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TRR 257 - Particle Physics Phenomenology
after the Higgs Discovery

Application of the nested soft-collinear subtraction scheme to deep-inelastic scattering

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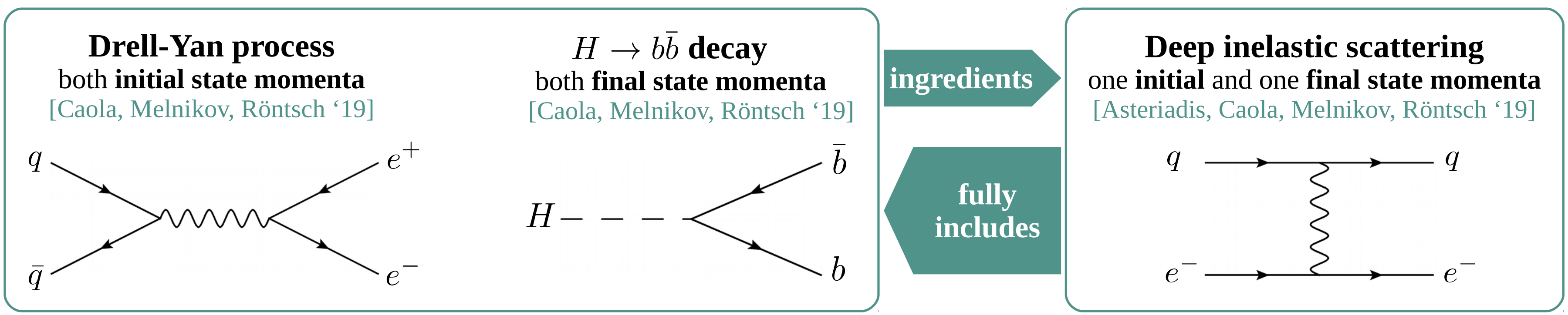
CRC workshop on Soft-Collinear QCD Dynamics, Siegen

Deep inelastic scattering

- Non-trivial parts of soft and collinear limits of scattering amplitudes relevant for NNLO calculations involve pairs of external momenta. In a complex process these momenta can be ***both incoming***, ***both outgoing*** or ***one*** can be ***incoming*** and ***one outgoing***.

- Therefore, before dealing with complex processes it is important to apply the subtraction scheme to simpler processes with only two external colour-charged particles; the results will provide useful ***building blocks*** for more complex processes.



- For these simple processes results for subtractions can be checked extensively against existing analytic results.

Deep inelastic scattering

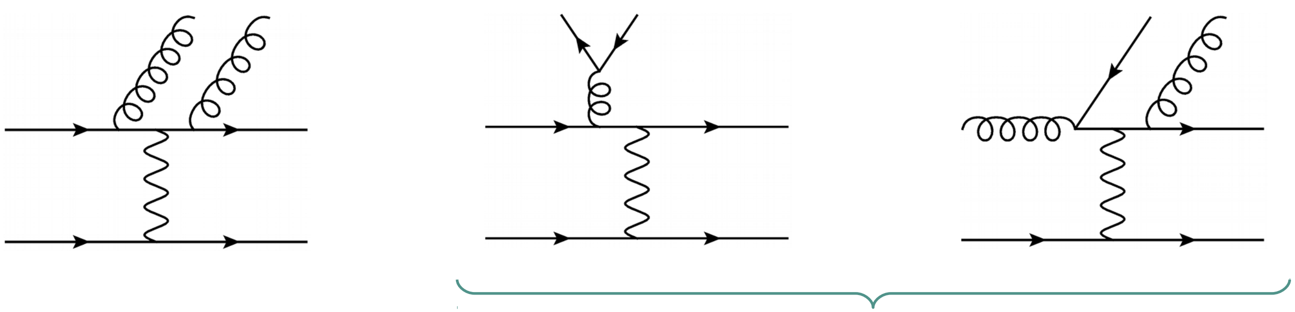
- Contributions to NNLO fully-differential partonic cross sections

contain ***explicit*** poles in 1/ε

$$\mathrm{d}\sigma_{\mathrm{NNLO}} = \underbrace{\mathrm{d}\sigma_{\mathrm{rr}} + \overbrace{\mathrm{d}\sigma_{\mathrm{rv}}}\!\!\!\!\!\!\!\!\!\!\!\!\!}\ \ + \mathrm{d}\sigma_{\mathrm{vv}} + \mathrm{d}\sigma_{\mathrm{pdf}}$$

contain singularities that need to be extracted and regulated

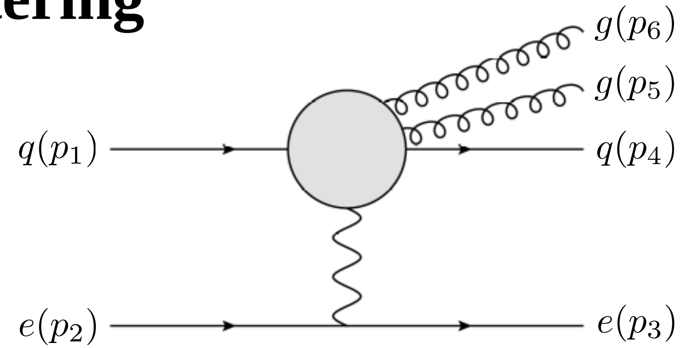
- We consider the double real emission contribution $\mathrm{d}\sigma_{\mathrm{rr}}$ that has the most complex singular structure. It includes different partonic channels



quark final states have a less singular structure

out of which we consider the channel $q + e \to q + e + gg$.

Deep inelastic scattering



- We write the differential cross section as

energy ordering

$$2s \cdot \mathrm{d}\sigma_{\mathrm{rr}} = \int [\mathrm{d}g_5][\mathrm{d}g_6] \boxed{\theta(E_5 - E_6)} F_{\mathrm{LM}}(1,4,5,6) \equiv \langle F_{\mathrm{LM}}(1,4,5,6) \rangle$$

with

$$F_{\mathrm{LM}}(1,4,5,6) = \mathcal{N} \int \mathrm{dLips}\ (2\pi)^d \delta^{(d)}(p_1 + p_2 - p_3 - p_4 - p_5 - p_6)$$
$$\times \, |M^{\mathrm{tree}}(\{p\}), p_5, p_6|^2 \times \mathcal{O}(p_3, p_4, p_5, p_6)$$

needs to be sufficiently large but otherwise arbitrary

$$[\mathrm{d}g_i] = \frac{\mathrm{d}^{d-1} p_i}{(2\pi)^{d-1} 2E_i} \boxed{\theta(E_{\max} - E_i)}$$

- The integral diverges and needs to be regulated. Due to the absence of entangled soft and collinear singularities all singularities can be subtracted ***iteratively*** using well understood soft & collinear limits [see Melnikov's talk].

