, k_5) - S_{jj}(k_4, k_5),$$

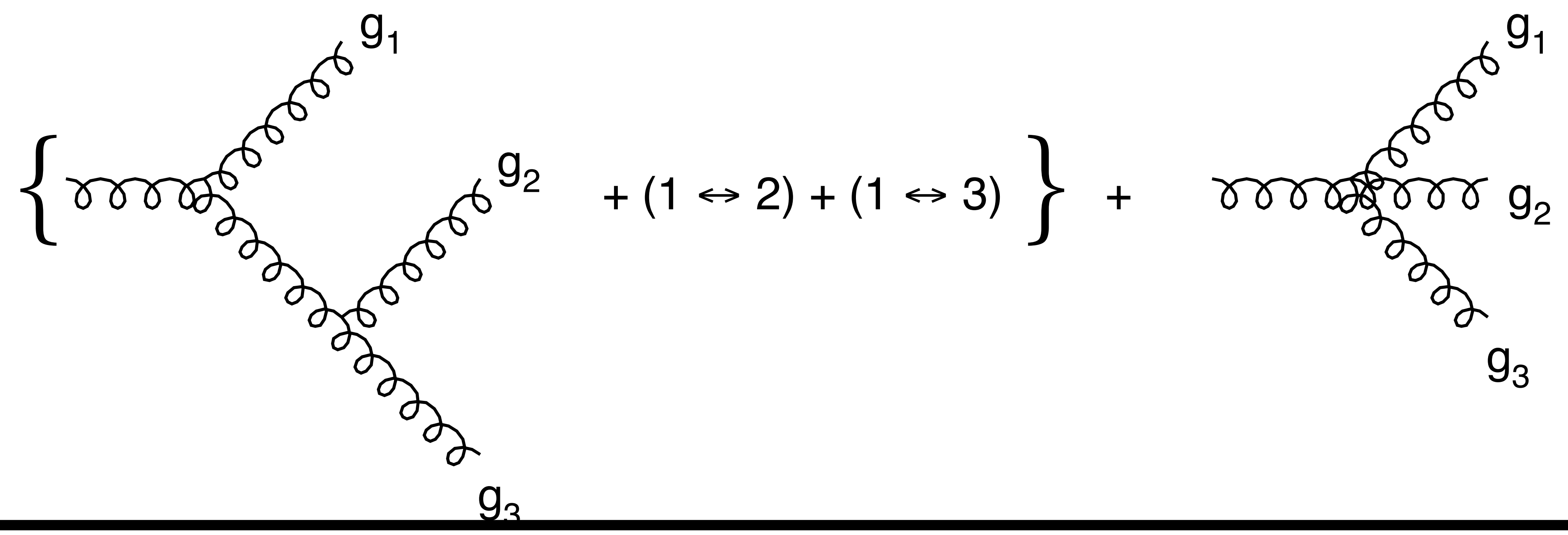
$$S_{ij}(k) = \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)},$$

$$S_{ij}(k_4, k_5) = S_{ij}^{\rm so}(k_4, k_5) - \frac{2 p_i \cdot p_j}{k_4 \cdot k_5 \left[p_i \cdot (k_4 + k_5)\right] \left[p_j \cdot (k_4 + k_5)\right]}$$
$$+ \frac{(p_i \cdot k_4)(p_j \cdot k_5) + (p_i \cdot k_5)(p_j \cdot k_4)}{\left[p_i \cdot (k_4 + k_5)\right] \left[p_j \cdot (k_4 + k_5)\right]} \left[\frac{(1-\epsilon)}{(k_4 \cdot k_5)^2} - \frac{1}{2} S_{ij}^{\rm so}(k_4, k_5)\right]$$

$$S_{ij}^{\rm so}(k_4, k_5) = \frac{p_i \cdot p_j}{k_4 \cdot k_5} \left(\frac{1}{(p_i \cdot k_4)(p_j \cdot k_5)} + \frac{1}{(p_i \cdot k_5)(p_j \cdot k_4)} \right) - \frac{(p_i \cdot p_j)^2}{(p_i \cdot k_4)(p_j \cdot k_4)(p_i \cdot k_5)(p_j \cdot k_5)},$$

Catani, Grazzini

Factorization for collinear gluons



$$
\begin{aligned}
\hat{P}^{\mu\nu}_{g_1 g_2 g_3} = \; & C_A^2 \left\{ \frac{(1-\epsilon)}{4 s_{12}^2} \left[-g^{\mu\nu} t_{12,3}^2 + 16 s_{123} \frac{z_1^2 z_2^2}{z_3(1-z_3)} \left(\frac{\widetilde{k}_2}{z_2} - \frac{\widetilde{k}_1}{z_1} \right)^{\mu} \left(\frac{\widetilde{k}_2}{z_2} - \frac{\widetilde{k}_1}{z_1} \right)^{\nu} \right] \right. \\
& - \frac{3}{4}(1-\epsilon) g^{\mu\nu} + \frac{s_{123}}{s_{12}} g^{\mu\nu} \frac{1}{z_3} \left[\frac{2(1-z_3) + 4 z_3^2}{1-z_3} - \frac{1 - 2 z_3 (1-z_3)}{z_1(1-z_1)} \right] \\
& + \frac{s_{123}(1-\epsilon)}{s_{12} s_{13}} \left[2 z_1 \left(\widetilde{k}_2^{\mu} \widetilde{k}_2^{\nu} \frac{1-2z_3}{z_3(1-z_3)} + \widetilde{k}_3^{\mu} \widetilde{k}_3^{\nu} \frac{1-2z_2}{z_2(1-z_2)} \right) \right. \\
& + \frac{s_{123}}{2(1-\epsilon)} g^{\mu\nu} \left(\frac{4 z_2 z_3 + 2 z_1 (1-z_1) - 1}{(1-z_2)(1-z_3)} - \frac{1 - 2 z_1 (1-z_1)}{z_2 z_3} \right) \\
& \left. \left. + \left(\widetilde{k}_2^{\mu} \widetilde{k}_3^{\nu} + \widetilde{k}_3^{\mu} \widetilde{k}_2^{\nu} \right) \left(\frac{2 z_2 (1-z_2)}{z_3(1-z_3)} - 3 \right) \right] \right\} + (5 \text{ permutations}) \; .
\end{aligned}
$$

Catani, Grazzini

How can we use the information on infra-red and collinear limits to construct a subtraction scheme?

At NLO, two lines of thought proved to be successful:

1) Catani-Seymour (CS) dipoles: write functions that interpolate between soft and collinear limits in a clever way, avoiding double-counting. Make sure that these functions (dipoles) can be analytically integrated.

@NNLO: Antenna, ColorFull

2) Frixione-Kunszt-Signer (FKS). Partition phase space into sectors to ensure that in each sector only one particle can be soft and only two collinear. Subtract soft singularities and collinear singularities sector by sector. Make sure that partitions are such that they simplify in the soft/collinear limits and make analytic integration of the subtraction terms possible.

@NNLO: Residue-improved, nested, geometric, local analytic

$$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int \left[|\mathcal{M}|^2 \mathcal{F}_J - \mathcal{S} \right] \mathrm{d}\phi_4 + \int \mathcal{S} \mathrm{d}\phi_d$$

Since soft and collinear limits relevant for NNLO computations are known since long ago and since phase-space simplifications in both limits can be worked out, it may appear that generalization of NLO subtraction schemes to NNLO is straightforward.

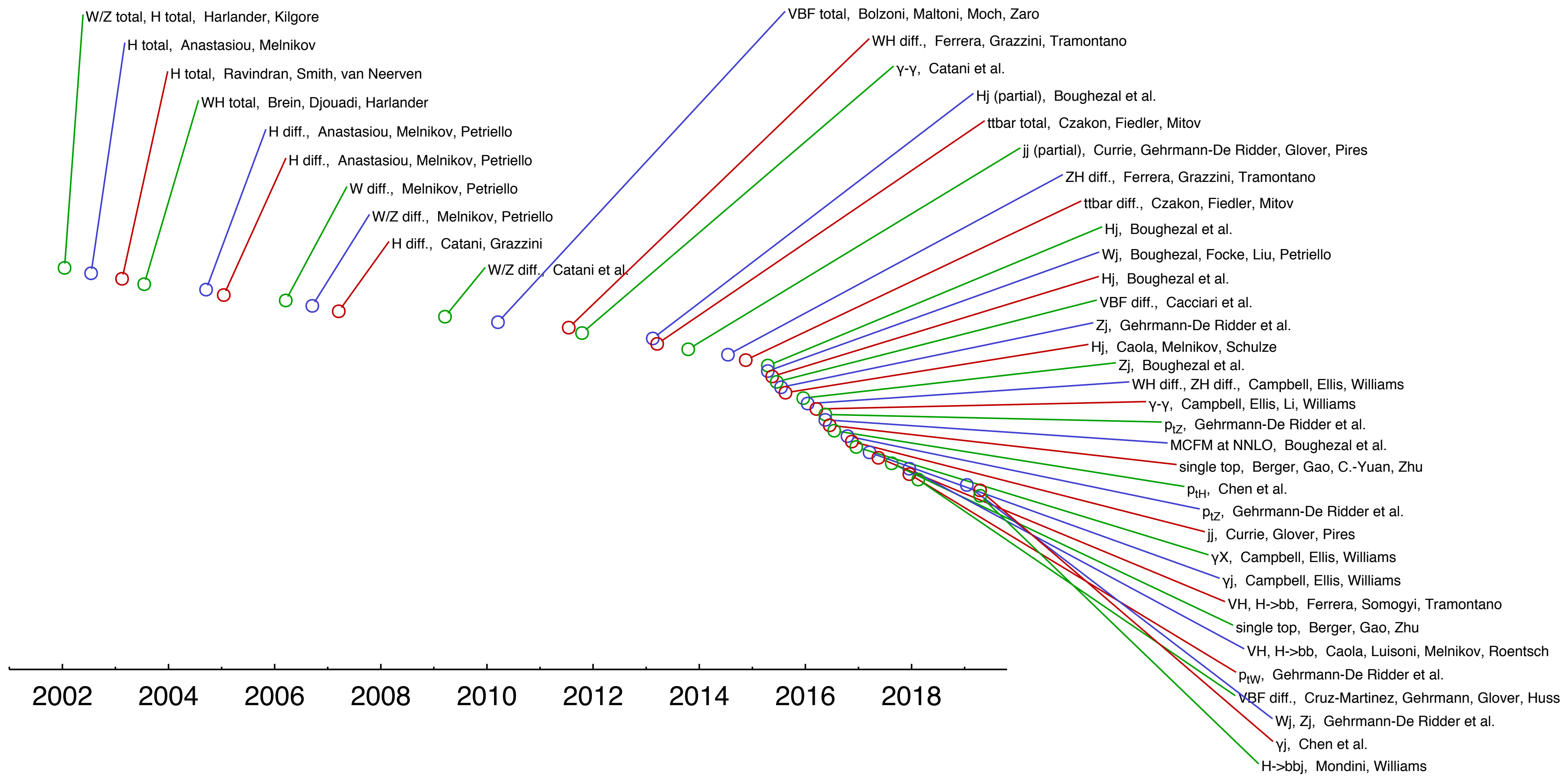
However, it did take almost twenty years for the field to get there … Why?

