TRR 257 - Particle Physics Phenomenology
after the Higgs Discovery

Integrated double-unresolved subtraction terms for the nested soft-collinear subtraction scheme

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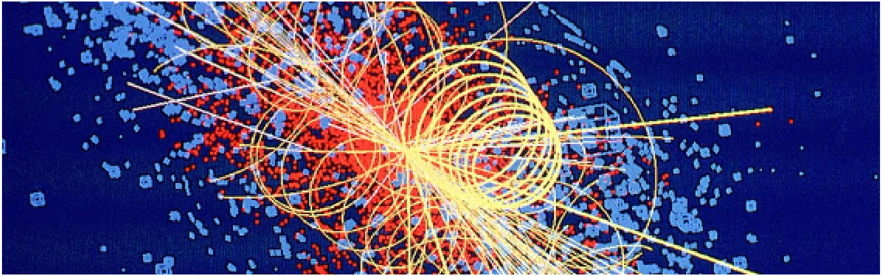
TTP



Introduction

- locally subtract and add back soft and collinear singularities

$$\mathrm{d}\sigma^{RR} \equiv \left\langle [\mathrm{d}k_{45}] F_{LM}(\{p\};4,5) \right\rangle$$
$$= \underbrace{\left\langle [\mathrm{d}k_{45}] \left(I - \hat{O} \right) F_{LM} \right\rangle}_{\text{regulated term}} + \underbrace{\left\langle [\mathrm{d}k_{45}] \hat{O} F_{LM} \right\rangle}_{\text{subtraction term}} = \ldots$$

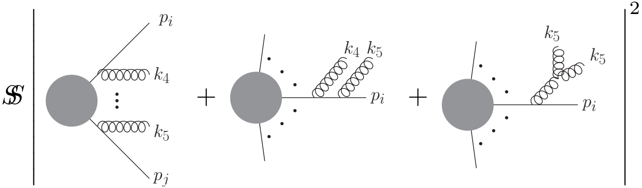
- fully regulated contributions are ready for numerical integration
- two genuinely double-unresolved subtraction terms

$$\mathcal{I}_{\mathrm{DS}} = \left\langle [\mathrm{d}k_{45}] \mathbb{S} F_{LM} \right\rangle \quad , \quad \mathcal{I}_{\mathrm{TC}} = \left\langle [\mathrm{d}k_{45}] \hat{O}^{\mathrm{reg}} \mathbb{C}_1 F_{LM} \right\rangle$$

- singular limits of F_{LM} factorize matrix elements and decouple infrared-safe observables
  $\Rightarrow$ analytic cancellation of poles and efficient evaluation of the finite parts of integrated subtraction terms

Factorization for double-soft gluons

$\mathbb{S} : E_4 \to 0,\ E_5 \to 0,\ E_4 \sim E_5$ [Catani,Grazzini '99]



$$= g_s^4 \Bigg\{ \frac{1}{2} \sum_{i,j,k,l} \underbrace{S(p_i, p_j, k_4) S(p_k, p_l, k_5)}_{(\text{NLO})^2} \underbrace{\left| \mathcal{M}_n^{(ij)(kl)}(\{p\}) \right|^2}_{\text{Born-like ME}}$$

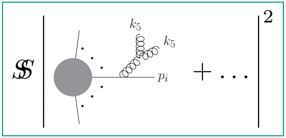
$$- C_A \sum_{i,j} \widetilde{S}(p_i, p_j, k_4, k_5) \underbrace{\left| \mathcal{M}_n^{(ij)}(\{p\}) \right|^2}_{\text{Born-like ME}} \Bigg\}$$

color-correlated

- several kinematic configurations of radiating partons i, j
  - massless or massive
  - arbitrary angle or aligned a back-to-back

Factorization for double-soft gluons

- non-abelian part is an intricate scalar function,e.g. massless case



$$\widetilde{S}(p_i, p_j, k_4, k_5)$$

$$= \frac{(1-\epsilon)}{(k_4 \cdot k_5)^2} \frac{[(p_i \cdot k_4)(p_j \cdot k_5) + i \leftrightarrow j]}{(p_i \cdot k_{45})(p_j \cdot k_{45})}$$

$$- \frac{(p_i \cdot p_j)^2}{2(p_i \cdot k_4)(p_j \cdot k_5)(p_i \cdot k_5)(p_j \cdot k_4)} \left[2 - \frac{[(p_i \cdot k_4)(p_j \cdot k_5) + i \leftrightarrow j]}{(p_i \cdot k_{45})(p_j \cdot k_{45})} \right]$$

$$+ \frac{(p_i \cdot p_j)}{2(k_4 \cdot k_5)} \left[\frac{2}{(p_i \cdot k_4)(p_j \cdot k_5)} + \frac{2}{(p_j \cdot k_4)(p_i \cdot k_5)} - \frac{1}{(p_i \cdot k_{45})(p_j \cdot k_{45})} \right.$$