Double-soft singularity

- To this end, we introduce an operator $\mathbb{S}$ that extracts the leading double soft singularity ($E_5 \sim E_6 \to 0$). The action on $F_{\rm LM}(1,4,5,6)$ is defined as

$$\mathbb{S}F_{\rm LM}(1,4,5,6) = \mathbb{S}\left[\mathcal{N}\int {\rm dLips}\ (2\pi)^d\delta^{(d)}\left(p_1+p_2-\sum_{i=3}^{6}p_i\right)|M^{\rm tree}(\{p\},p_5,p_6)|^2\mathcal{O}(p_3,p_4,p_5,p_6)\right]$$

$$\equiv g_{s,b}^4 \times \underbrace{{\rm Eikonal}(1,4,6,7)}_{} \times \underbrace{\mathcal{N}\int {\rm dLips}\ (2\pi)^d\delta^{(d)}\left(p_1+p_2-p_3-p_4\right)|M^{\rm tree}(\{p\})|^2\mathcal{O}(p_3,p_4)}_{\equiv F_{\rm LM}(1,4)}$$

independent of the hard matrix element and the observable
[for explicit formula see Delto's talk]

LO differential cross-section, independent of gluons 5 & 6

- We insert the identity operator $I = (I-\mathbb{S})+\mathbb{S}$ into the phase space

$$\langle F_{\rm LM}(1,4,5,6)\rangle = \underbrace{\langle (I-\mathbb{S})F_{\rm LM}(1,4,5,6)\rangle}_{} + \underbrace{\langle \mathbb{S}F_{\rm LM}(1,4,5,6)\rangle}_{}$$

double-soft singularity regulated

subtraction term, contains the 1/ε pole

- In the first term: the double-soft singularity is regulated ***locally*** at any point of the phase space.

Double-soft singularity

- In the subtraction term the soft gluons decouple from the **matrix element,** the **observable**, the **LO phase space** and the **momentum conservation condition.**

$$\langle \mathbb{S} F_{\mathrm{LM}}(1,4,5,6)\rangle = g_{s,b}^4 \times \int [\mathrm{d}g_5][\mathrm{d}g_6]\ \theta(E_5 - E_6) \times \mathrm{Eik}(1,4,5,6) \times F_{\mathrm{LM}}(1,4)$$

- It can be integrated ***analytically*** over the phase space of the two gluons 5 & 6 [see Delto's talk; Caola, Delto, Frellesvig, Melnikov, Röntsch '18].

- Since they decouple from the momentum conservation condition the upper energy cut-off E_{max} is necessary to avoid artificial "UV" divergences.

- The double soft regulated term $\langle (I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6)\rangle$ still contains unregulated single-soft and collinear singularities; we will now regulate them **iteratively**.

Single-soft singularity

- The differential cross section $\langle (I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6) \rangle$ contains only one single-soft singularity ($E_6 \to 0$) because of energy ordering $E_5 > E_6$.

- We introduce an operator S_6 that extracts the leading single-soft singularity

$$S_6 F_{\mathrm{LM}}(1,4,5,6) = g_{s,b}^2 \times \frac{1}{E_6^2} \left[(2C_F - C_A) \frac{\rho_{14}}{\rho_{16}\rho_{46}} + C_A \left(\frac{\rho_{15}}{\rho_{16}\rho_{56}} + \frac{\rho_{45}}{\rho_{46}\rho_{56}} \right) \right] \times F_{\mathrm{LM}}(1,4,5) \,,$$

where

$$\rho_{ij} = 1 - \vec{n}_i \cdot \vec{n}_j = 1 - \cos\theta_{ij} \,.$$

- We decompose an identity operator $I = (I - S_6) + S_6$ and insert it into the phase space

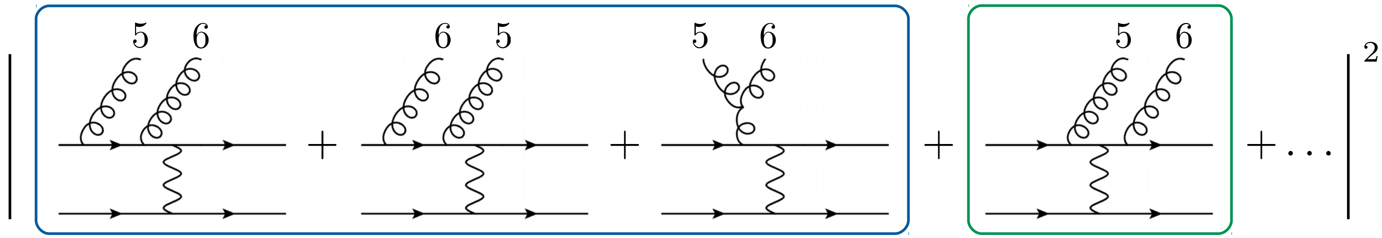
$$\langle (I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6) \rangle = \underbrace{\langle (I - S_6)(I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6) \rangle}_{\text{all soft singularities regulated}} + \underbrace{\langle S_6 (I - \mathbb{S}) F_{\mathrm{LM}}(1,4,5,6) \rangle}_{\text{extracted } 1/\varepsilon \text{ pole}}$$

- In the first term: all soft singularities are now regulated ***locally*** at any phase-space point of the resolved phase space. However, collinear singularities remain unregulated.

- In the subtraction term: the soft gluon decouples from the **matrix element** and the **observable**. Hence we can integrate ***analytically*** over the phase space of the gluon 6 [more details later].

Collinear singularities

$$\left|M^{\text{tree}}(\{p\}, p_5, p_6)\right|^2 =$$



- In the collinear limits, many different singular configurations exist.