Main problems:

1) difficulties with overlapping singularities (limits, where several particles become collinear or soft at the same time);

2) difficulties with finding a suitable interpolating functions (for CS-like schemes);

3) concerns about possible existence of singular limits beyond the (conventional) soft and collinear ones;



The appearance of overlapping singularities at NNLO has a clear origin.

At NLO: only 2-particle invariants can vanish, $s_{ij} \sim E_j\,[1-\cos(\theta_{ij})] \equiv x_E\, x_\theta$. Singularities completely factorized.

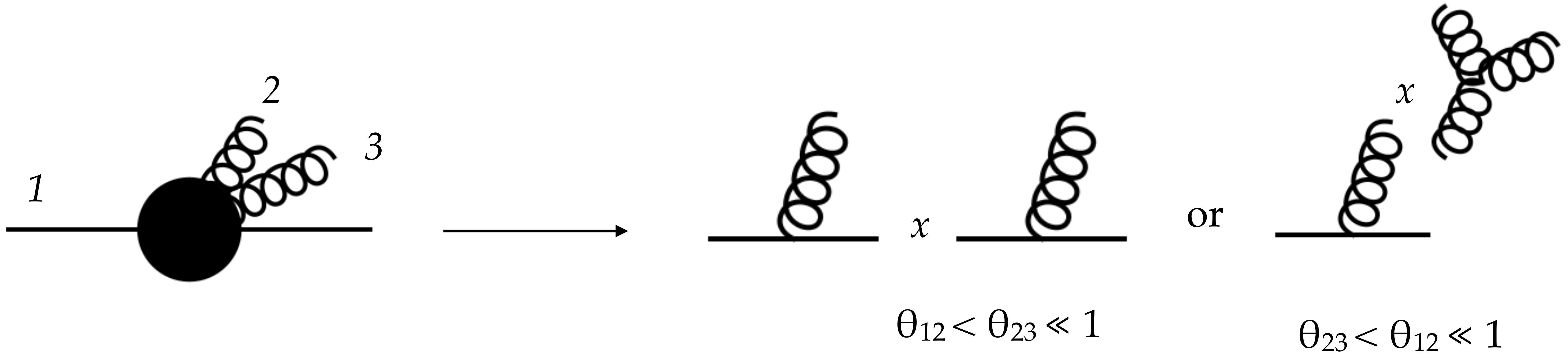
$$\int |\mathcal{M}|^2 \mathrm{d}\phi \sim \int \left[x_E^2 x_\theta |\mathcal{M}|^2\right] \frac{\mathrm{d}x_E}{x_E^{1+2\epsilon}} \frac{\mathrm{d}x_\theta}{x_\theta^{1+\epsilon}}$$

At NNLO, 3-particle invariants can vanish, $s_{ijk} \sim \Sigma\, E_i E_j \cos(\theta_{ij})$. This leads to overlapping singularities. Schematically:

$$\int |\mathcal{M}|^2 \mathrm{d}\phi \sim \int \left[x_i^a x_j^b |\mathcal{M}|^2\right] \frac{\mathrm{d}x_i}{x_i^{1+\alpha\epsilon}} \frac{\mathrm{d}x_j}{x_j^{1+\beta\epsilon}} \frac{1}{(x_i + x_j)^{\gamma\epsilon}}$$

This means that the $x_i \to 0$ limit at fixed x_j and the $x_j \to 0$ limit at fixed x_i are different. In general, this feature reflects physics and is not a mathematical artefact.

In CS-like approaches (i.e. antenna, colour-full), overlapping singularities must be reproduced by the subtraction function. It is non-trivial to devise simple enough subtraction function that don't contain extra spurious singularities. In FKS-like approaches, the $x_i > x_j$ and $x_j > x_i$ cases are dealt with separately (``*sector decomposition*'' or *additional partitioning)*.



The other problem is the (possible) existence of singular limits that go beyond conventional soft and collinear ones. For example, limits where energies and emission angles appear in non-factorizable combinations are present in Feynman propagators that appear in individual diagrams.

$$\frac{1}{E_i\theta_{ik} + E_j\theta_{jk}}$$

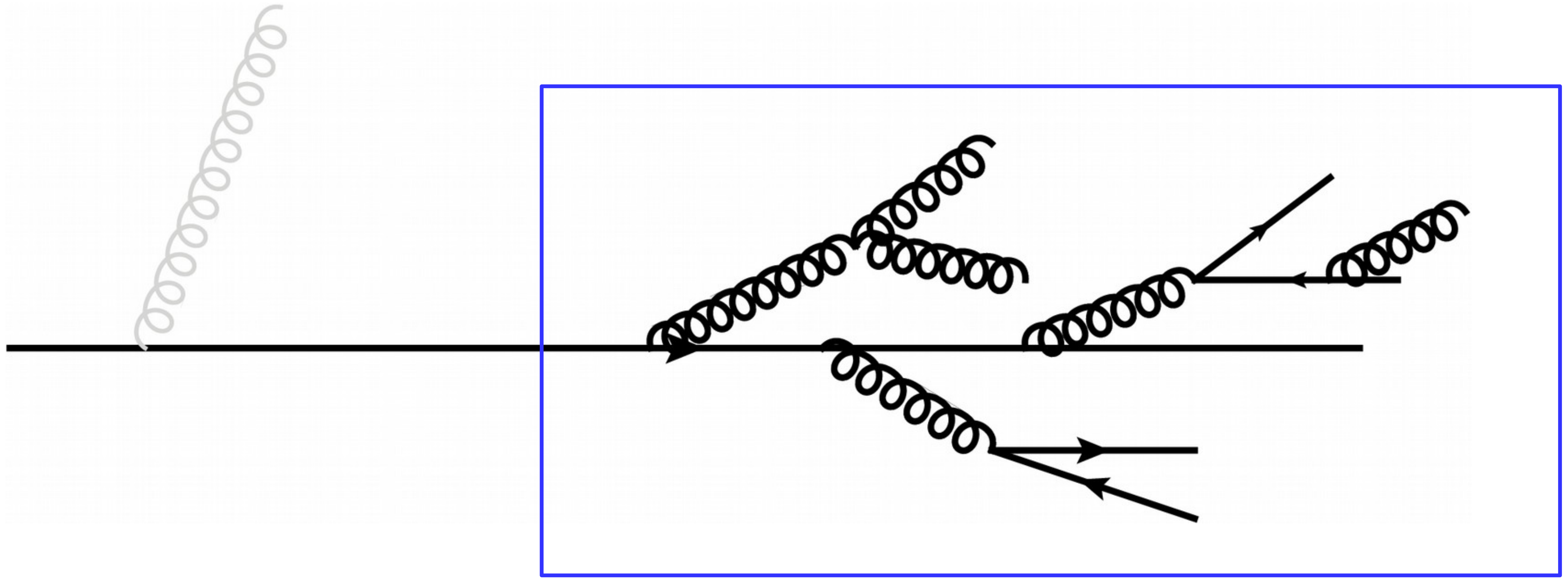
If one assumes the ordering $E_i > E_j$, such that $E_j = x_1 E_i$, and $\theta_{jk} > \theta_{ik}$, such that $\theta_{ik} = x_2\, \theta_{jk}$, this becomes

$$\frac{1}{E_i\theta_{jk}} \times \frac{1}{x_1 + x_2}$$

The $x_1 + x_2$ term leads to a non-trivial soft/collinear overlapping singularity.

Such singularities do appear in propagator of individual Feynman diagrams. However, when diagrams are combined into on-shell gauge-invariant matrix elements, such overlapping singularities must disappear.

Indeed, existence of such singularities violates the color coherence of QCD, which means that if any number of partons become collinear, soft gluons can only resolve the total color charge of the collinear system; as such soft energy can't be correlated with any of the (would be) collinear directions.



This implies that no non-trivial soft/collinear overlaps can appear in physical scattering amplitudes so that knowledge of soft and collinear limits, taken independently, should be sufficient.

The idea behind the so-called nested soft-collinear subtraction scheme is to use universal soft and collinear limits of scattering amplitudes to subtract non-integrable singularities in a ``nested'' way.

$$I = (I - S) + S = (I - C)(I - S) + C(I - S) + S$$

Specifically, the subtraction proceeds as follows.

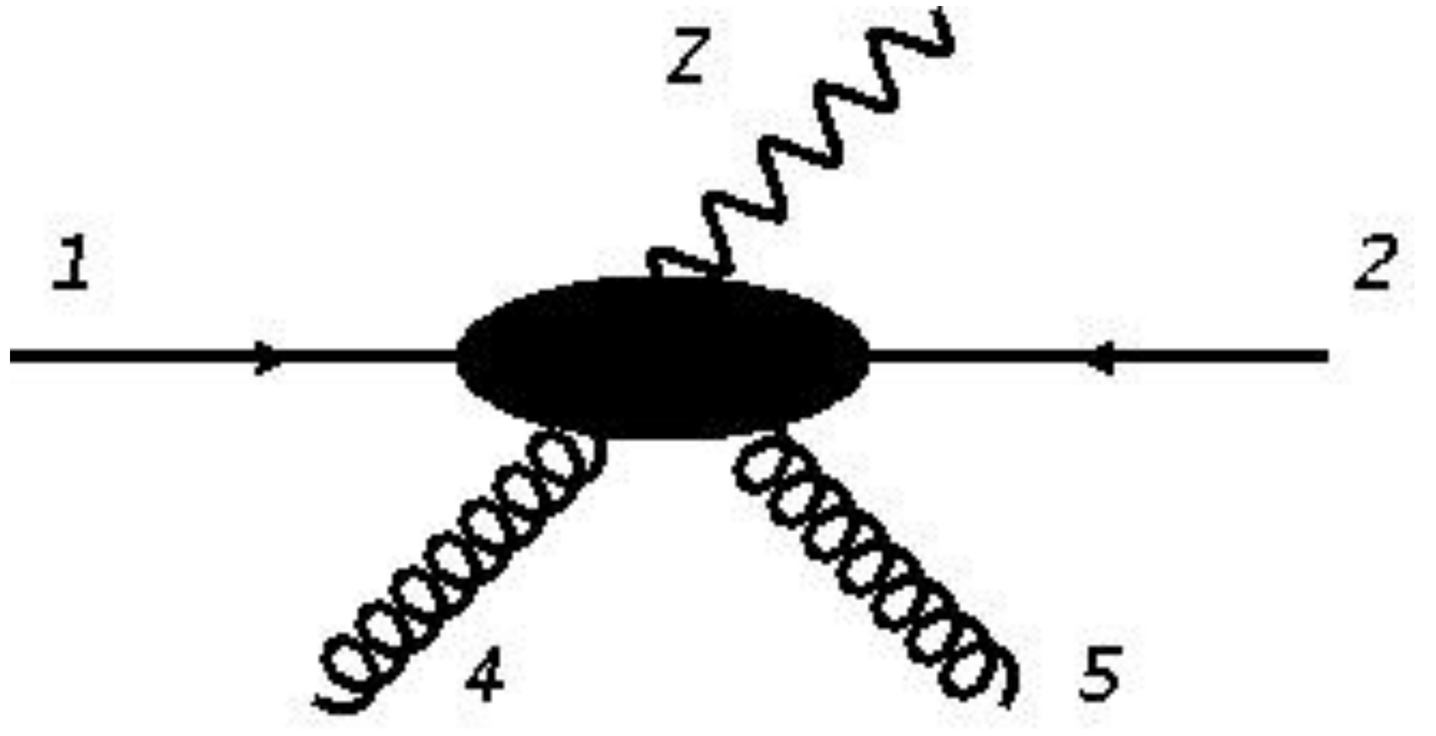
1. all soft singularities are subtracted globally.
   The soft counterterms are simple enough that they can be analytically integrated for arbitrary processes with massless partons [Caola, Delto, Frellesvig, KM (2018)];
2. FKS partitions are introduced to separate different collinear regions. For each partition, (physical) collinear overlaps are removed using sector decomposition.