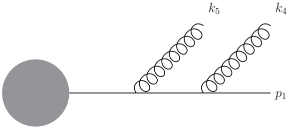
$$\left. \times \left(4 + \frac{[(p_i \cdot k_4)(p_j \cdot k_5) + i \leftrightarrow j]^2}{(p_i \cdot k_4)(p_j \cdot k_5)(p_i \cdot k_5)(p_j \cdot k_4)} \right) \right],$$

$$k_{45} = k_4 + k_5$$

Triple-collinear gluon emission

$\mathbb{C}_1 : \rho_{14} \to 0,\ \rho_{15} \to 0,\ \rho_{14} \sim \rho_{15} \sim \rho_{45},\ \rho_{ij} = 1 - \boldsymbol{n}_i \boldsymbol{n}_j$ [Catani,Grazzini '99]

$\mathbb{C}_1 \omega^{14,15} \Bigg|$



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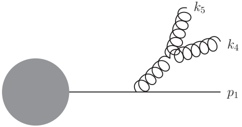

$+ \ldots \Bigg|^2$

$$= g_s^4 \frac{P_{g_4,g_5,q_1}(s_{14}, s_{15}, s_{45}, z_4, z_5)}{(s_{14}+s_{15}+s_{45})^2} \times \underbrace{\left| \mathcal{M}\left(\frac{E_1+E_4+E_5}{E_1} p_1, \ldots \right) \right|^2}_{\text{Born-like ME}}$$

- two distinct configurations for the splitting
  - final-state $\rightarrow$ Born process fully decouples in suitable parameterization
  - initial-state $\rightarrow$ Born process still depends on total radiated energy

Phase-space integration

- two double-unresolved subtraction terms, universal functions of Born kinematics for generic NNLO contributions

$$\mathcal{I}_{\mathrm{DS}} = \left\langle [\mathrm{d}k_{45}]\, \widetilde{S}_{f_4 f_5}(p_i, p_j, k_4, k_5) \times F_{LM}(\{p\}) \right\rangle \sim 1/\epsilon^4$$

$$\mathcal{I}_{\mathrm{TC}} = \left\langle [\mathrm{d}k_{45}] \hat{O}^{\mathrm{reg}}\, \frac{P_{f_4,f_5,f_r}(\{s_{ij}\},\{z_i\})}{s_{r45}^2} \times F_{LM}(\{E' \cdot p_r, \{p\}\}) \right\rangle \sim 1/\epsilon$$

- emission phase-space (energy-ordering and cut-off ⇒ not LI)

$$[\mathrm{d}k_{45}] = \mathrm{d}E_{45} (E_{45})^{1-2\epsilon}\, \theta_{0<E_5<E_4<E_{\max}}\; \mathrm{d}^{\mathrm{d}-1}\Omega_{45}$$

- perform angular integrals suitable for a straightforward integration in $\boldsymbol{x}$

$$\int_0^1 \mathrm{d}\boldsymbol{x} \int \mathrm{d}\Omega_{45}\; K(\boldsymbol{n}_4, \boldsymbol{n}_5, \boldsymbol{x}, \{p\}) \quad K = \left\{ \widetilde{S}_{f_4 f_5}, \frac{P_{f_4,f_5,f_r}}{s_{r45}^2} \right\}$$

Reverse unitarity

- introduce constrained momenta

$$\mathrm{d}^{\mathrm{d}-1}\Omega_i = 2x_i^{-1+2\epsilon}\,\underbrace{\delta^+\left(k_i^2\right)}_{\text{on-shell}}\,\underbrace{\delta(k_i\cdot N - x_i)}_{\text{fixed energy}}\,\mathrm{d}^{\mathrm{d}}k_i\;, \qquad N=(1,\mathbf{0})$$

- then write δ-functions as propagators [Anastasiou, Melnikov '02]

$$(2\pi\mathrm{i})\,\delta\left(q^2\right) \to \left[\frac{1}{q^2+\mathrm{i}0} - \frac{1}{q^2-\mathrm{i}0}\right]$$

- finally

$$\Rightarrow \int \mathrm{d}\Omega_{45}\,K(\boldsymbol{n}_4,\boldsymbol{n}_5,\boldsymbol{x},\{p\}) = \int \frac{\mathrm{d}^{\mathrm{d}}k_4\,\mathrm{d}^{\mathrm{d}}k_5\;K(k_4,k_5,\boldsymbol{x},\{p\})}{D_{\mathrm{cut}}}$$

where

$$D_{\mathrm{cut}} \equiv \left[k_4^2\right]_c\left[k_4\cdot N - x_4\right]_c\left[k_5^2\right]_c\left[k_5\cdot N - x_5\right]_c$$

Multi-loop techniques

- reduce to master integrals $\boldsymbol{I}$ using IBP identities [Chetyrkin,Tkachov '81]

$$0 = \int \mathrm{d}^d k \frac{\partial}{\partial k_\mu} \left[\frac{v^\mu}{\prod_i D_i^{\alpha_i}} \right] \Rightarrow \int \frac{\mathrm{d}^d k_4 \, \mathrm{d}^d k_5 \; K}{D_{\text{cut}}} = \underbrace{\boldsymbol{R}(\epsilon, \boldsymbol{x}))}_{\text{red. coeff.}} \times \boldsymbol{I}(\epsilon, \boldsymbol{x})$$

- use reduction relations to derive closed system of DEQ [Kotikov '90]

$$\frac{\partial}{\partial p^2} \boldsymbol{I} = \hat{m}(\epsilon, p^2, \dots) \boldsymbol{I}$$