- However, collinear singularities factorize on external legs, therefore either *three partons* become collinear (triple-collinear singularity) or *two pairs of partons* become collinear (double-collinear singularity) at once.

- To control which partons develop collinear singularities, the different configurations are separated by ***introducing partition functions***.

- Different double collinear singularities in *triple collinear partitions* are further isolated in the angular phase space. We separate them by ***splitting the phase space*** into different ***sectors***.

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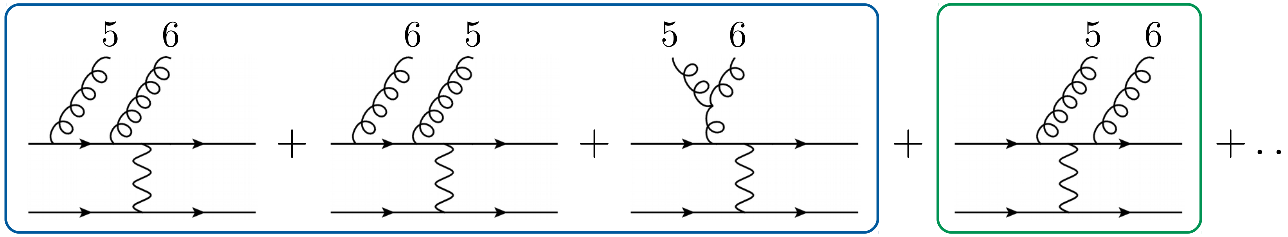
Partition functions

$$\left|M^{\text{tree}}(\{p\}, p_5, p_6)\right|^2 = \Big| \quad \cdots \quad \Big|^2$$



- The different collinear configurations are separated by **introducing partition functions** in the phase space

$$1 = \boxed{w^{51,61}} + w^{54,64} + \boxed{w^{51,64}} + w^{54,61}\,,$$

where

$$\lim_{5\|l} w^{5i,6j} \sim \delta_{li}\,, \quad \lim_{6\|l} w^{5i,6j} \sim \delta_{lj} \quad \text{and} \quad \lim_{5\|i}\lim_{6\|j} w^{5i,6j} = 1\,.$$

- One possible choice (that we use) is

$$w^{51,61} = \frac{\rho_{54}\rho_{64}}{d_5 d_6}\left(1 + \frac{\rho_{51}}{d_{5641}} + \frac{\rho_{61}}{d_{5614}}\right), \quad w^{51,64} = \frac{\rho_{54}\rho_{61}\rho_{56}}{d_5 d_6 d_{5614}}\,,$$

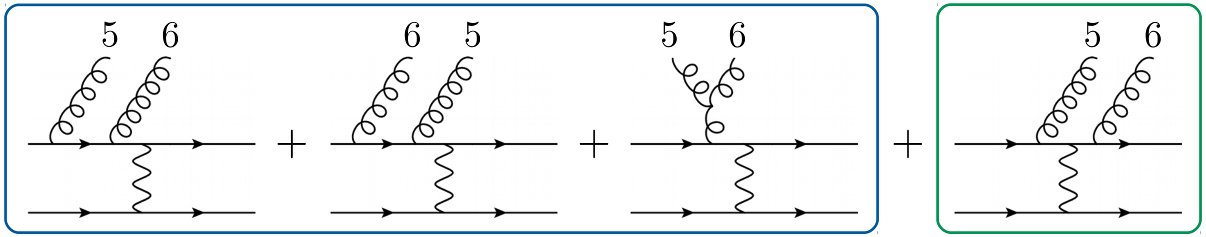
$$w^{54,64} = \frac{\rho_{51}\rho_{61}}{d_5 d_6}\left(1 + \frac{\rho_{64}}{d_{5641}} + \frac{\rho_{54}}{d_{5614}}\right), \quad w^{54,61} = \frac{\rho_{51}\rho_{64}\rho_{56}}{d_5 d_6 d_{5641}}\,,$$

with

$$d_{i=5,6} \equiv \rho_{1i} + \rho_{4i}\,, \quad d_{5614} \equiv \rho_{56} + \rho_{51} + \rho_{64}\,, \quad d_{5641} \equiv \rho_{56} + \rho_{54} + \rho_{61}\,.$$

Partition functions

$$\left|M^{\text{tree}}(\{p\},p_5,p_6)\right|^2 = \Big|$$



$$+\ \ldots\ \Big|^2$$

- We then write the cross section as

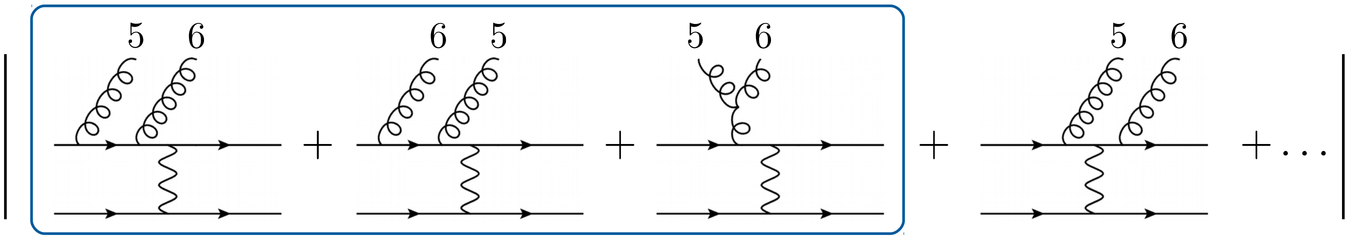
$$\langle (I-S_6)(I-\mathbb{S})F_{\text{LM}}(1,4,5,6)\rangle = \big\langle (I-S_6)(I-\mathbb{S})\big(\boxed{w^{51,61}} + w^{54,64} + \boxed{w^{51,64}} + w^{54,61}\big)F_{\text{LM}}(1,4,5,6)\big\rangle$$

- $\boxed{w^{51,64}|M|^2}$ is $\begin{cases} \text{singular when } (5\|1) \text{ and } (6\|4), \\ \text{finite when } (5\|4), (6\|1), (5\|6), (5\|6\|1) \text{ and } (5\|6\|4). \end{cases}$