   Also in this case, the counterterms are very simple and have been analytically integrated for all possible cases [Delto, Melnikov (2019)].

The emerging structure of the subtraction terms is very transparent and relatively compact.

Once the splitting into the different sectors/partitions is established, writing down subtraction formulas is straightforward. To this end, it is useful to introduce operators that extract leading singularities acting on the matrix element and phase spaces. We need to use these operators to construct fully regulated contribution. For concreteness, imagine that we deal with the production of a Z boson in quark anti-quark collisions and consider Zgg final state.

$\mathbb{S}$	Double-soft: E_4, E_5 -> 0
S_5	Single-soft: E_5 -> 0
$\mathbb{C}_{1,2}$	Triple-collinear: 4|5|1 and 4|5|2
C_{4i}, C_{5i}	Double-collinear 4|i, 5|i, i=1,2
C_{45}	Double-collinear 4|5



$$2s \cdot \mathrm{d}\hat{\sigma}^{\mathrm{RR}}_{q\bar{q}} = \int [df_4][df_5]\theta(E_4 - E_5) F_{\mathrm{LM},q\bar{q}}(1,2,4,5) \equiv \langle F_{\mathrm{LM},q\bar{q}}(1,2,4,5) \rangle,$$

$$[df_i] = \frac{\mathrm{d}^{d-1}p_i}{(2\pi)^{d-1} 2E_i} \theta(E_{\max} - E_i).$$

A fully-regulated real emission contribution (Drell-Yan, pp -> Z) is shown below. It is finite and can be directly computed in four dimensions; this is the only contribution that contains Z+2 partons final state.

Each subtracted term needs to be added back somewhere else and integrated over the unresolved phase space.

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{\mathrm{NNLO}}_{1245,f_a f_b} = & \sum_{(ij)\in dc} \Big\langle \Big[(I - C_{5j})(I - C_{4i}) \Big] \big[I - \mathbb{S} \big] \big[I - S_5 \big] \times \\
& \times [df_4][df_5] w^{4i,5j} F_{\mathrm{LM},f_a f_b}(1,2,4,5) \Big\rangle \\
+ \sum_{i\in tc} \Big\langle \Big[& \theta^{(a)} \big[I - \mathbb{C}_i \big] \big[I - C_{5i} \big] + \theta^{(b)} \big[I - \mathbb{C}_i \big] \big[I - C_{45} \big] \\
& + \theta^{(c)} \big[I - \mathbb{C}_i \big] \big[I - C_{4i} \big] + \theta^{(d)} \big[I - \mathbb{C}_i \big] \big[I - C_{45} \big] \Big] \\
& \times \big[I - \mathbb{S} \big] \big[I - S_5 \big] [df_4][df_5] w^{4i,5i} F_{\mathrm{LM},f_a f_b}(1,2,4,5) \Big\rangle .
\end{aligned}
$$

Double-collinear sectors

Triple-collinear sectors

Caola, K.M., Röntsch

To deal with subtraction terms, analytic integration of soft and collinear limits of scattering amplitudes over restricted (or approximate) phase space is required, to obtain unresolved subtraction contributions.

$$I = \int \prod [\mathrm{d}f_i] \otimes \mathrm{Limit} \otimes \mathrm{Constraint}, \qquad [\mathrm{d}f_i] = \frac{\mathrm{d}^d k_i}{(2\pi)^d}(2\pi)\delta_+(k_i^2)$$

The limiting functions are eikonal factors and splitting functions; this is a universal statement. Phase space constraints can be quite different and depend on the details of the subtraction/slicing.

For example, for NNLO computations within the soft-collinear nested subtraction scheme, the following double-unresolved integrals are needed

$$I_{ij} \sim \int \prod_{i=4}^{i=5} [\mathrm{d}k_i]\ \theta(E_{\max} - E_4)\theta(E_{\max} - E_5) S_{ij}(p_i, p_j, k_4, k_5) \qquad R_{i\to jk} \sim \int \prod_{i=4}^{i=5} [\mathrm{d}k_i]\ P_{i\to jk}(z_4, z_5, ...)$$

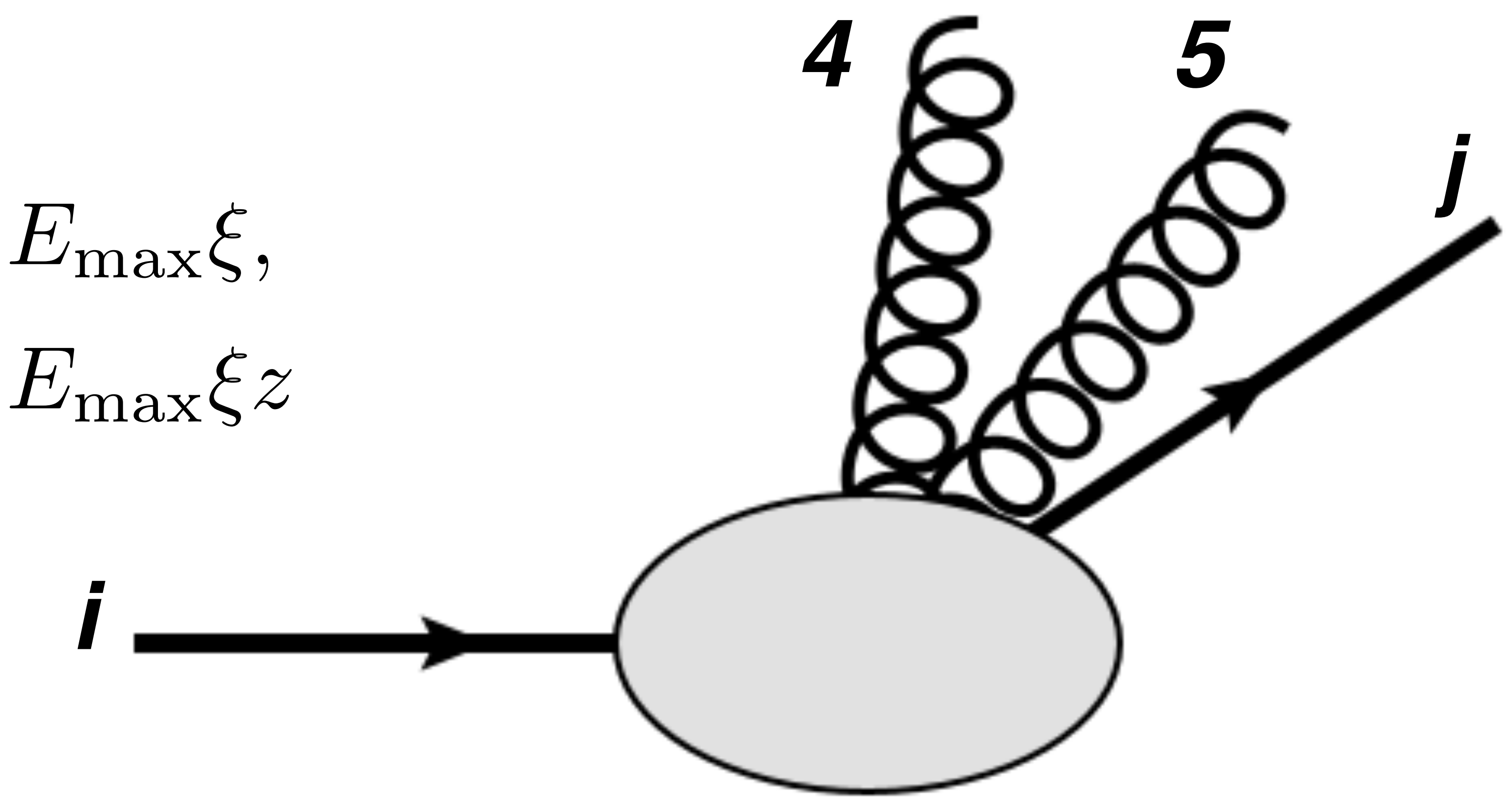
Although these quantities are neither amplitudes nor loop integrals, their computation is greatly simplified if "loop technology", including reverse unitarity, integration by parts, differential equations etc., is used. The results are expressed through functions that are very similar to what one gets from loop integrals.

There are two NNLO subtraction terms that appear in each and every calculation; one is the double-soft and the other is triple-collinear. Both need to be calculated.

Double-soft contribution; emitters at an arbitrary angle to each other.

$$\mathbb{S}_{ij}^{(gg)} = \int [dk_4][dk_5]\theta(E_{\max} - k_4^0)\theta(k_4^0 - k_5^0)\widetilde{S_{ij}}(k_4, k_5),$$

$$k_4^0 = E_{\max}\xi,$$
$$k_5^0 = E_{\max}\xi z$$



Fix the ratio of energies of two soft gluons. Use reverse unitarity. Derive differential equations w.r..t to the ratio of energies of emitted gluons at fixed emittors angle. (Almost) canonical form. Boundary conditions in the limit when gluon 5 has zero energy, for arbitrary angle.

A complicated computation produces ungodly result, given by a very large number of Goncharov polylogarithms with complicated arguments and huge rational prefactors.