- for rational $\hat{m}$, the ϵ-expanded solution can be expressed in GPLs

$$\mathsf{G}_{a,\vec{w}}(x) = \int_0^x \frac{\mathrm{d}u}{u-a} \mathsf{G}_{\vec{w}}(u) \ , \ \mathsf{G}_{\{\}}(x) = 1 \ , \ \mathsf{G}_a(x) = \ln(1 - x/a)$$

- boundary conditions have to be determined by evaluating $\boldsymbol{I}$ in an appropriate limit

Massless double-soft subtraction term

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- 19 MI $\boldsymbol{I}$, functions of $z = E_5/E_4$, $\sin^2\delta = \eta_{ij} = (1 - \boldsymbol{n}_i\boldsymbol{n}_j)/2$

$$\mathcal{I}_{\rm DS} = \int_0^1 {\rm d}z \int \frac{{\rm d}^d k_4\, {\rm d}^d k_5\; \widetilde{S}}{D_{\rm cut}} = \frac{1}{\epsilon}\int_0^1 {\rm d}z\; \underbrace{\boldsymbol{R}(\epsilon,z,\delta)\times\boldsymbol{I}(\epsilon,z,\delta)}_{\equiv G(\epsilon,z,\delta)\sim 1/\epsilon^2} \sim 1/\epsilon^4$$

- DEQ for $\boldsymbol{I}$ contains a square root $\sqrt{(1-z)^2\sin^2\delta + 4z}$, rationalize

$$z \to \frac{(1-t\cos\delta)(\cos\delta - t)}{t\sin^2\delta}$$

- after careful ϵ-expansion, integrate only DEQ in t

$${\rm d}\boldsymbol{I} = \left[\hat{M}_t(\epsilon,t,\delta)\,{\rm d}t + \hat{M}_\delta(\epsilon,t,\delta)\,{\rm d}\delta\right]\times\boldsymbol{I}$$

and express result in terms of rational functions an $\mathrm{G}_{\vec{\omega}(\delta)}(t)$

Massless double-soft subtraction term

- expand MI $\boldsymbol{I}$ in the limit $z \to 0$ as functions of δ, e.g. $\boldsymbol{I}_{13}(\epsilon, z, \delta)$

$$\lim_{z\to 0} \int \frac{\mathrm{d}^{\mathrm{d}-1}\Omega_{45}}{\left[\eta_{14} + z\eta_{15}\right]^2 \left[\eta_{24} + z\eta_{25}\right]}$$

$$= 16(1-2\epsilon)^2 \times \left\{ \frac{-(1+2\epsilon)}{(1+\epsilon)(2+\epsilon)(1-2\epsilon)}\, {}_2F_1\left[\{1,2\},\{3+\epsilon\};\sin^2\delta\right] \right.$$

$$+ \left[\sin^2\delta\right]^{-2-\epsilon} \frac{(1+2\epsilon)}{\epsilon(1-2\epsilon)} \Gamma(1+\epsilon)\,\Gamma(1-\epsilon) \cdot \left(1 - \epsilon\sin^2\delta - \left[\cos^2\delta\right]^{\epsilon}\right)$$

$$- \left[\sin^2\delta\right]^{-1} z^{-\epsilon} \cdot \frac{(1+2\epsilon)}{(1-3\epsilon)} I_\Gamma + \left[\sin^2\delta\right]^{-1-\epsilon} \Gamma(1-\epsilon)\,\Gamma(1+\epsilon)\, \frac{(1+2\epsilon)}{(1-2\epsilon)}$$

$$- \left[\sin^2\delta\right]^{-2-\epsilon} \Gamma(1-\epsilon)\,\Gamma(1+\epsilon)\, \frac{(1+2\epsilon)}{\epsilon(1-2\epsilon)} - \left[\sin^2\delta\right]^{-1} z^{-1-\epsilon} \frac{I_\Gamma}{4\epsilon}$$

$$\left. + z^{-\epsilon} \frac{I_\Gamma}{\epsilon(1-3\epsilon)} \left[\sin^2\delta\right]^{-2} \left(1 + \frac{\left[\sin^2\delta\right]}{4}\left(1+3\epsilon+6\epsilon^2\right)\right) \right\} + \mathcal{O}\left(z^2\right) \; .$$

where $\quad I_\Gamma \equiv \dfrac{\Gamma(1+\epsilon)\,\Gamma^3(1-2\epsilon)}{\Gamma(1-3\epsilon)\,\Gamma^2(1-\epsilon)}$

Massless double-soft subtraction term

- split integrand

$$G(\epsilon, z, \delta) = G^{\text{reg}}(\epsilon, z, \delta) + G^{\text{sing}}(\epsilon, z, \delta)$$

  - obtain $G^{\text{sing}}(\epsilon, z, \delta) \sim z^{-1-2\epsilon} G_0(\epsilon, \delta)$ from the limit $\boldsymbol{I}(z \approx 0)$,
  - construct $G^{\text{reg}}(\epsilon, t, \delta)$ from full solution, which contains $\mathrm{G}_{\vec{\omega}(\delta)}(t)$

$$\Rightarrow \mathcal{I}_{\text{DS}} = \frac{1}{\epsilon} \int_{\cos\delta}^{\frac{1-\sin\delta}{\cos\delta}} \mathrm{d}t \; G^{\text{reg}}(t) \; - \underbrace{\frac{G_0(\epsilon, \delta)}{2\epsilon^2}}_{\checkmark}$$