- However, $\boxed{w^{51,61}|M|^2}$ is $\begin{cases} \text{still singular when } (5\|1), (6\|1), (5\|6) \text{ and } (5\|6\|1), \\ \text{finite for } (5\|4), (6\|4) \text{ and } (5\|6\|4). \end{cases}$

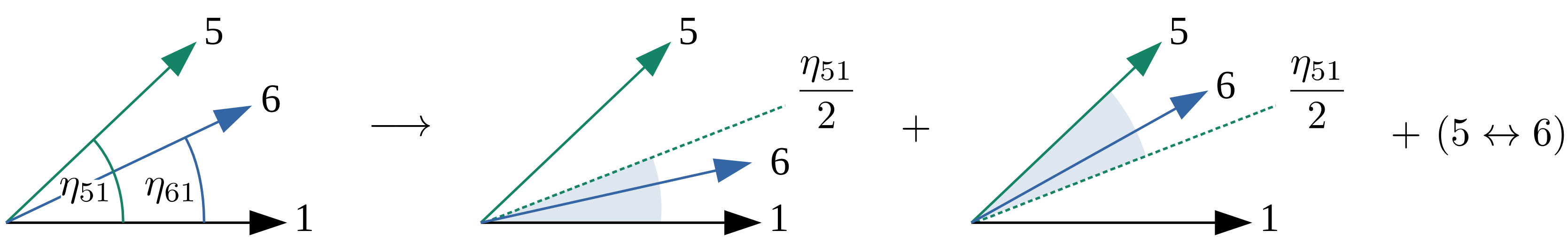
Phase space sectors

$$\left|M^{\text{tree}}(\{p\},p_5,p_6)\right|^2 =$$



- To separate the double collinear singularities in *triple collinear partitions* we **split the angular phase space** into different **sectors**.

- As an example we consider partition $w^{51,61}$ which describe triple-collinear emissions along p_1. The angular phase space is split into regions with definite collinear singularities (with $\eta_{ij} = (1-\cos\theta_{ij})/2$)



double-collinear ***(6||1)***
triple-collinear ***(6||5||1)***

double-collinear ***(6||5)***
triple-collinear ***(6||5||1)***

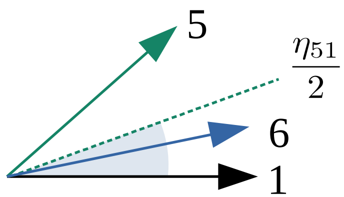
- In practice this is done by introducing the partition of the unity

$$1 = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right) + \theta\left(\frac{\eta_{51}}{2} < \eta_{61} < \eta_{51}\right) + \theta\left(\eta_{51} < \frac{\eta_{61}}{2}\right) + \theta\left(\frac{\eta_{61}}{2} < \eta_{51} < \eta_{61}\right) \equiv \theta^{(a)} + \theta^{(b)} + \theta^{(c)} + \theta^{(d)}\,.$$

Collinear singularities

- In each partition and sector exactly two collinear singularities are present that are uniquely defined. It is now straightforward to regulate them.

- As an example, we consider the partition $w^{51,61}$ and the sector $\theta^{(a)} = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right)$.



$$\left\langle (I - S_6)(I - \mathbb{S})w^{51,61}\theta^{(a)}F_{\mathrm{LM}}(1,4,5,6)\right\rangle$$

- By construction there are the **two collinear singularities**: a double-colinear when **(6||1)** and and a triple-collinear when **(5||6||1)**. Introducing operators C_{61} and $\mathbb{C}_1$ that extract the corresponding leading singularities. We regulate the singularities iteratively by writing

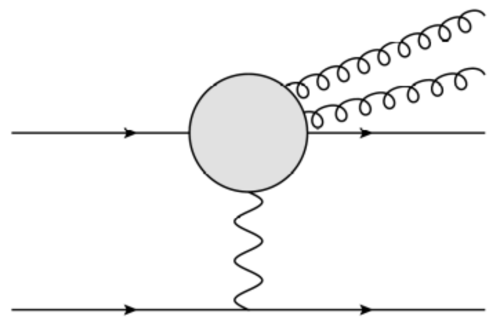
$$\begin{aligned}\left\langle (I - S_6)(I - \mathbb{S})w^{51,61}\theta^{(a)}F_{\mathrm{LM}}(1,4,5,6)\right\rangle &= \left\langle (I - C_{61})(I - \mathbb{C}_1)(I - S_6)(I - \mathbb{S})w^{51,61}\theta^{(a)}F_{\mathrm{LM}}(1,4,5,6)\right\rangle \\ &+ \left\langle C_{61}(I - \mathbb{C}_1)(I - S_6)(I - \mathbb{S})w^{51,61}\theta^{(a)}F_{\mathrm{LM}}(1,4,5,6)\right\rangle \\ &+ \left\langle \mathbb{C}_1(I - S_6)(I - \mathbb{S})w^{51,61}\theta^{(a)}F_{\mathrm{LM}}(1,4,5,6)\right\rangle\end{aligned}$$

- In partition $w^{51,61}$ and sector $\theta^{(a)}$ all singularities are now regulated.

- We deal with remaining partitions and sectors in a similar way.

Fully regulated double-real contribution

- To regularize the real emission contribution



we need the following operators
(acting on matrix elements and the phase space)

$\mathbb{S}$	Double-soft: $E_5, E_6 \to 0$
S_6	Single-soft: $E_6 \to 0$
$\mathbb{C}_{1,4}$	Triple-collinear: $(5 \parallel 6 \parallel 1)$ and $(5 \parallel 6 \parallel 4)$
$C_{51}, C_{54}, C_{61}, C_{64}$	Double-collinear: $(5 \parallel 1)$, $(5 \parallel 4)$ and $(6 \parallel 1)$ $(6 \parallel 1)$
C_{56}	Double-collinear: $(5 \parallel 6)$

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Fully regulated double-real contribution