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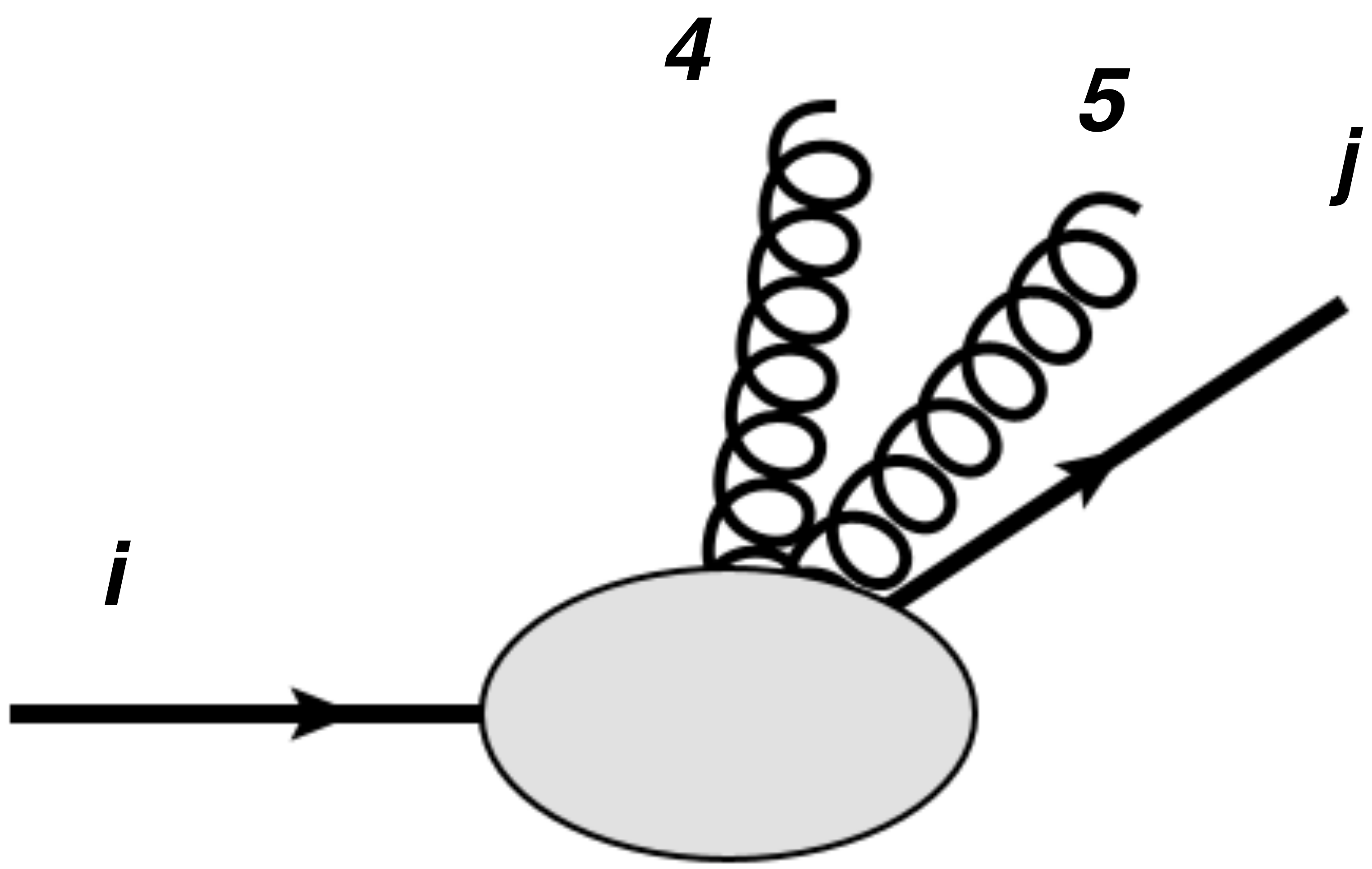
The symbol of that expression is however incredibly simple; computing it and integrating back, produces the compact (17 lines !) expression on the right…

$$\mathcal{S}_{ij}^{(gg)} = \int [dk_4][dk_5]\theta(E_{\max} - k_4^0)\theta(k_4^0 - k_5^0)\widetilde{S_{ij}}(k_4, k_5),$$



$$\delta = \frac{\theta_{ij}}{2}$$

$$s = \sin\frac{\theta_{ij}}{2}$$

$$c = \cos\frac{\theta_{ij}}{2}$$

$$\mathrm{Ci}_n(z) = \frac{(\mathrm{Li}_n(e^{iz}) + \mathrm{Li}_n(e^{-iz}))}{2}, \qquad \mathrm{Si}_n(z) = \frac{(\mathrm{Li}_n(e^{iz}) - \mathrm{Li}_n(e^{-iz}))}{2i}$$

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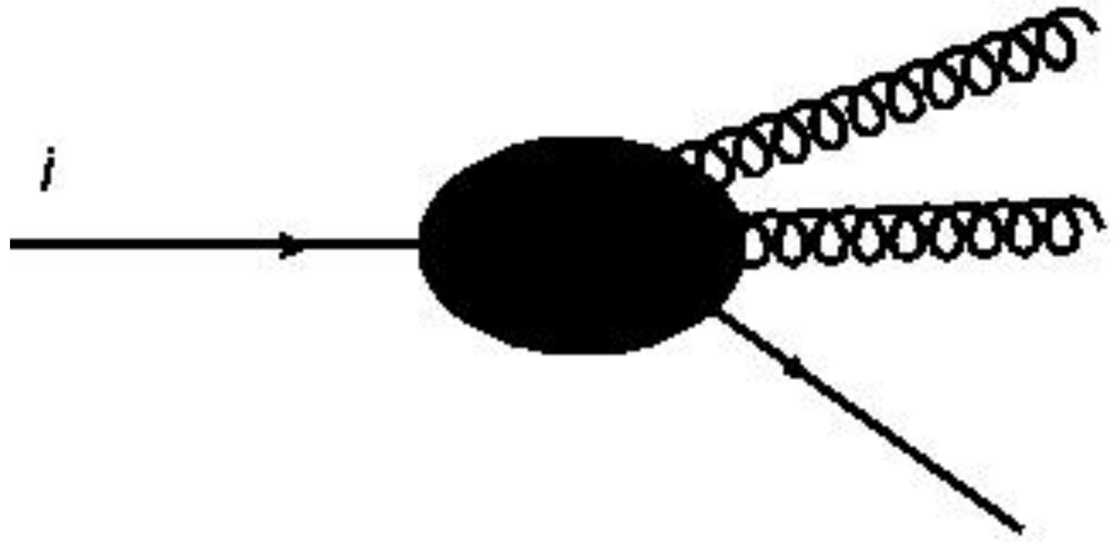
$$\begin{aligned}
\mathcal{S}_{ij}^{(gg)} = {} & (2E_{\max})^{-4\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2 \Bigg\{ \frac{1}{2\epsilon^4} + \frac{1}{\epsilon^3}\left[\frac{11}{12} - \ln(s^2)\right] \\
& + \frac{1}{\epsilon^2}\left[2\mathrm{Li}_2(c^2) + \ln^2(s^2) - \frac{11}{6}\ln(s^2) + \frac{11}{3}\ln 2 - \frac{\pi^2}{4} - \frac{16}{9}\right] \\
& + \frac{1}{\epsilon}\Bigg[6\mathrm{Li}_3(s^2) + 2\mathrm{Li}_3(c^2) + \left(2\ln(s^2) + \frac{11}{3}\right)\mathrm{Li}_2(c^2) - \frac{2}{3}\ln^3(s^2) \\
& \qquad + \left(3\ln(c^2) + \frac{11}{6}\right)\ln^2(s^2) - \left(\frac{22}{3}\ln 2 + \frac{\pi^2}{2} - \frac{32}{9}\right)\ln(s^2) \\
& \qquad - \frac{45}{4}\zeta_3 - \frac{11}{3}\ln^2 2 - \frac{11}{36}\pi^2 - \frac{137}{18}\ln 2 + \frac{217}{54}\Bigg] \\
& + 4\mathrm{G}_{-1,0,0,1}(s^2) - 7\mathrm{G}_{0,1,0,1}(s^2) + \frac{22}{3}\mathrm{Ci}_3(2\delta) + \frac{1}{3\tan(\delta)}\mathrm{Si}_2(2\delta) \\
& + 2\mathrm{Li}_4(c^2) - 14\mathrm{Li}_4(s^2) + 4\mathrm{Li}_4\left(\frac{1}{1+s^2}\right) - 2\mathrm{Li}_4\left(\frac{1-s^2}{1+s^2}\right) \\
& + 2\mathrm{Li}_4\left(\frac{s^2-1}{1+s^2}\right) + \mathrm{Li}_4(1-s^4) + \Bigg[10\ln(s^2) - 4\ln\left(1+s^2\right) \\
& + \frac{11}{3}\Bigg]\mathrm{Li}_3(c^2) + \left[14\ln(c^2) + 2\ln(s^2) + 4\ln\left(1+s^2\right) + \frac{22}{3}\right]\mathrm{Li}_3(s^2) \\
& + 4\ln(c^2)\mathrm{Li}_3(-s^2) + \frac{9}{2}\mathrm{Li}_2^2(c^2) - 4\mathrm{Li}_2(c^2)\mathrm{Li}_2(-s^2) + \Bigg[7\ln(c^2)\ln(s^2) \\
& - \ln^2(s^2) - \frac{5}{2}\pi^2 + \frac{22}{3}\ln 2 - \frac{131}{18}\Bigg]\mathrm{Li}_2(c^2) + \left[\frac{2}{3}\pi^2 - 4\ln(c^2)\ln(s^2)\right]\times \\
& \mathrm{Li}_2(-s^2) + \frac{\ln^4(s^2)}{3} + \frac{\ln^4\left(1+s^2\right)}{6} - \ln^3(s^2)\left[\frac{4}{3}\ln(c^2) + \frac{11}{9}\right] \\
& + \ln^2(s^2)\left[7\ln^2(c^2) + \frac{11}{3}\ln(c^2) + \frac{\pi^2}{3} + \frac{22}{3}\ln 2 - \frac{32}{9}\right] - \frac{\pi^2}{6}\ln^2\left(1+s^2\right) \\
& + \zeta_3\left[\frac{17}{2}\ln(s^2) - 11\ln(c^2) + \frac{7}{2}\ln\left(1+s^2\right) - \frac{21}{2}\ln 2 - \frac{99}{4}\right] + \ln(s^2)\times \\
& \left[-\frac{7\pi^2}{2}\ln(c^2) + \frac{22}{3}\ln^2 2 - \frac{11}{18}\pi^2 + \frac{137}{9}\ln 2 - \frac{208}{27}\right] - 12\mathrm{Li}_4\left(\frac{1}{2}\right) \\
& + \frac{143}{720}\pi^4 - \frac{\ln^4 2}{2} + \frac{\pi^2}{2}\ln^2 2 - \frac{11}{6}\pi^2\ln 2 + \frac{125}{216}\pi^2 + \frac{22}{9}\ln^3 2 \\
& + \frac{137}{18}\ln^2 2 + \frac{434}{27}\ln 2 - \frac{649}{81} + \mathcal{O}(\epsilon)\Bigg\},
\end{aligned}$$

Another double-unresolved limit involves integrals of the triple collinear splitting functions, for both initial and final state splittings, keeping the energy of parton that goes into a hard process fixed. Again, application of "loop technology" leads to analytic results.