- final result consists of $\mathrm{G}_{\vec{\omega}(\delta)}\left(\frac{1-\sin\delta}{\cos\delta}\right)$ and $\mathrm{G}_{\vec{\omega}(\delta)}(\cos\delta)$ up to weight four with entries drawn from

$$\left\{ a_{\pm}\left[\frac{\cos(\delta)}{2}\right], a_{\pm}\left[\frac{\cos(\delta)(3-\cos^2(\delta))}{2}\right], a_{\pm}\left[\cos(\delta)\right], \cos(\delta)^{\pm 1}, 0, \pm 1 \right\}$$

$$a_{\pm}(u) = u \pm i\sqrt{1-u^2}$$

Massless double-soft subtraction term

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( (4 (-844 + 33 \[Pi]^2))/(27 ep) - (2 (-820 + 33 \[Pi]^2))/(27 ep) + (  11 (62 + E^(-2 I delta) + E^(2 I delta)) GN[0, a61, 1]^2)/(12 ep) - (  32 GN[0, a61, 1]^3)/ep - (16 GN[0, aZ1, 1]^3)/ep
+ (  33 (6 + E^(-2 I delta) + E^(2 I delta)) GN[a62, a61, 1]^2)/ep + (  12 GN[a62, a61, 1]^3)/ep -   GN[a41, a61,   1]^2 (-(176/(3 ep)) - (16 GN[a61, aZ1, 1])/ep - (    16 GN[a62, aZ1, 1])/ep)
-  GN[a42, a61,    1]^2 (-(176/(3 ep)) - (16 GN[a61, aZ1, 1])/ep - (    16 GN[a62, aZ1, 1])/ep) -  GN[a62, a61,    1]^2 ((33 (6 + E^(-2 I delta) + E^(2 I delta)))/(2 ep)
+ (     12 GN[a62, aZ1, 1])/ep)- ( 11 (6 + E^(-2 I delta) + E^(2 I delta)) GN[0, 0, a61, 2])/(6 ep) + ( E^(-2 I delta) (11 - 486 E^(2 I delta) - 1162 E^(4 I delta)
-    486 E^(6 I delta) + 11 E^(8 I delta)) GN[0, 0, a61, 2])/( 3 (1 + E^(2 I delta))^2 ep) + ( 14 (19 + 50 E^(2 I delta) + 19 E^(4 I delta)) GN[0, 0, aZ1, 2])/( 3 (1 + E^(2 I delta))^2 ep)
(88 GN[0, a41, a61, 2])/ep + ( 88 GN[0, a42, a61, 2])/ep - ( 77 (6 + E^(-2 I delta) + E^(2 I delta)) GN[0, a62, a61, 2])/( 6 ep) - ((726 + 77 E^(-2 I delta)
+ 77 E^(2 I delta)) GN[0, 4/(   2 + E^(-2 I delta) + E^(2 I delta)), 1, 2])/(6 ep)+ ( 88 GN[a41, 0, a61, 2])/ep - (176 GN[a41, a41, a61, 2])/(3 ep) - ( 176 GN[a41, a42, a61, 2])/(3 ep)
- (88 GN[a41, a62, a61, 2])/( 3 ep) + (88 GN[a42, 0, a61, 2])/ep - (176 GN[a42, a41, a61, 2])/( 3 ep) - (176 GN[a42, a42, a61, 2])/(3 ep) - ( 88 GN[a42, a62, a61, 2])/(3 ep)
+ (220 GN[a62, 0, a61, 2])/(3 ep) - ( 88 GN[a62, a41, a61, 2])/(3 ep) - (88 GN[a62, a42, a61, 2])/(3 ep) + GN[a31, a61,    1] (-((16 GN[a42, a61, 1]^2)/ep)
+     GN[a42, a61,       1] ((16 GN[a61, a61, 1])/ep + (16 GN[a62, a61, 1])/ep) - (    16 GN[a41, 0, a61, 2])/ep + (16 GN[a41, a41, a61, 2])/ep
+ (    16 GN[a41, a42, a61, 2])/ep - (16 GN[a42, 0, a61, 2])/ep + (    16 GN[a42, a41, a61, 2])/ep + (16 GN[a42, a42, a61, 2])/ep + (    16 GN[a61, 0, a61, 2])/ep
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+  GN[a32, a61,    1] (-((16 GN[a42, a61, 1]^2)/ep) +     GN[a42, a61,       1] ((16 GN[a61, a61, 1])/ep + (16 GN[a62, a61, 1])/ep) - (    16 GN[a41, 0, a61, 2])/ep
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 + (       16 GN[a62, aZ1, 1])/ep) - (16 GN[a41, 0, aZ1, 2])/ep + (    16 GN[a41, a41, aZ1, 2])/ep + (16 GN[a41, a42, aZ1, 2])/ep - (    16 GN[a42, 0, aZ1, 2])/ep