- We obtain a very compact formula for the fully-regulated double-real contribution to DIS

$$
\begin{aligned}
2s \cdot \mathrm{d}\sigma_{\mathrm{rr}}^{\mathrm{FR}} = & \sum_{\substack{i,j=1,4 \\ i \neq j}} \Big\langle [I - \mathbb{S}][I - S_6][I - C_{6j}][I - C_{5i}][\mathrm{d}g_5][\mathrm{d}g_6] w^{5i,6j} F_{\mathrm{LM}}(1,4,5,6) \Big\rangle \\
& + \sum_{i=1,4} \Big\langle [I - \mathbb{S}][I - S_6] \Big[\theta^{(a)} [I - \mathbb{C}_i][I - C_{6i}] + \theta^{(b)} [I - \mathbb{C}_i][I - C_{56}] \\
& \qquad + \theta^{(c)} [I - \mathbb{C}_i][I - C_{5i}] + \theta^{(d)} [I - \mathbb{C}_i][I - C_{56}] \Big] \\
& \qquad \times [\mathrm{d}g_5][\mathrm{d}g_6] w^{5i,6j} F_{\mathrm{LM}}(1,4,5,6) \Big\rangle .
\end{aligned}
$$

- Actions of all operators are well defined and lead to analytic expressions that consists of matrix elements, splitting functions & phase-space weights whose numerical implementation is straightforward.

- The 16 terms in each contribution describe all physical singular limits that may occur at NNLO.

- $\mathrm{d}\sigma_{\mathrm{rr}}^{\mathrm{FR}}$ is finite and can be used to numerically ***compute arbitrary infra-red safe observables*** in $d = 4$ dimensions.

Subtraction terms

- Analytic integration of the subtraction terms was discussed in Delto's talk.
- We find simplifications if we recombine all subtractions terms.
- For instance recombining all subtraction terms from single collinear final state emission

$$\left\langle [I-\mathbb{S}][I-S_6]\Big[C_{54}w^{54,61}+C_{64}w^{51,64}+\Big(\theta^{(a)}C_{64}+\theta^{(c)}C_{54}\Big)w^{54,64}\Big][\mathrm{d}g_5][\mathrm{d}g_6]F_{\mathrm{LM}}(1,4,5,6)\right\rangle$$
$$=\frac{[\alpha_s]C_F}{\epsilon}\left\langle\Big[\Big(\frac{1}{\epsilon}+Z^{2,2}\Big)(2E_4)^{-2\epsilon}-(2E_5)^{-2\epsilon}\Big]\Big(w_{\mathrm{DC}}^{51}+w_{\mathrm{TC}}^{54}\Big(\frac{\rho_{54}}{4}\Big)^{-\epsilon}\Big)\boxed{F_{\mathrm{LM}}(1,4,5)}\right\rangle$$
$$-\frac{[\alpha_s]^2C_F^2}{\epsilon^3}\Big(\frac{1}{2\epsilon}+Z^{2,4}\Big)\Big\langle\langle\Delta_{51}\rangle_{S_5}(2E_4)^{-4\epsilon}F_{LM}(1,4)\Big\rangle\,.$$

NLO differential cross section

where

$$\int_0^1\mathrm{d}z\;z^{-n\epsilon}(1-z)^{-m\epsilon}P_{qq}(z)=-\Big(\frac{2}{m\epsilon}+Z^{n,m}\Big)=-\frac{2}{m\epsilon}-\frac{3}{2}-\frac{1}{12}\left[6+21m+15n-4n\pi^2\right]\epsilon+\mathcal{O}(\epsilon^2)\,,$$

$$\langle\Delta_{51}\rangle_{S_5}=\left(-\frac{1}{\epsilon}\left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]2^{-2\epsilon}\right)^{-1}\int\mathrm{d}\Omega_5^{(d-1)}\,\frac{\rho_{14}}{\rho_{15}\rho_{45}}\left[w_{\mathrm{DC}}^{51}+w_{\mathrm{TC}}^{54}\Big(\frac{\rho_{54}}{4}\Big)^{-\epsilon}\right]=\frac{3}{2}+\epsilon\left(\frac{\ln 2}{2}-2\ln\eta_{14}\right)+\mathcal{O}(\epsilon^2)\,,$$

$$w_{\mathrm{DC}}^{51}=C_{64}w^{51,64}\,,$$

$$w_{\mathrm{TC}}^{54}=C_{64}w^{54,64}\,.$$

- The subtraction terms contains the **NLO differential cross-section** with ***NLO singularities***.

NLO singularities

- They are regulated using the NLO FKS method [Frixione, Kunszt, Signer '95]

$$\langle F_{\mathrm{LM}}(1,4,5)\rangle = \underbrace{\langle \hat{O}_{\mathrm{NLO}} F_{\mathrm{LM}}(1,4,5)\rangle}_{\text{fully regulated}} + \underbrace{\langle [C_{51}+C_{54}][I-S_5]F_{\mathrm{LM}}(1,4,5)\rangle + \langle S_5 F_{\mathrm{LM}}(1,4,5)\rangle}_{\text{subtraction terms, extracted } 1/\varepsilon \text{ pole}}$$

- Final state emission fully regulated subtraction term

$$\Big\langle [I-\mathbb{S}][I-S_6]\Big[C_{54}w^{54,61} + C_{64}w^{51,64} + \Big(\theta^{(a)}C_{64} + \theta^{(c)}C_{54}\Big)w^{54,64}\Big][\mathrm{d}g_5][\mathrm{d}g_6]F_{\mathrm{LM}}(1,4,5,6)\Big\rangle$$

$$= \frac{[\alpha_s]C_F}{\epsilon}\Big\langle \hat{O}_{NLO}\Big[\Big(\frac{1}{\epsilon}+Z^{2,2}\Big)(2E_4)^{-2\epsilon} - \frac{1}{\epsilon}(2E_5)^{-2\epsilon}\Big]\Big[w_{\mathrm{DC}}^{51} + w_{\mathrm{TC}}^{54}\Big(\frac{\rho_{54}}{4}\Big)^{-\epsilon}\Big]F_{\mathrm{LM}}(1,4,5)\Big\rangle$$

$$+ \frac{[\alpha_s]^2C_F^2}{\epsilon^3}\Big\langle\Big[\Big(\frac{1}{\epsilon}+Z^{2,2}\Big)(2E_4)^{-2\epsilon}(2E_{\mathrm{max}})^{-2\epsilon} - \frac{1}{2\epsilon}(2E_{\mathrm{max}})^{-4\epsilon}\Big]$$