$q \to q^* + gg$

$$\mathcal{I}_{\rm TC} = [\alpha_s]^2 E_1^{-4\epsilon} \int_0^1 {\rm d}z \left[R_\delta\, \delta(1-z) + \frac{R_+}{[(1-z)^{1+4\epsilon}]_+} + R_{\rm reg}(z) \right] \left\langle \frac{F_{\rm LM}(z\cdot 1, 2)}{z} \right\rangle$$

$$R_{\delta,+,{\rm reg}} = C_F^2 R^{\rm A}_{\delta,+,{\rm reg}} + C_F C_A\, R^{\rm NA}_{\delta,+,{\rm reg}}$$

$$\begin{aligned}
R_\delta^{\rm A} &= \frac{1}{\epsilon}\left(\frac{\pi^2}{3}\ln(2)\right) - \frac{7\pi^2}{6}\ln^2(2) + 8\zeta_3\ln(2),\\
R_\delta^{\rm NA} &= \frac{1}{\epsilon}\left(-\frac{1571}{216} + \frac{11\pi^2}{36} + \frac{3}{8}\zeta_3 + \frac{\pi^2}{3}\ln(2) + \frac{11}{2}\ln^2(2) + \left(-\frac{32}{9} + \frac{\pi^2}{6} - \frac{11\ln(2)}{3}\right)\ln(E_{\rm max}/E_1)\right)\\
&- \frac{1}{12}\ln^4(2) - \frac{176}{9}\ln^3(2) - \left(\frac{79}{9} + \frac{11\pi^2}{12}\right)\ln^2(2) + \frac{513\zeta_3 + 913 + 165\pi^2}{108}\ln(2)\\
&+ \left(\frac{64}{9} - \frac{\pi^2}{3} + \frac{22\ln(2)}{3}\right)\ln(E_{\rm max}/E_1)\\
&+ \left(\frac{11\zeta_3}{2} + \frac{383}{54} - \frac{22\pi^2}{9} - 11\ln^2(2) + \frac{\ln(2)}{3} - \frac{2}{3}\pi^2\ln(2)\right)\ln^2(E_{\rm max}/E_1),\\
R_+^A &= -\frac{4\pi^2}{3}\ln(2),\\
R_+^{\rm NA} &= \frac{1}{\epsilon}\left(\frac{11}{3}\ln(2) - \frac{\pi^2}{6} + \frac{32}{9}\right) - 11\ln^2(2) - \frac{1+2\pi^2}{3}\ln(2) - 7\zeta_3 + \frac{11\pi^2}{9} + 22.
\end{aligned}$$

$$\begin{aligned}
R_{\rm reg}^{\rm A} &= \frac{1}{\epsilon}\left(-\frac{z+1}{2}\ln(2)\ln(z) + (1-z)\ln(2) + \frac{(z^2+3)}{4(z-1)}\ln^2(z) - \ln(z)z + \frac{3(z-1)}{2}\right)\\
&+ \frac{z^2\left(-36\zeta_3 + 33 + 4\pi^2\right) - 2\left(33 + 2\pi^2\right)z - 60\zeta_3 + 33}{6(z-1)} + \frac{7(z-1)}{2}\ln^2(2)\\
&+ \left(-6z + \pi^2(z+1) + 6\right)\ln(2) + \frac{\left(3(z-1)z - \pi^2\left(3z^2+5\right)\right)}{3(z-1)}\ln(z)\\
&+ \frac{z}{2}\ln^2(z) + \frac{(9z^2+19)}{12(1-z)}\ln^3(z) + \frac{7(z+1)}{4}\ln^2(2)\ln(z) + \frac{(z^2+7)}{2(1-z)}\ln(2)\ln^2(z)\\
&+ (3z-1)\ln(2)\ln(z) + 6(1-z)\ln(1-z) - 4(1-z)\ln(1-z)\ln(2)\\
&+ \left(-2(z+1)\ln(2) - \frac{2\left(z^2+1\right)}{z-1}\ln(z) - 4z\right){\rm Li}_2(z) + \left(\frac{2\left(3z^2+5\right)}{z-1}\right){\rm Li}_3(z),
\end{aligned}$$

Delto, K.M.

Putting all fully-unresolved parts together, we obtain the Born-like contribution to the NNLO cross-section for the Drell-Yan process. This is the combination of virtual corrections, the double-soft emissions and the soft-collinear contributions.

$$
\begin{aligned}
\mathrm{d}\hat{\sigma}^{\rm NNLO}_{(1,2),f_a f_b} = \left\langle F_{\mathrm{LM},f_a f_b}(1,2)\right\rangle \times \left(\frac{\alpha_s(\mu)}{2\pi}\right)^2 \Bigg\{ C_F^2\left[\frac{8\pi^4}{45} - (2\pi^2+16\zeta_3)\ln\left(\frac{\mu^2}{s}\right)\right. \\
\left. + \left(\frac{9}{2}-\frac{2\pi^2}{3}\right)\ln^2\left(\frac{\mu^2}{s}\right)\right] + C_A C_F\left[\frac{739}{81}+\frac{209\pi^2}{72}-\frac{7\pi^4}{80}+\ln 2\times\right. \\
\left(\frac{4}{3}+\frac{11\pi^2}{9}-\frac{7}{2}\zeta_3\right) + (\zeta_2-2)\ln^2 2 - \frac{\ln^4 2}{6} - \frac{407}{36}\zeta_3 - 4\mathrm{Li}_4\left(\frac{1}{2}\right) \\
\left. + \ln\left(\frac{\mu^2}{s}\right)\left(-\frac{199}{54}+\frac{23\pi^2}{24}-7\zeta_3\right) - \frac{11}{4}\ln^2\left(\frac{\mu^2}{s}\right)\right] + C_F n_f\left[-\frac{214}{81}\right. \\
- \frac{7\pi^2}{18} - \ln 2\left(\frac{4}{3}+\frac{2\pi^2}{9}\right) + 2\ln^2 2 + \frac{37}{18}\zeta_3 + \ln\left(\frac{\mu^2}{s}\right)\left(\frac{17}{27}-\frac{\pi^2}{12}\right) \\
\left. + \frac{1}{2}\ln^2\left(\frac{\mu^2}{s}\right)\right] + \Theta_{bd}\left[\frac{23}{36}C_F n_f + C_A C_F\left(\frac{\pi^2}{3}-\frac{131}{36}\right) + (2\ln 2)C_F\beta_0\right]\Bigg\}
\end{aligned}
$$

$$\Theta_{bd} = 2 - \frac{\pi^2}{3}$$

Caola, K.M., Röntsch

Nested soft-collinear subtraction scheme has several attractive features:

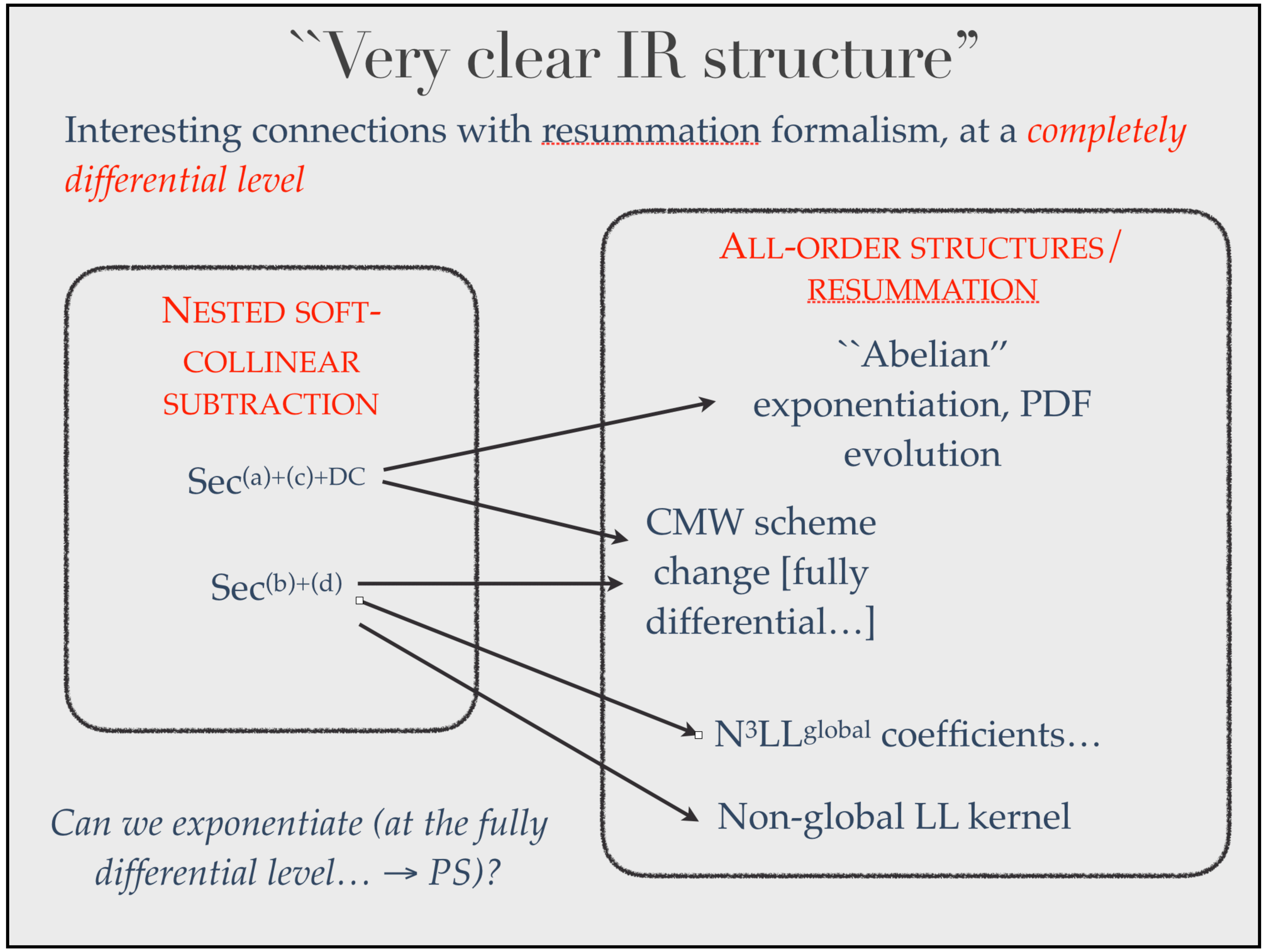
- it is based on universal soft and collinear limits of QCD amplitudes and, in this sense, it is physically transparent;
- it is fully analytic, fully local and, in principle, it can be applied to arbitrary processes;
- it is ``minimal'' in a sense that it contains the smallest number of subtractions needed to regulate QCD amplitudes (this minimalistic nature of the nested subtraction scheme may become relevant for high multiplicity processes;
- it works in practice and can be extensively checked against known analytic results;
- it is flexible in that it is not tied to a particular phase-space parametrization; this point deserves further exploration.

  Similarly, overlapping singularities do not necessarily require sector decomposition, but can be dealt with using e.g. additional partitioning as in the local analytic scheme [Magnea et al.].

Currently, the nested scheme has been fully validated for initial-initial , final-final and initial-final "dipole" configurations [Asteriadis, Caola, Röntsch, KM].

Apart from their theoretical appeal, ``transparent'' subtraction schemes can be useful to develop a better understanding of the infra-red structure of perturbative QCD. Indeed, on the one hand, subtraction schemes are informed by resummations and, on the other hand, they provide interesting fully-differential information to them.

Better understanding of subtraction schemes could also lead to more physical matching/merging schemes and perhaps even more accurate showers.