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 + (    16 GN[a62, 0, aZ1, 2])/ep - (16 GN[a62, a41, aZ1, 2])/ep - (    16 GN[a62, a42, aZ1, 2])/ep) + (44 GN[a62, a62, a61, 2])/(3 ep) - GN[a62, a61,    1] (-((5442 + 368 E^(-2 I delta)
 + 368 E^(2 I delta) -       33 E^(-4 I delta) - 33 E^(4 I delta))/(24 ep)) + (    11 (6 + E^(-2 I delta) + E^(2 I delta)) GN[a62, aZ1, 1])/(    2 ep) + (8 GN[0, a41, aZ1, 2])/ep
 + (8 GN[0, a42, aZ1, 2])/ep - (    16 GN[0, a61, aZ1, 2])/ep - (16 GN[0, a62, aZ1, 2])/ep - ( 72 GN[a41, 0, aZ1, 2])/ep + (32 GN[a41, a41, aZ1, 2])/ep
 + (    32 GN[a41, a42, aZ1, 2])/ep + (40 GN[a41, a61, aZ1, 2])/ep + (    40 GN[a41, a62, aZ1, 2])/ep - (72 GN[a42, 0, aZ1, 2])/ep + (    32 GN[a42, a41, aZ1, 2])/ep
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 - (40 GN[a62, a42, aZ1, 2])/ep - (    24 GN[a62, a61, aZ1, 2])/ep - (24 GN[a62, a62, aZ1, 2])/ep) - ( 16 GN[0, 0, a41, a61, 3])/ep + (16 GN[0, 0,a41, aZ1, 3])/ep
 - ( 16 GN[0, 0, a42, a61, 3])/ep + (16 GN[0, 0, a42, aZ1, 3])/ep - ( 80 GN[0, 0, a61, a61, 3])/ep + (80 GN[0, 0, a61, aZ1, 3])/ep - ( 80 GN[0, 0, a62, a61, 3])/ep
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 + ( 56 GN[0, a61, a61, a61, 3])/ep - (56 GN[0, a61, a61, aZ1, 3])/ep + ( 120 GN[0, a61, a62, a61, 3])/ep - ( 120 GN[0, a61, a62, aZ1, 3])/ep - (136 GN[0, a62, 0, a61, 3])/ep
 + (  136 GN[0, a62, 0, aZ1, 3])/ep + (16 GN[0, a62, a21, a61, 3])/ep - ( 16 GN[0, a62, a21, aZ1, 3])/ep + (16 GN[0, a62, a22, a61, 3])/ep - ( 16 GN[0, a62, a22, aZ1, 3])/ep
 + (120 GN[0, a62, a61, a61, 3])/ep - ( 120 GN[0, a62, a61, aZ1, 3])/ep + (56 GN[0, a62, a62, a61, 3])/ep - (  56 GN[0, a62, a62, aZ1, 3])/ep + (24 GN[a41, 0, 0, a61, 3])/ep
 - ( 24 GN[a41, 0, 0, aZ1, 3])/ep + (16 GN[a41, 0, a31, a61, 3])/ep - ( 16 GN[a41, 0, a31, aZ1, 3])/ep + (16 GN[a41, 0, a32, a61, 3])/ep - ( 16 GN[a41, 0, a32, aZ1, 3])/ep
 - (16 GN[a41, 0, a41, a61, 3])/ep + ( 16 GN[a41, 0, a41, aZ1, 3])/ep - (16 GN[a41, 0, a42, a61, 3])/ep + ( 16 GN[a41, 0, a42, aZ1, 3])/ep - (24 GN[a41, 0, a61, a61, 3])/ep
 + ( 24 GN[a41, 0, a61, aZ1, 3])/ep - (88 GN[a41, 0, a62, a61, 3])/ep + ( 88 GN[a41, 0, a62, aZ1, 3])/ep - (48 GN[a41, a41, 0, a61, 3])/ep + (  48 GN[a41, a41, 0, aZ1, 3])/ep
 + ( 16 GN[a41, a41, a21, a61, 3])/ep - ( 16 GN[a41, a41, a21, aZ1, 3])/ep + ( 16 GN[a41, a41, a22, a61, 3])/ep - ( 16 GN[a41, a41, a22, aZ1, 3])/ep - ( 16 GN[a41, a41, a31, a61, 3])/ep
 + ( 16 GN[a41, a41, a31, aZ1, 3])/ep - ( 16 GN[a41, a41, a32, a61, 3])/ep + ( 16 GN[a41, a41, a32, aZ1, 3])/ep + ( 48 GN[a41, a41, a61, a61, 3])/ep - ( 48 GN[a41, a41, a61, aZ1, 3])/ep
 + ( 48 GN[a41, a41, a62, a61, 3])/ep - ( 48 GN[a41, a41, a62, aZ1, 3])/ep - ( 48 GN[a41, a42, 0, a61, 3])/ep + (48 GN[a41, a42, 0, aZ1, 3])/ep + (  16 GN[a41, a42, a21, a61, 3])/ep
 - ( 16 GN[a41, a42, a21, aZ1, 3])/ep + ( 16 GN[a41, a42, a22, a61, 3])/ep - ( 16 GN[a41, a42, a22, aZ1, 3])/ep - ( 16 GN[a41, a42, a31, a61, 3])/ep + ( 16 GN[a41, a42, a31, aZ1, 3])/ep
 - ( 16 GN[a41, a42, a32, a61, 3])/ep + (  16 GN[a41, a42, a32, aZ1, 3])/ep + ( 48 GN[a41, a42, a61, a61, 3])/ep - ( 48 GN[a41, a42, a61, aZ1, 3])/ep + ( 48 GN[a41, a42, a62, a61, 3])/ep
