Numerical methods and beam functions at NNLO and beyond

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Outline

1. At hadron colliders, there are many interesting final states to observe. Among these, the production of colour-neutral final states has gotten much attention.

2. There are many methods to compute cross sections at NLO and NNLO. However, at N^3LO the implementation of these methods is involved.

3. I shall present the methods to compute the soft function for top pair production at NNLO which have already been computed and report on progress that has been made to apply them to the computation of beam functions at N^3LO.

4. This is the last ingredient to implement the q_T slicing method at N^3LO.

The q_T slicing method

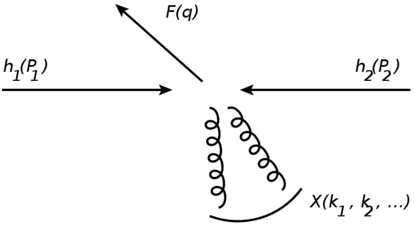
[Catani, Grazzini '07, '15]

$$p + p \to F\,(q_T) + X$$

$$\sigma^{F}_{\mathrm{N}^m\mathrm{LO}} = \int_0^{q_{T,\mathrm{cut}}} dq_T \, \frac{d\sigma^{F}_{\mathrm{N}^m\mathrm{LO}}}{dq_T} + \int_{q_{T,\mathrm{cut}}}^{\infty} dq_T \, \frac{d\sigma^{F}_{\mathrm{N}^m\mathrm{LO}}}{dq_T}$$

$$= \underbrace{\int_0^{q_{T,\mathrm{cut}}} dq_T \, \frac{d\sigma^{F}_{\mathrm{N}^m\mathrm{LO}}}{dq_T}}_{\text{enough to know in small-}q_T\text{ approximation}} + \underbrace{\int_{q_{T,\mathrm{cut}}}^{\infty} dq_T \, \frac{d\sigma^{F+\mathrm{jet}}_{\mathrm{N}^{m-1}\mathrm{LO}}}{dq_T}}_{\text{known}}$$

Factorization



where $F = H, Z, W, ZZ, WW, t\bar{t}, \ldots$
If not colourless the final state must be at least massive

$q^2 \sim q_T^2 \gg \Lambda_{\mathrm{QCD}}$ collinear factorization

$$\frac{d\sigma_F}{d\Phi} = \phi_1 \otimes \phi_2 \otimes C + \mathcal{O}\left(\frac{1}{q^2}\right)$$

$q^2 \gg q_T^2 > \Lambda_{\mathrm{QCD}}$ small-q_T factorization

$$\frac{d\sigma_F}{d\Phi} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{H} \otimes \mathcal{S} + \mathcal{O}\left(\frac{q_T^2}{q^2}\right)$$

All those functions

To get the cross section at N^{m}LO, we need to know all those functions at N^{m}LO

$$\frac{d\sigma_F^{\mathrm{N}^m\mathrm{LO}}}{d\Phi} = \mathcal{B}_1^{\mathrm{N}^m\mathrm{LO}} \otimes \mathcal{B}_2^{\mathrm{N}^m\mathrm{LO}} \otimes \mathcal{H}^{\mathrm{N}^m\mathrm{LO}} \otimes \mathcal{S}^{\mathrm{N}^m\mathrm{LO}}$$

$\mathcal{B}$	-	beam function	-	radiation collinear to the beam, process-independent, known up to NNLO
$\mathcal{H}$	-	hard function	-	virtual corrections, process-dependent
$\mathcal{S}$	-	soft function	-	soft, real radiation, process-dependent

Today, I will focus on $\mathcal{S}$ and $\mathcal{B}$.

Renormalization

separately divergent

finite

$$\frac{d\sigma_F}{d\Phi} = \mathcal{B}_1^{(\text{bare})} \otimes \mathcal{B}_2^{(\text{bare})} \otimes \text{Tr}\left[\mathcal{H}^{(\text{bare})} \otimes \mathcal{S}^{(\text{bare})}\right]$$

$$= Z_B \mathcal{B}_1^{(\text{bare})} \otimes Z_B \mathcal{B}_2^{(\text{bare})} \otimes \text{Tr}\left[\boldsymbol{Z}_H^\dagger \mathcal{H}^{(\text{bare})} \boldsymbol{Z}_H \otimes \boldsymbol{Z}_S^\dagger \mathcal{S}^{(\text{bare})} \boldsymbol{Z}_S\right]$$

$$= \mathcal{B}_1(\mu) \otimes \mathcal{B}_2(\mu) \otimes \text{Tr}\left[\mathcal{H}(\mu) \otimes \mathcal{S}(\mu)\right]$$

separately finite

$$\frac{d}{d\mu}\frac{d\sigma_F}{d\Phi} = 0 \quad \rightarrow \quad \text{Renormalization Group Equations for } \mathcal{B}, \mathcal{H} \text{ and } \mathcal{S}$$