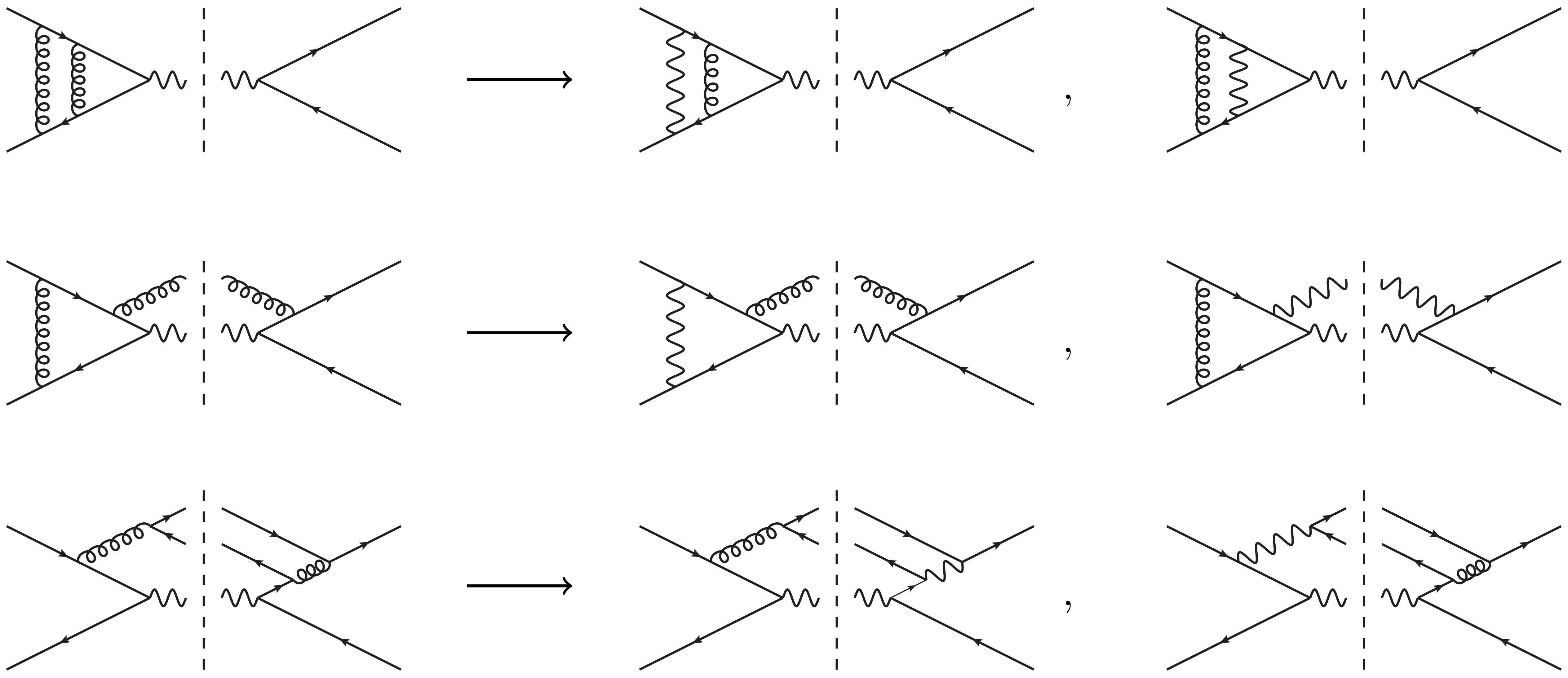
SLIDE BY F. CAOLA

So far, discussion mostly focused on finding ``a'' subtraction scheme, but interest in a better understanding of the underlying IR structure (see e.g. [Magnea et al (2018)])

If a subtraction scheme is transparent, it is straightforward to work with it / modify it to do (interesting) physics. For example, since many years there are ongoing attempts to extend existing theoretical predictions for Z and W boson production to include mixed QCD/EW corrections. This is especially relevant for the case of W-bosons because of the high-precision W-mass determination at the LHC.

Working within a transparent NNLO QCD subtraction scheme, it should be straightforward to modify it to enable computations of mixed QCD/EW corrections since soft and collinear singularities in mixed QCD/QED amplitudes are not much different from singularities of QCD amplitudes.

The nested soft-collinear subtraction subtraction scheme makes such modifications possible and, in fact, rather straightforward to implement to obtain mixed QCD/QED corrections to Z production at a fully differential level [Delto, Jaquier, Melnikov, Röntsch (2019)].



$$C_F^2 \to 2C_F e_q^2, \quad T_R \to 0, \quad C_A \to 0.$$

Partonic Channel	$[\Delta_{\alpha_s\alpha}]_{P\otimes P} \cdot 10^4$
$q\bar{q}$	5.60
qq	0.13
$qg + gq$	-7.01
$q\gamma + \gamma q$	-0.32
γg	0.06
Total	-1.54

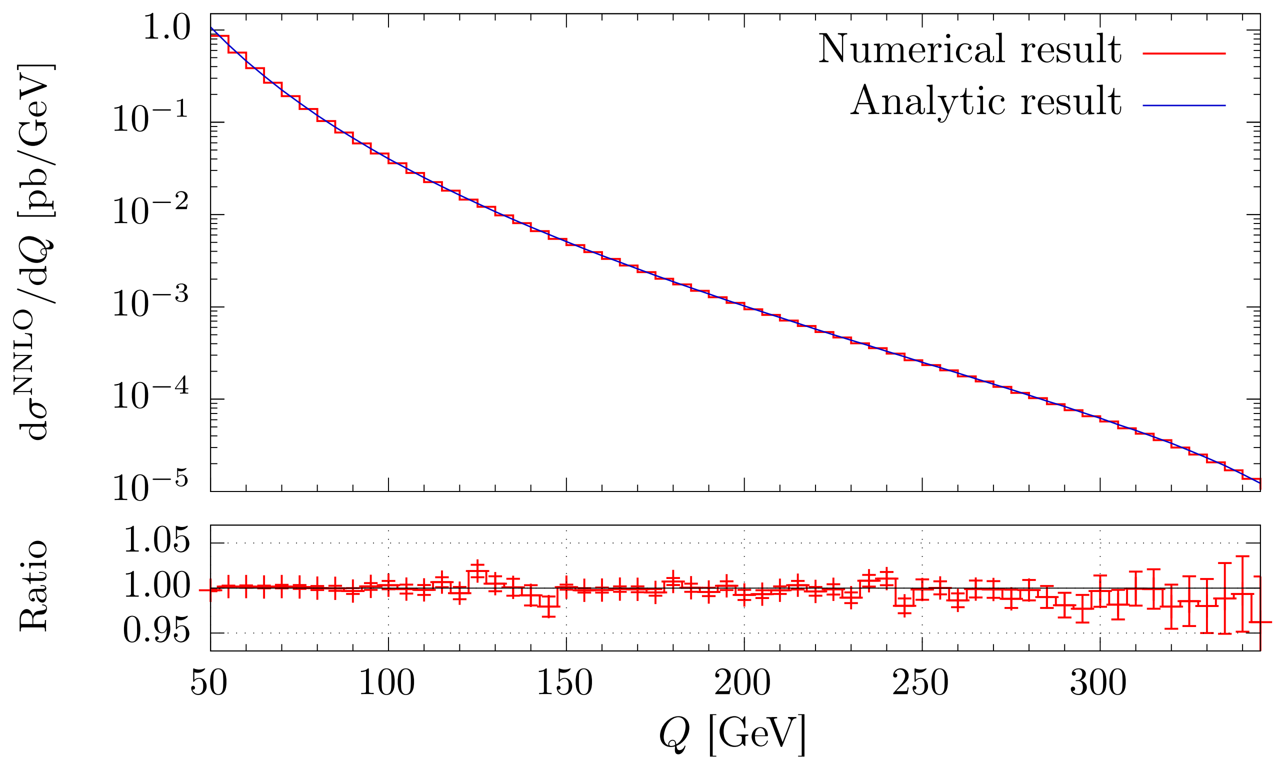
Relative contributions of various partonic channels to QCD-QED corrections to Z production cross section.

From a practical point of view, a good NNLO framework should be:

1. reliable (corrections should be computable to high numerical precision, to ensure that results are correct).
2. efficient.

It is reasonable to expect that a fully local and analytic scheme could fulfil these requirements. Within the nested soft-collinear scheme, one can test it in simple processes.

1. *reliable* → pick a process where analytic results are known, and compare to very high accuracy



Channel	Color structures	Numerical result (nb)	Analytic result (nb)
$q_i\bar{q}_i \to gg$	–	8.351(1)	8.3516
$q_i\bar{q}_i \to q_j\bar{q}_j$	$C_F T_R n_{up}$, $C_F T_R n_{dn}$	-2.1378(5)	-2.1382
	$C_F(C_A - 2C_F)$	$-4.8048(3) \cdot 10^{-2}$	$-4.8048 \cdot 10^{-2}$
	$C_F T_R$	$5.441(7) \cdot 10^{-2}$	$5.438 \cdot 10^{-2}$
$q_i q_j \to q_i q_j$ $(i \neq -j)$	$C_F T_R$	0.4182(5)	0.4180
	$C_F(C_A - 2C_F)$	$-9.26(1) \cdot 10^{-4}$	$-9.26 \cdot 10^{-4}$
$q_i g + g q_i$	–	-9.002(9)	-8.999
gg	–	1.0772(1)	1.0773

Very accurate results for DY NNLO *corrections*. Similar results for $X \to q\bar{q}$, gg decay processes.

2. *efficient*: Higgs and DY production in 1 hour on a standard laptop (1 core)

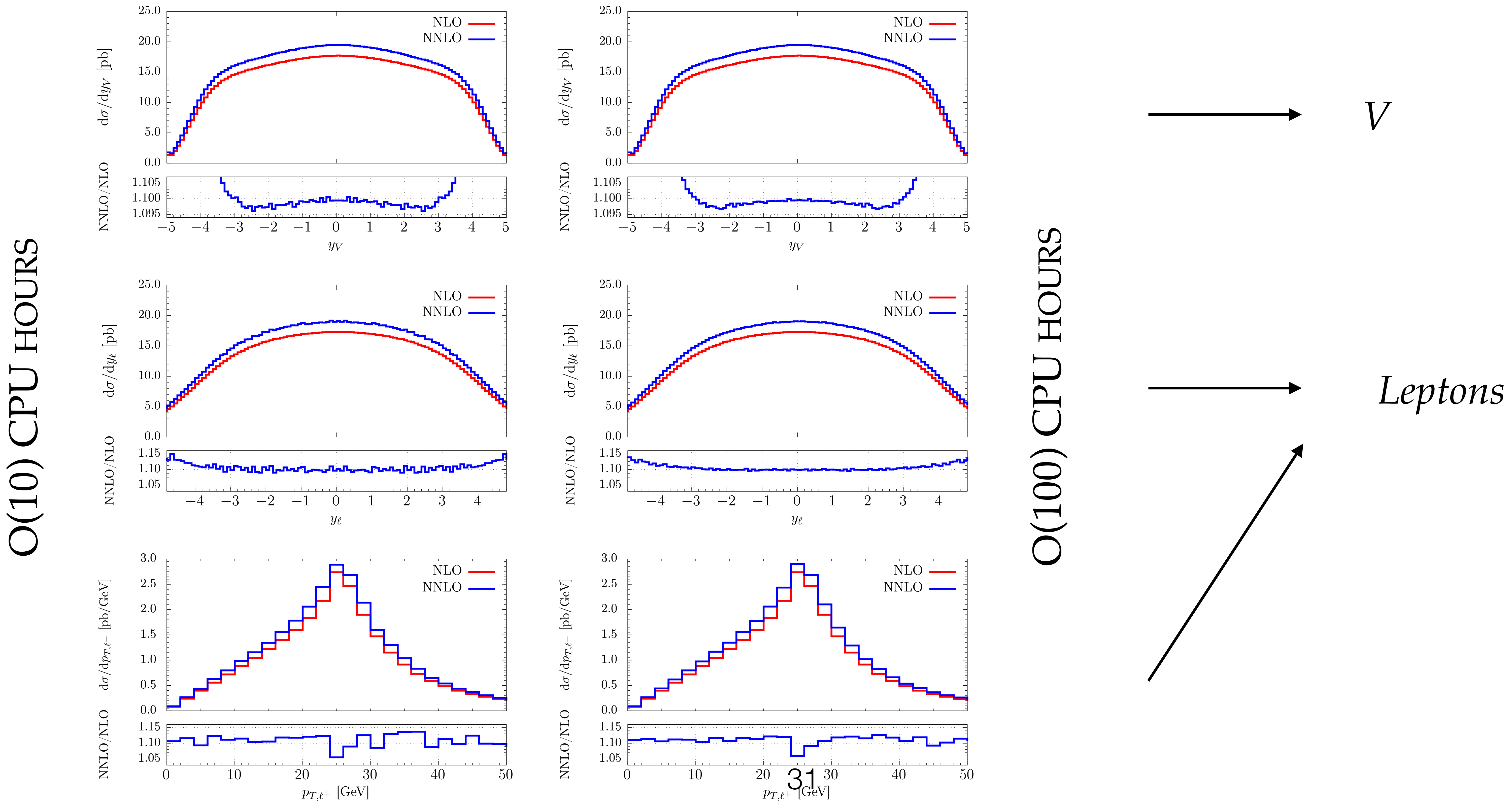
Higgs total cross section:

$$\sigma_{\rm H}^{\rm LO} = 15.42(1)\ \text{pb}; \qquad \sigma_{\rm H}^{\rm NLO} = 30.25(1)\ \text{pb}; \qquad \sigma_{\rm H}^{\rm NNLO} = 39.96(2)\ \text{pb}.$$