 - ( 48 GN[a41, a42, a62, aZ1, 3])/ep - ( 40 GN[a41, a61, 0, a61, 3])/ep + (40 GN[a41, a61, 0, aZ1, 3])/ep - ( 16 GN[a41, a61, a21, a61, 3])/ep + ( 16 GN[a41, a61, a21, aZ1, 3])/ep
 - ( 16 GN[a41, a61, a22, a61, 3])/ep + ( 16 GN[a41, a61, a22, aZ1, 3])/ep + ( 16 GN[a41, a61, a41, a61, 3])/ep - ( 16 GN[a41, a61, a41, aZ1, 3])/ep + ( 16 GN[a41, a61, a42, a61, 3])/ep
 - ( 16 GN[a41, a61, a42, aZ1, 3])/ep + (  40 GN[a41, a61, a61, a61, 3])/ep - ( 40 GN[a41, a61, a61, aZ1, 3])/ep - ( 24 GN[a41, a61, a62, a61, 3])/ep + ( 24 GN[a41, a61, a62, aZ1, 3])/ep
 + ( 24 GN[a41, a62, 0, a61, 3])/ep - (24 GN[a41, a62, 0, aZ1, 3])/ep - ( 16 GN[a41, a62, a21, a61, 3])/ep + (16 GN[a41, a62, a21, aZ1, 3])/ep - (16 GN[a41, a62, a22, a61, 3])/ep
 + ( 16 GN[a41, a62, a22, aZ1, 3])/ep + ( 16 GN[a41, a62, a41, a61, 3])/ep - ( 16 GN[a41, a62, a41, aZ1, 3])/ep + ( 16 GN[a41, a62, a42, a61, 3])/ep - ( 16 GN[a41, a62, a42, aZ1, 3])/ep
 - ( 24 GN[a41, a62, a61, a61, 3])/ep + (  24 GN[a41, a62, a61, aZ1, 3])/ep + ( 40 GN[a41, a62, a62, a61, 3])/ep - ( 40 GN[a41, a62, a62, aZ1, 3])/ep + (24 GN[a42, 0, 0, a61, 3])/ep
 - ( 24 GN[a42, 0, 0, aZ1, 3])/ep + (16 GN[a42, 0, a31, a61, 3])/ep   - ( 16 GN[a42, 0, a31, aZ1, 3])/ep + (16 GN[a42, 0, a32, a61, 3])/ep - ( 16 GN[a42, 0, a32, aZ1, 3])/ep
```

Massless double-soft subtraction term

- simplifies from several kB of text-file by symbol calculus [Duhr '12]

$$
\begin{aligned}
\mathcal{I}_{\rm DS}^{gg} = {} & (2E_{\max})^{-4\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2 \Bigg\{\frac{1}{2\epsilon^4}+\frac{1}{\epsilon^3}\left[\frac{11}{12}-\ln(s^2)\right] \\
& +\frac{1}{\epsilon^2}\left[2{\rm Li}_2(c^2)+\ln^2(s^2)-\frac{11}{6}\ln(s^2)+\frac{11}{3}\ln 2-\frac{\pi^2}{4}-\frac{16}{9}\right] \\
& +\frac{1}{\epsilon}\Bigg[6{\rm Li}_3(s^2)+2{\rm Li}_3(c^2)+\left(2\ln(s^2)+\frac{11}{3}\right){\rm Li}_2(c^2)-\frac{2}{3}\ln^3(s^2) \\
& \quad +\left(3\ln(c^2)+\frac{11}{6}\right)\ln^2(s^2)-\left(\frac{22}{3}\ln 2+\frac{\pi^2}{2}-\frac{32}{9}\right)\ln(s^2) \\
& \quad -\frac{45}{4}\zeta_3-\frac{11}{3}\ln^2 2-\frac{11}{36}\pi^2-\frac{137}{18}\ln 2+\frac{217}{54}\Bigg] \\
& +4\mathsf{G}\left(\{-1,0,0,1\};s^2\right)-7\mathsf{G}\left(\{0,1,0,1\};s^2\right)+\frac{22}{3}{\rm Ci}_3(2\delta) \\
& +\frac{1}{3\tan(\delta)}{\rm Si}_2(2\delta)+2{\rm Li}_4(c^2)-14{\rm Li}_4(s^2)+4{\rm Li}_4\left(\frac{1}{1+s^2}\right)-2{\rm Li}_4\left(\frac{1-s^2}{1+s^2}\right) \\
& +2{\rm Li}_4\left(\frac{s^2-1}{1+s^2}\right)+{\rm Li}_4(1-s^4)+\Bigg[10\ln(s^2)-4\ln\left(1+s^2\right) \\
& +\frac{11}{3}\Bigg]{\rm Li}_3(c^2)+\left[14\ln(c^2)+2\ln(s^2)+4\ln\left(1+s^2\right)+\frac{22}{3}\right]{\rm Li}_3(s^2) \\
& +4\ln(c^2){\rm Li}_3(-s^2)+\frac{9}{2}{\rm Li}_2^2(c^2)-4{\rm Li}_2(c^2){\rm Li}_2(-s^2)+\Bigg[7\ln(c^2)\ln(s^2) \\
& -\ln^2(s^2)-\frac{5}{2}\pi^2+\frac{22}{3}\ln 2-\frac{131}{18}\Bigg]{\rm Li}_2(c^2)+\left[\frac{2}{3}\pi^2-4\ln(c^2)\ln(s^2)\right]\times \\