$$\times\Big[\langle\Delta_{51}\rangle_{S_5} - \frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} - \frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\Big]F_{\mathrm{LM}}(1,4)\Big\rangle$$

$$+ \frac{[\alpha_s]^2C_F^2}{\epsilon^2}\Big[\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\Big]\Big[\frac{1}{\epsilon}+Z^{2,2}\Big]\Big[\frac{1}{\epsilon}+Z^{4,2}\Big]\Big\langle(2E_4)^{-4\epsilon}F_{\mathrm{LM}}(1,4)\Big\rangle$$

$$- \frac{[\alpha_s]^2C_F^2}{\epsilon^3}\Big[\frac{1}{2\epsilon}+Z^{2,4}\Big]\Big\langle\Big[\langle\Delta_{51}\rangle_{S_5} + \Big(\frac{2^\epsilon}{2}\frac{\Gamma(1-\epsilon)\Gamma(1-2\epsilon)}{\Gamma(1-3\epsilon)}\Big)\Big](2E_4)^{-4\epsilon}F_{\mathrm{LM}}(1,4)\Big\rangle$$

$$- \frac{[\alpha_s]^2C_F^2}{\epsilon^2}\Big[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\Big]\int \mathrm{d}z\,\Big\langle\Big[\Big(\frac{1}{\epsilon}+Z^{2,2}\Big)(2E_4)^{-2\epsilon} - \frac{1}{\epsilon}(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\Big]$$

$$\times(2E_1)^{-2\epsilon}(1-z)^{-2\epsilon}\bar{P}_{qq}(z)\frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\Big\rangle.$$

NLO kinematics, all singularities regulated, needs to be calculated numerically

Born kinematics, contain poles

boosted born kinematics

Combining contributions

- We combine the result with **real-virtual**, **double-virtual** and contributions from **collinear renormalization**.

- All ϵ - poles are known ***analytically*** and we can check the cancellation ***explicitly***.

- For instance for the ϵ^{-4} - pole we find the contributions (no $\mathrm{d}\sigma_{\mathrm{pdf}} = \mathcal{O}(\epsilon^{-2})$)

$$\mathrm{d}\sigma_{\mathrm{rr}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \times \frac{1}{\epsilon^4}\left(\frac{C_A C_F}{2} + 2C_F^2\right) \times \langle F_{\mathrm{LM}}(1,4)\rangle + \mathcal{O}(\epsilon^{-3})$$

$$\mathrm{d}\sigma_{\mathrm{rv}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \times \frac{1}{\epsilon^4}\left(-\frac{C_A C_F}{2} - 4C_F^2\right) \times \langle F_{\mathrm{LM}}(1,4)\rangle + \mathcal{O}(\epsilon^{-3})$$

$$\mathrm{d}\sigma_{\mathrm{vv}} = \left(\frac{\alpha_s}{2\pi}\right)^2 \times \frac{1}{\epsilon^4}\left(2C_F^2\right) \times \langle F_{\mathrm{LM}}(1,4)\rangle + \mathcal{O}(\epsilon^{-3})$$

$$\Rightarrow \mathrm{d}\sigma_{\mathrm{NNLO}} = \mathrm{d}\sigma_{\mathrm{rr}} + \mathrm{d}\sigma_{\mathrm{rv}} + \mathrm{d}\sigma_{\mathrm{vv}} + \mathrm{d}\sigma_{\mathrm{pdf}} = \mathcal{O}(\epsilon^{-3})$$

- We can take the $\epsilon \to 0$ explicitly and obtain an analytical 4- dimensional formula for the fully-differential cross section.

Validation of results

- Our results have been extensively tested against known analytic results [Kazakov et al. ‘90; Zijlstra, van Neerven ‘92; Moch, Vermaseren ‘00; …].

- In the case of photon-induced deep-inelastic scattering with only up-quarks and gluons in the initial state and $\sqrt{s} = 100\,\mathrm{GeV},\; 10\,\mathrm{GeV} < Q < 100\,\mathrm{GeV},\; \mu_R = \mu_F = 100\,\mathrm{GeV}$ we obtain permille agreement for the ***NNLO contribution***

$$\sigma = \sigma_{\mathrm{LO}} + \sigma_{\mathrm{NLO}} + \boxed{\sigma_{\mathrm{NNLO}}}$$

channel	numerical result	analytic result
$\sigma^{\mathrm{NNLO}}_{\mathrm{q,ns}}$	$[33.1(2) - 2.18(1)\cdot n_f]\,\mathrm{pb}$	$[33.1 - 2.17\cdot n_f]\,\mathrm{pb}$
$\sigma^{\mathrm{NNLO}}_{\mathrm{q,s}}$	$9.19(2)\,\mathrm{pb}$	$9.18\,\mathrm{pb}$
$\sigma^{\mathrm{NNLO}}_{\mathrm{g}}$	$-142.4(4)\,\mathrm{pb}$	$-142.7\,\mathrm{pb}$

[Asteriadis, Caola, Melnikov, Röntsch ‘19]

- In general, we find that we can get permill precision on the NNLO total cross section, corresponding to a few percent precision on the NNLO coefficient, running for a few hours on an 8-core machine.

Conclusion

- We applied the nested soft-collinear subtraction scheme to DIS.
- The double real emission contribution is regulated ***locally*** and finite in any point of the phase space.
- The poles are extracted ***analytically*** and the cancellation of the ε – poles between different contributions was checked explicitly.
- We obtained a 4 - dimensional formula for the fully-differential DIS cross section that can be used to calculate arbitrary infra-red safe observables numerically.
- The formulas are checked numerically against analytic results. We found that the total NNLO cross section can be calculated in only a few CPU hours to permille precission.
- The analytic formulas can be used as ***building blocks*** to design subtractions for more complex LHC processes.