$pp \to 2l$, symmetric cuts:

$$\sigma_{\rm DY}^{\rm LO} = 650.4 \pm 0.1\ \text{pb}; \qquad \sigma_{\rm DY}^{\rm NLO} = 700.2 \pm 0.3\ \text{pb}; \qquad \sigma_{\rm DY}^{\rm NNLO} = 734.8 \pm 1.4\ \text{pb}.$$

$pp \to \gamma^* \to 2l$, differential distributions *[old semi-numerical code]*



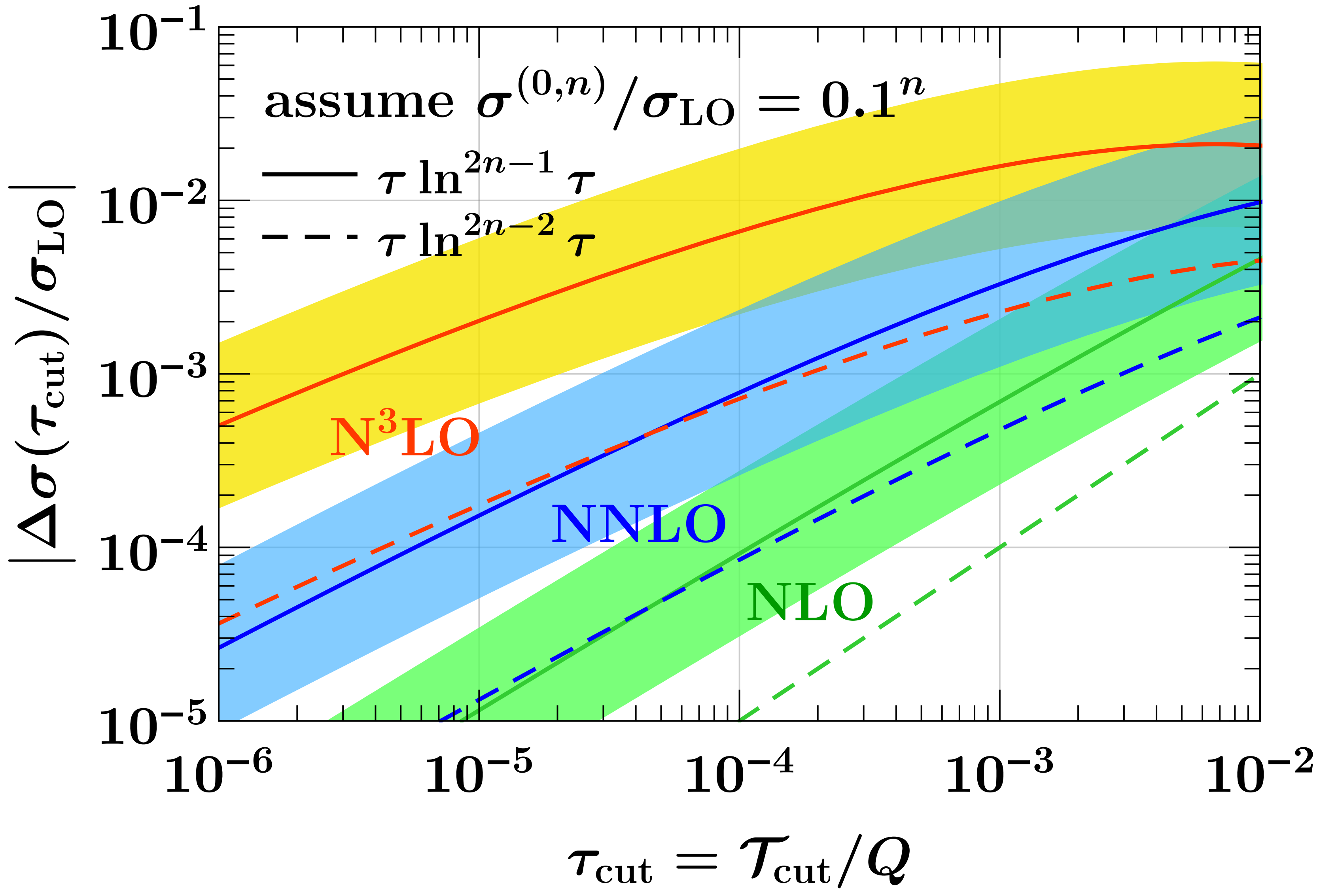
A few words on slicing

A slicing calculation requires a suitable variable δ that separates resolved and unresolved parts of a relevant phase space. For example, this parameter can be a transverse momentum of a ``Born'' final state q_t [Catani, Grazzini] or its jettiness [Boughezal et al, Gaunt et al].

In the unresolved region, we use soft-collinear approximations to integrate over unresolved momenta. In the resolved region, we only need $N^{k-1}LO$ corrections to X+J.

$$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int_0^\delta \left[|\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d\right]_{\mathrm{s.c.}} + \int_\delta^1 |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_4 + \mathcal{O}(\delta)$$

[Moult et al (2017)]



Challenge: individual contributions are logarithmically divergent ~ $\ln^{2k-1} \delta$.

Power corrections of order $\delta \ln^{2k-1} \delta$. At higher order, very small δ required.

Good control at small δ is difficult to achieve.

Because of this issue, slicing was eventually abandoned at NLO in favour of subtractions.

Despite not being widely used in modern NLO calculations, slicing techniques have reappeared in NNLO calculations!

This is due to the following reasons:

1. within the slicing approach, one can use existing NLO results for the CPU-intensive ``X+J'' part of the calculation. The availability of very efficient tools for NLO calculations allowed to obtain sufficiently stable results for several key processes like V/H production, VV production, single-top production, top decay, H/V+J production, H→bb decay, top pair production;

2. access to large computing facilities mitigates extremely high CPU requirements of the slicing method;

3. NNLO corrections are typically small, as the result, often an O(20%-50%) error on the NNLO coefficient only results in a percent-level error on the total cross-section;

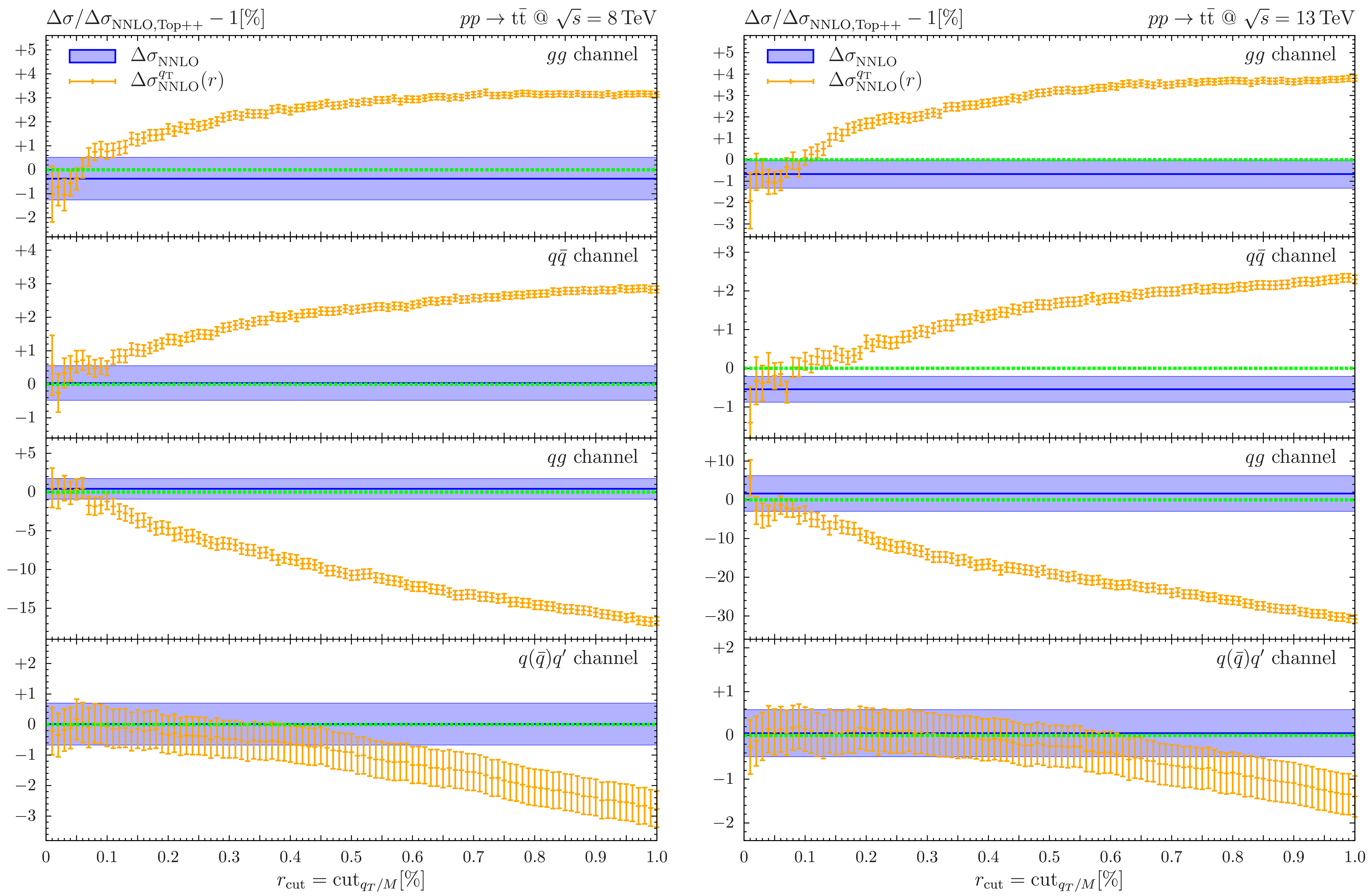
4. at NNLO, a simple-enough subtraction framework analogous to Catani-Seymour or FKS has yet to emerge.

$$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int_0^\delta \left[|\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d\right]_{\mathrm{s.c.}} + \int_\delta^1 |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_4 + \mathcal{O}(\delta)$$

Slicing techniques are very delicate and if one wants to use them it is very important to ensure that power corrections are under control. In general, their impact becomes more difficult to control in processes with a non-trivial color structure (see e.g. Campbell et al. (2019)), and in delicate fiducial regions (e.g. isolation).