& {\rm Li}_2(-s^2)+\frac{\ln^4(s^2)}{3}+\frac{\ln^4\left(1+s^2\right)}{6}-\ln^3(s^2)\left[\frac{4}{3}\ln(c^2)+\frac{11}{9}\right] \\
& +\ln^2(s^2)\left[7\ln^2(c^2)+\frac{11}{3}\ln(c^2)+\frac{\pi^2}{3}+\frac{22}{3}\ln 2-\frac{32}{9}\right]-\frac{\pi^2}{6}\ln^2\left(1+s^2\right) \\
& +\zeta_3\left[\frac{17}{2}\ln(s^2)-11\ln(c^2)+\frac{7}{2}\ln\left(1+s^2\right)-\frac{21}{2}\ln 2-\frac{99}{4}\right]+\ln(s^2)\times \\
& \left[-\frac{7\pi^2}{2}\ln(c^2)+\frac{22}{3}\ln^2 2-\frac{11}{18}\pi^2+\frac{137}{9}\ln 2-\frac{208}{27}\right]-12{\rm Li}_4\left(\frac{1}{2}\right) \\
& +\frac{143}{720}\pi^4-\frac{\ln^4 2}{2}+\frac{\pi^2}{2}\ln^2 2-\frac{11}{6}\pi^2\ln 2+\frac{125}{216}\pi^2+\frac{22}{9}\ln^3 2 \\
& +\frac{137}{18}\ln^2 2+\frac{434}{27}\ln 2-\frac{649}{81}+\mathcal{O}(\epsilon)\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I}_{\rm DS}^{q\bar{q}} = {} & (2E_{\max})^{-4\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]^2 \Bigg\{-\frac{1}{3\epsilon^3}+\frac{1}{\epsilon^2}\Bigg[\frac{2}{3}\ln(s^2)-\frac{4}{3}\ln 2 \\
& +\frac{13}{18}\Bigg]+\frac{1}{\epsilon}\Bigg[-\frac{4}{3}{\rm Li}_2(c^2)-\frac{2}{3}\ln^2(s^2)+\ln(s^2)\left(\frac{8}{3}\ln 2-\frac{13}{9}\right)+\frac{\pi^2}{9} \\
& +\frac{4}{3}\ln^2 2+\frac{35}{9}\ln 2-\frac{125}{54}\Bigg]-\frac{8}{3}{\rm Ci}_3(2\delta)-\frac{2}{3\tan(\delta)}{\rm Si}_2(2\delta)-\frac{4}{3}{\rm Li}_3(c^2) \\
& -\frac{8}{3}{\rm Li}_3(s^2)+{\rm Li}_2(c^2)\left[\frac{29}{9}-\frac{8}{3}\ln 2\right]+\frac{4}{9}\ln^3(s^2)+\ln^2(s^2)\Bigg[-\frac{4}{3}\ln(c^2) \\
& -\frac{8}{3}\ln 2+\frac{13}{9}\Bigg]+\ln(s^2)\left[-\frac{8}{3}\ln^2 2-\frac{70}{9}\ln 2+\frac{2}{9}\pi^2+\frac{107}{27}\right]+9\zeta_3 \\
& +\frac{2\pi^2}{3}\ln 2-\frac{8}{9}\ln^3 2-\frac{23}{108}\pi^2-\frac{35}{9}\ln^2 2-\frac{223}{27}\ln 2+\frac{601}{162}+\mathcal{O}(\epsilon)\Bigg\}
\end{aligned}
$$

[Caola, MD, Frellesvig, Melnikov '18]

Triple collinear subtraction term

- consider initial- and final-state splittings for arbitrary flavors $f_4 f_5 f_r$
- four MI $\boldsymbol{I}$ as functions of $\omega_i = E_i/E_1 \; i = 4, 5$.
- apply transformation $\hat{T}$ into ϵ-homogeneous form [Henn '13]

$$\boldsymbol{I} = \hat{T}\boldsymbol{J} \quad \Rightarrow \quad \frac{\partial}{\partial y}\boldsymbol{J} = \epsilon \sum_{y-y_0 \in \mathcal{A}_y} \frac{\hat{m}_{y_0}}{y - y_0}\boldsymbol{J}$$

- find T via algorithmic approach [Lee '15], applied sequentially in both variables

$$\Rightarrow \mathcal{A} = \{\omega_4, \omega_4 - 1, \omega_5, \omega_5 - 1, \omega_4 + \omega_5, \omega_4 + \omega_5 - 1\}$$

- compute one boundary condition in the limit $\omega_4 = \omega_5 = \omega \to 0$

$$\boldsymbol{I}_2 \sim \int \frac{\mathrm{d}\Omega_{45}}{\omega\left[2 - \boldsymbol{n}_1 \cdot (\boldsymbol{n}_4 + \boldsymbol{n}_5)\right]} = 4\ln(2) + \epsilon\left(\frac{\pi^2}{6} - 4\ln^2(2)\right) + \mathcal{O}\left(\epsilon^2\right)$$

Triple collinear subtraction term

- decoupling final-state energy parameterization