Backup slides

Single-soft subtraction terms

- We consider the double-soft regulated single-soft subtraction term:

Double-soft regulated single-soft

$$\langle (1-\mathbb{S})F_{\rm LM}(1,4,5,6)\rangle = \langle (1-S_6)(1-\mathbb{S})F_{\rm LM}(1,4,5,6)\rangle + \langle S_6(1-\mathbb{S})F_{\rm LM}(1,4,5,6)\rangle$$

NLO kinematics, all singularities regulated, needs to be calculated numerically

$$\Big\langle [1-\mathbb{S}]S_6 F_{LM}(1,4,5,6)\Big\rangle = \Big\langle \hat{O}_{NLO} J_{145} F_{LM}(1,4,5)\Big\rangle$$

$$-\frac{[\alpha_s]^2 C_F}{\epsilon^3}\Big\langle \left[2C_F\left(\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right)\eta_{14}^{-\epsilon}K_{14} + C_A\left(\frac{\Gamma^4(1-\epsilon)\Gamma(1+\epsilon)}{2\Gamma(1-3\epsilon)}\right)\right]$$

$$\times\left[\left(\frac{1}{2\epsilon}\left((2E_{\rm max})^{-4\epsilon}-(2E_4)^{-4\epsilon}\right)-Z^{2,4}(2E_4)^{-4\epsilon}\right)F_{LM}(1,4)\right.$$

$$\left.+\frac{1}{2\epsilon}(2E_{\rm max})^{-4\epsilon}F_{LM}(1,4)+(2E_1)^{-4\epsilon}\int {\rm d}z(1-z)^{-4\epsilon}\bar{P}_{qq}(z)\frac{F_{LM}(z\cdot 1,4)}{z}\right]\Big\rangle .$$

Born kinematics, contain poles explicitly

with

$$K_{ij} = \left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\eta_{ij}^{1+\epsilon}\,{}_2F_1(1,1,1-\epsilon,1-\eta_{ij}) = 1+\left[{\rm Li}_2(1-\eta_{ij})-\frac{\pi^2}{6}\right]\epsilon^2+\mathcal{O}(\epsilon^3)$$

$$J_{145} = \frac{[\alpha_s]}{\epsilon^2}\left[(2C_F-C_A)\eta_{14}^{-\epsilon}K_{14}+C_A\left[\eta_{15}^{-\epsilon}K_{15}+\eta_{45}^{-\epsilon}K_{45}\right]\right](2E_5)^{-2\epsilon}$$

- Explicit $E_{\rm max}$ dependence cancels against implicit dependence in the fully regulated double real contribution. This provides a powerful check on the implementation.

Phase space parametrization

- We parametrize the directions of gluons 5 and 6 as

$$n_5^\mu = t^\mu + \cos\theta_5 \epsilon_3^\mu + \sin\theta_5 b^\mu \,,$$
$$n_6^\mu = t^\mu + \cos\theta_6 \epsilon_3^\mu + \sin\theta_6 (\cos\varphi_6 b^\mu + \sin\varphi_6 a^\mu) \,,$$

and write the angular phase space as

$$\mathrm{d}\Omega_5 \mathrm{d}\Omega_6 = \mathrm{d}\Omega_{56} = \frac{\mathrm{d}\Omega_b^{(d-2)} \mathrm{d}\Omega_a^{(d-3)}}{2^{6\epsilon}(2\pi)^{2d-2}} [\eta_5(1-\eta_5)]^{-\epsilon} [\eta_6(1-\eta_6)]^{-\epsilon} \frac{|\eta_5-\eta_6|^{1-2\epsilon}}{D^{1-2\epsilon}} \frac{\mathrm{d}\eta_5 \mathrm{d}\eta_6 \mathrm{d}\lambda}{[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}$$

where

$$D = \eta_5\eta_6 - 2\eta_5\eta_6 + 2(2-1)\sqrt{\eta_5\eta_6(1-\eta_5)(1-\eta_6)}$$

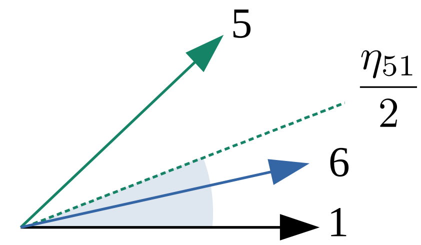
and

$$\eta_{56} = \frac{|\eta_5-\eta_6|^2}{D} \qquad \sin^2\varphi_{56} = 4\lambda(1-\lambda)\frac{|\eta_5-\eta_6|^2}{D^2}$$

- In the different sectors we perform the substitutions

$$\begin{array}{lll} \text{(a)} & \eta_5 = x_3 & \eta_6 = \frac{x_3 x_4}{2} \\ \text{(b)} & \eta_5 = x_3 & \eta_6 = x_3\left(1-\frac{x_4}{2}\right) \\ \text{(c)} & \eta_5 = \frac{x_3 x_4}{2} & \eta_6 = x_3 \\ \text{(d)} & \eta_5 = x_3\left(1-\frac{x_4}{2}\right) & \eta_6 = x_3 \end{array}$$

Phase space parametrization



- For instance in sector (a) $\eta_5 = x_3$ $\eta_6 = \frac{x_3 x_4}{2}$ we then obtain

$$\mathrm{d}\Omega_{56}^{(a)} = \left[\frac{1}{8\pi^2}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\right]\left[\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\right]\frac{\mathrm{d}\Omega_b^{(d-2)}}{\Omega^{d-2}}\frac{\mathrm{d}\Omega_a^{(d-3)}}{\Omega^{d-3}}\boxed{\frac{\mathrm{d}x_3}{x_3^{1+2\epsilon}}\frac{\mathrm{d}x_4}{x_4^{1+2\epsilon}}}\frac{\mathrm{d}\lambda}{\pi[\lambda(1-\lambda)]^{\frac{1}{2}+\epsilon}}(256F_\epsilon)^{-\epsilon}4F_0x_3^2x_4$$

where

$$F_\epsilon = \frac{(1-x_3)(1-\frac{x_3x_4}{2})(1-\frac{x_4}{2})^2}{2N(x_3,x_4,\lambda)^2} \qquad F_0 = \frac{1-\frac{x_4}{2}}{2N(x_3,\frac{x_4}{2},\lambda)}$$

and

$$N(x_3,x_4,\lambda) = 1 + x_4(1-2x_3) - 2(1-2\lambda)\sqrt{x_4(1-x_3)(1-x_3x_4)}$$

- This parametrization accounts for the angular ordering of sector $\theta^{(a)} = \theta\left(\eta_{61} < \frac{\eta_{51}}{2}\right)$ by construction.
- The double (6||1) and triple (5||6||1) collinear singularities in this sector are $x_4 = 0$ and $x_3 = 0$; they are factored out explicitly.
- The same happened for sectors $\theta^{(b)}$ to $\theta^{(d)}$.
- For a simpler analytic integration we define the single collinear limits to also act on the phase space.