Recently, the q_t-slicing method was extended to processes involving coloured massive particles. This allowed for the calculation of NNLO corrections to top pair production within this approach [Catani et al (2019)] confirming earlier results by [Czakon et al.]

[Catani et al (2019)]



For the total cross section, a detailed study showed that power corrections are under good control.

Use of slicing techniques for complex processes or in higher perturbative orders would likely require improvements. Roughly, they can be divided in three categories:

1. Devise ``optimal'' slicing parameters, as they can lead to better performances. For example, even within N-jettiness slicing it is well-known that different N-jettiness definitions perform very differently. In general, one would want a variable that restricts *all* radiation to be soft and collinear;
2. devise more differential slicing approaches. In principle, a fully differential slicing technique can be easily upgraded to a fully-fledged subtraction;
3. develop a better understanding of power corrections.

Recently, there has been a lot of activity on (3). Conceptually, this is non trivial because it corresponds to understanding factorization properties of QCD at next-to-leading power.

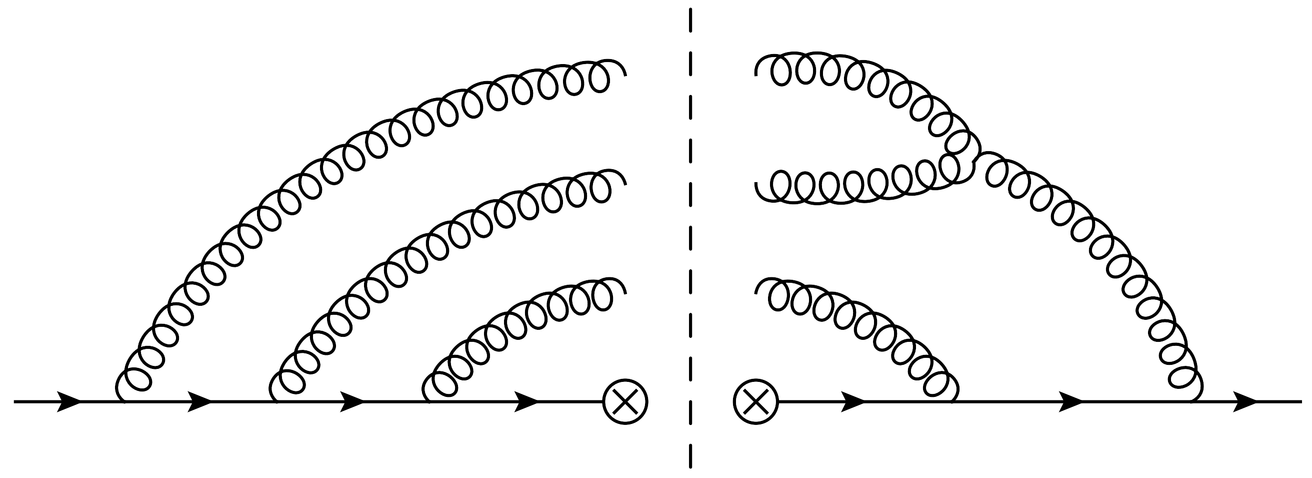
The first power corrections have been obtained using (SCET-assisted) fixed-order calculations. So far, results are only known for color singlet production [Boughezal et al, Moult et al, Ebert et al], and also in this case their structure is not completely understood. Analogous results for q_t have been obtained [Ebert et al].

Integration of soft and collinear limits of scattering amplitudes is a problem that is common to all subtraction AND slicing schemes. The ``unresolved'' phase-spaces, however, differ and are defined by a choice of a slicing variables or by particularities of a subtraction scheme.

Interestingly, slicing methods offer a (reasonably) clear path to a fully-differential description of color-singlet production in hadron collisions at N3LO: one needs N3LO soft and beam functions and an efficient way to compute V+j or H+j at NNLO.

Recent progress in three-loop quark beam function computations defined with both zero-jettiness variable [Behring, Melnikov, Riietkerk, Tancredi, Wever; Baranowski; large Nc only] and q_t slicing variable [Luo, Yang, Zhu, Zhu].

$$I_N(t,z) \sim \int \prod_{i=1}^{N} [\mathrm{d}k_i] \delta(s(1-z) - \sum_{i=1}^{N} 2p_2 \cdot k_i) \delta(t - z \sum_{i=1}^{N} 2p_1 \cdot k_i) P_{\text{split}}(z_1, z_2, ..)$$



$$\mathcal{T} = \sum_j \min_{i\in\{1,2\}} \left[\frac{2p_i \cdot k_j}{Q_i} \right]$$

An important open question is whether the three-loop soft function can be computed for zero-jettiness.

$$S_{RRR}(\tau) = \int \prod_{i=1}^{3} \frac{\mathrm{d}^d k_i}{(2\pi)^d} \, \delta(k_i^2) \, \delta(\mathcal{T} - \tau) \, \mathrm{Eik}(\{k_i\}, p_1, p_2)$$

$$\delta(\mathcal{T} - \tau) = \theta(\alpha_1 - \beta_1)\theta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)\delta(\beta_1 + \beta_2 + \beta_3 - \tau) + \theta(\beta_1 - \alpha_1)\theta(\alpha_2 - \beta_2)\theta(\alpha_3 - \beta_3)\delta(\alpha_1 + \beta_2 + \beta_3 - \tau) + \ldots$$

Conclusion

NNLO calculations are at the core of the precision program at the LHC. They require multi-loop amplitudes and efficient subtraction schemes. In the recent past, there has been a lot of progress on both points.

In particular, there exist several subtraction schemes for fully differential NNLO calculations that appear to be generic (*sector decomposition+FKS, antenna, jettiness slicing*).

As processes of interest become more and more complex, very efficient subtraction schemes are required. A "perfect" subtraction scheme at NNLO is yet to emerge.

Several proposals for ``better'' schemes appeared recently (*geometric, local analytic subtraction, nested subtraction*). They are promising, but not yet fully developed.

Finding local and analytic subtraction schemes is also an interesting theoretical problem in QCD. Indeed, a clear organization of soft/collinear information is needed not only for fixed-order computations but also for resummations and parton shower algorithms.

Perhaps, these approaches, traditionally addressed from very different perspectives, learn from each other in a way that will improve all of them.

Slicing problems are overcome with subtractions: one devises a subtraction term *S* that *a)* reproduces the matrix element in the soft-collinear limit and *b)* is simple enough to be integrated for generic configurations of the resolved phase space.

$$\int |\mathcal{M}|^2 \mathcal{F}_J \mathrm{d}\phi_d = \int \left[|\mathcal{M}|^2 \mathcal{F}_J - \mathcal{S}\right] \mathrm{d}\phi_4 + \int \mathcal{S} \mathrm{d}\phi_d$$

At least in principle, subtraction are local: singularities are subtracted point-by-point in the phase space and not on average.

Because of this, power corrections are not an issue and cancellations of very large weights never happen; as a consequence, subtraction techniques should perform much better than slicing (confirmed in NLO computations).

Subtraction schemes however suffer from other drawbacks:

1. it is non trivial to identify a good function to subtract;

2. integration of the subtraction terms can be (prohibitively) difficult

3. it is non-straightforward to re-use existing NLO results for the ``X+J'' part of the calculation

In the recent past, several new schemes have been proposed with the goal to obtain a generic, local and analytic framework for NNLO calculations.

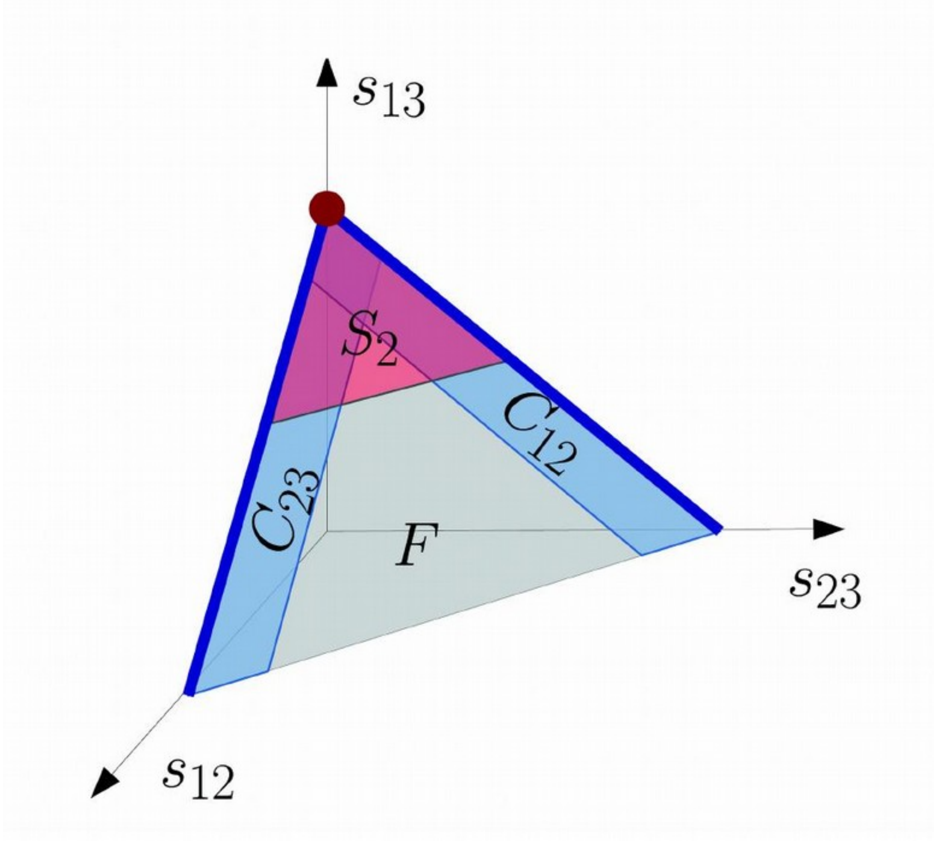
One example is the ``geometric'' approach [F. Herzog]. It is based on the following:

1. identify singular regions directly in the s_{ij} space;

2. use this to construct a local slicing scheme in the s_{ij} space;

3. promote the slicing to a full subtraction.

In this framework, overlapping singularities are removed by explicitly ordering all possible limits.



A nice feature of this approach is that the integration of the countersterms is simple.

Its main drawback is that it is based on looking at the structure of individual Feynman diagrams. Every diagram is treated differently. This makes the underlying IR structure of the amplitude hidden.

Currently, this approach is at the proof-of-concept stage. So far, it has been used to reproduce the known pole-structure of the double-real correction to H→gg in pure gluodynamics ($n_f = 0$).

Another example of new schemes is the so-called ``local analytic sector subtraction'' scheme [Magnea et al].

The origina idea of this approach is to combine FKS partition with Catani-Seymour parametrization of the phase space. It is constructed in the following way:

1. the phase space is partitioned in different regions in a FKS-like approach. Partitions are engineered to automatically remove overlapping singularities: different orderings are treated in different partitions;
2. in each partition, a simple CS-like parametrization is used. This leads to very simple counterterms, whose analytic integration is straightforward.

The method is fully local and fully analytic, and combine different approaches in an interesting way.

The final result has the expected dipole-like structure, although soft radiation is not treated globally but split into the various FKS sectors.

Currently, the framework has been used to reproduce the n_f contribution to $V \to jj$ decay, although extension to the more complex initial-state case is underway.