$$\{\omega_4,\omega_5\} = \frac{x_1\{1,x_2\}}{1-x_1-x_1x_2} \;\Rightarrow\; \mathcal{I}_{\mathrm{TC}} = \int_0^1 \mathrm{d}x_1\mathrm{d}x_2\;\hat{O}^{\mathrm{reg}}\;\theta_{1-x_1-x_1x_2}\;\underbrace{\boldsymbol{R}.\hat{T}.\boldsymbol{J}}_{\sim 1/\epsilon}$$

- solve DEQ such that $\boldsymbol{J}$ consists of $\mathsf{G}_{\vec{\omega}(x_1)}(x_2)$,$\mathsf{G}_{\vec{\omega}}(x_2)$.
  Allows for a straightforward integration in x_2

$$\Rightarrow \left\{\mathsf{G}_{\vec{R}(x_1)}(1)\,,\mathsf{G}_{\vec{R}(x_1)}\left(\frac{1-x_1}{x_1}\right),\mathsf{G}_{\vec{c}}\left(\frac{1-x_1}{x_1}\right)\right\}$$

- move variable to the argument (iteratively form lower to higher weight)

$$\underbrace{\mathsf{G}_{\vec{R}(x_1)}\left(\frac{1-x_1}{x_1}\right)}_{\text{weight } n} = \int_0^{x_1}\mathrm{d}t\,\underbrace{\left[\frac{\partial}{\partial t}\mathsf{G}_{\vec{R}(t)}\left(\frac{1-t}{t}\right)\right]}_{\text{weight } n-1} + \text{const}$$

- example

$$\mathsf{G}_{-1,1/x_1,1/x_1}(1/x_1-1) = \zeta_3 - \frac{\pi^2}{6}\mathsf{G}_{-1}(x_1) - 2\mathsf{G}_{-1,-1,0}(x_1) - \mathsf{G}_{-1,0,0}(x_1)$$

Triple collinear subtraction term

- emission of two collinear gluons off a final-state quark

$$
\begin{aligned}
\mathcal{I}_{\mathrm{TC}}^{qgg} = C_A C_F \Bigg\{ & \frac{1}{\epsilon}\Bigg[-\frac{1015}{108} + \frac{19\zeta_3}{8} + \frac{\pi^2}{8} + \frac{11}{2}\ln^2(2) - \frac{11}{4}\ln(2) + \frac{1}{3}\pi^2\ln(2) \Bigg] \\
& + \Bigg[-\frac{2281}{48} - 2\,\mathrm{Li}_4(1/2) + \frac{25\zeta_3}{24} - \frac{13}{4}\zeta_3\ln(2) - \frac{119\pi^2}{144} + \frac{173\pi^4}{480} - \frac{\ln^4(2)}{12} \\
& - \frac{176}{9}\ln^3(2) - \frac{19}{36}\ln^2(2) - \frac{11}{12}\pi^2\ln^2(2) - \frac{1247}{108}\ln(2) + \frac{161}{36}\pi^2\ln(2) \Bigg] \Bigg\} \\
+ C_F^2 \Bigg\{ & \frac{1}{\epsilon}\Bigg[\frac{31}{16} - 2\zeta_3 + \frac{9}{8}\ln(2) + \frac{1}{3}\pi^2\ln(2) \Bigg] + \Bigg[\frac{715}{32} + 16\zeta_3\ln(2) - \frac{7\pi^4}{30} \\
& - \frac{63}{16}\ln^2(2) - \frac{7}{6}\pi^2\ln^2(2) + \frac{17}{8}\ln(2) + \pi^2\ln(2) \Bigg] \Bigg\}
\end{aligned}
$$

[MD,Melnikov ’19]

Conclusion

- Analytic computation of the two double-unresolved subtraction terms
  - double-soft: massless radiators at an arbitrary angle;
  - triple-collinear: arbitrary flavors, initial- and final-state splittings;
  - double-soft: two massive radiators in back-to-back alignment.

  Using reverse unitarity makes the problem amenable to well-known loop-techniques such as IBP reduction and differential equations.

- appear in double-real NNLO corrections to generic processes, i.e. they are part of the analytic and fully-differential formulation of
  - color-singlet production [Caola,Melnikov,Roentsch '19];
  - color-singlet decay [Caola,Melnikov,Roentsch '19];
  - deep-inelastic scattering [Asteriadis,Caola,Melnikov,Roentsch '19];

  which are building blocks for arbitrary processes. [c.f. K.Asteriadis talk]

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Thanks!