Double-soft limit

- The double soft limit of the amplitude is given by [Catani, Grazzini '99]

$$\lim_{E_5,E_6\to 0} |M^{\rm tree}(\{p\},p_5,p_6)|^2 = g_{s,b}^2 \times {\rm Eikonal}(1,4,5,6) \times |M^{\rm tree}(\{p\})|^2\,,$$

where

$$\begin{aligned}
{\rm Eikonal}(1,4,6,7) &= 4C_F^2 S_{14}(6)S_{14}(7) + C_A C_F\left[2S_{12}(6,7) - S_{11}(6,7) - S_{22}(6,7)\right]\,,\\
S_{ij}(k) &= \frac{p_i\cdot p_j}{[p_i\cdot p_k][p_j\cdot p_k]}\,,\\
S_{ij}(k,l) &= S_{ij}^{\rm so}(k,l) - \frac{2[p_i\cdot p_j]}{[p_k\cdot p_l][p_i\cdot(p_k+p_l)][p_j\cdot(p_k+p_l)]}\\
&\quad + \frac{[p_i\cdot p_k][p_j\cdot p_l]+[p_i\cdot p_l][p_j\cdot p_k]}{[p_i\cdot(p_k+p_l)][p_j\cdot(p_k+p_l)]}\left(\frac{1-\epsilon}{[p_k\cdot p_l]^2} - \frac{1}{2}S_{ij}^{\rm so}(k,l)\right)\,,\\
S_{ij}^{\rm so}(k,l) &= \frac{p_i\cdot p_j}{p_k\cdot p_l}\left(\frac{1}{[p_i\cdot p_k][p_j\cdot p_l]} + \frac{1}{[p_i\cdot p_l][p_j\cdot p_k]}\right) - \frac{[p_i\cdot p_j]^2}{[p_i\cdot p_k][p_j\cdot p_k][p_i\cdot p_l][p_j\cdot p_l]}\,.
\end{aligned}$$

- The limit is known independent of the hard matrix element.

Collinear subtraction terms

- Consider a double-collinear limit of the amplitude

$$|M(\{p\},p_5,p_6)|^2 \underset{p_6||p_1}{\approx} -g_{s,b}^2 \times \frac{1}{E_1 E_6} P_{qq}\left(\frac{E_1}{E_1-E_6}\right) \times \frac{1}{\rho_{16}} \times \left| M\left(\left\{\frac{E_1-E_6}{E_1}\cdot p_1,\dots\right\},p_5\right)\right|^2 ,$$

where

$$P_{qq}(z) = C_F\left[\frac{1+z^2}{1-z} - \epsilon(1-z)\right] .$$

- To calculate the corresponding subtraction term analytically it is crucial that the d - dimensional phase space is parametrized in such a way that the collinear singularity factorizes

$$\int [\mathrm{d}g_6]\ |M(\{p\},p_5,p_6)|^2 \sim \int \mathrm{d}\rho_{16}\ \rho_{61}^{-\epsilon} \times \frac{1}{\rho_{16}} \sim -\frac{2^{-\epsilon}}{\epsilon} .$$

- Using the phase space parametrization from [Czakon ‘10, Phys.Lett. B693 (2010) 259-268] the singularities in all sectors are made explicit.

Contributions to the ε^{-3} - pole

- No contributions from $\mathrm{d}\sigma_{\mathrm{pdf}} = \mathcal{O}(\epsilon^{-2})$.
- $\mathrm{d}\sigma_{\mathrm{rr}}$ only has contributions from $\langle \mathbb{S} F_{\mathrm{LM}}(1,4,5,6)\rangle$ and $\langle S_6(I-\mathbb{S})F_{\mathrm{LM}}(1,4,5,6)\rangle$.
- $\left(\frac{\alpha_s}{2\pi}\right)^2 \times \frac{1}{\epsilon^3}$ neglected.
- Contributions proportional to the ***LO differential cross section***

$$
\begin{aligned}
\mathrm{d}\sigma_{\mathrm{rr}}:&\quad \left\langle\left(\frac{20}{12}C_AC_F + 3C_F^2 - C_AC_F\ln\left(\frac{s_{14}}{\mu^2}\right) - 4C_F^2\ln\left(\frac{s_{14}}{\mu^2}\right)\right)\times F_{\mathrm{LM}}(1,4)\right\rangle \\
\mathrm{d}\sigma_{\mathrm{rv}}:&\quad \left\langle\left(-\frac{9}{12}C_AC_F - 9C_F^2 + C_AC_F\ln\left(\frac{s_{14}}{\mu^2}\right) + 8C_F^2\ln\left(\frac{s_{14}}{\mu^2}\right)\right)\times F_{\mathrm{LM}}(1,4)\right\rangle \\
\mathrm{d}\sigma_{\mathrm{vv}}:&\quad \left\langle\left(-\frac{11}{12}C_AC_F + 6C_F^2 - 4C_F^2\ln\left(\frac{s_{14}}{\mu^2}\right)\right)\times F_{\mathrm{LM}}(1,4)\right\rangle
\end{aligned}
$$

- Contributions proportional to the ***boosted LO differential cross section***

$$
\begin{aligned}
\mathrm{d}\sigma_{\mathrm{rr}}:&\quad \left\langle\left(\frac{1}{2}C_AC_F(1+z) + 2C_F^2(1+z) - C_AC_F\left[\frac{1}{1-z}\right]_+ - 4C_F^2\left[\frac{1}{1-z}\right]_+\right)\times \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\right\rangle \\
\mathrm{d}\sigma_{\mathrm{rv}}:&\quad \left\langle\left(-\frac{1}{2}C_AC_F(1+z) - 2C_F^2(1+z) + C_AC_F\left[\frac{1}{1-z}\right]_+ + 4C_F^2\left[\frac{1}{1-z}\right]_+\right)\times \frac{F_{\mathrm{LM}}(z\cdot 1,4)}{z}\right\rangle
\end{aligned}
$$