Soft Collinear Effective Theory (SCET)

$$\text{SCET} \simeq \text{QCD}\Big|_{\text{IR limit}}$$

Hard degrees of freedom are integrated out into Wilson coefficients, which are then used to adjust new couplings of the (effective) theory.

QCD fields written as sums of collinear, anti-collinear and soft components:

$$\phi(x) = \phi_c(x) + \phi_{\bar{c}}(x) + \phi_s(x)$$

The new fields decouple in the Lagrangian

$$\mathcal{L}_{\text{SCET}} = \mathcal{L}_c + \mathcal{L}_{\bar{c}} + \mathcal{L}_s$$

The separation of fields in the Lagrangian into collinear, anti-collinear and soft sectors, facilitates proofs of factorization theorems

Small-q_T factorization in SCET

Gluons' momenta in light-cone coordinates

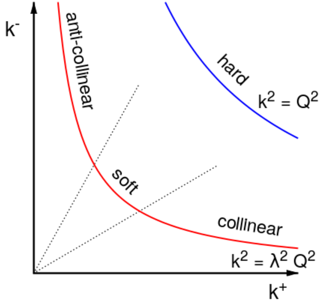
$$k_i^\mu = \left(k_i^+, k_i^-, \boldsymbol{k}_i^\perp\right) \qquad \text{where} \qquad k^\pm = k^0 \pm k^3$$

Expansion parameter

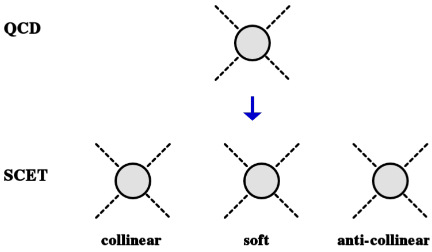
$$\lambda = \sqrt{\frac{q_T^2}{q^2}} \ll 1$$

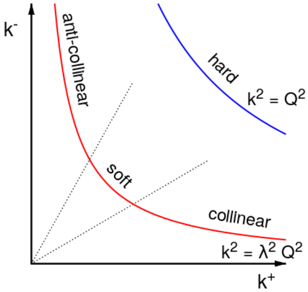
Phase space regions

collinear	$k_i^\mu \sim (1, \lambda^2, \lambda)\, Q^2$	$\mathcal{B}_1$
anti-collinear	$k_i^\mu \sim (\lambda^2, 1, \lambda)\, Q^2$	$\mathcal{B}_2$
hard	$k_i^\mu \sim (1, 1, 1)\, Q^2$	$\mathcal{H}$
soft	$k_i^\mu \sim (\lambda, \lambda, \lambda)\, Q^2$	$\mathcal{S}$



Rapidity divergences and analytic regulator





Modification of the measure [Becher, Bell '12]

$$\int d^d k\, \delta^+(k^2) \to \int d^d k \left(\frac{\nu}{k_+}\right)^{\alpha} \delta^+(k^2)$$

The regulator is necessary at intermediate steps of the calculation.

Rapidity divergences do not appear in QCD, hence, the complete SCET result has to stay finite in the limit $\alpha \to 0$.

NNLO soft function for top pair production

[Angeles-Martinez, Czakon, Sapeta, '18]

[Catani, Devoto, Grazzini, Kallweit, Mazzitelli, '19]

Motivation

1. This calculation has been carried out in the small q_T-factorization framework and we are using it as a basis to carry out computations for beam functions.

2. In general, the IR divergences appearing are overlapping. To seperate them, sector decomposition is employed.

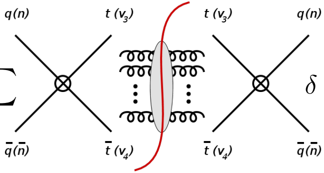
3. After this decomposition into sectors, the poles can be made explicit in terms of a Laurent series in ϵ where the coefficients can be computed fully numerically.

Soft function

Represents corrections coming from exchanges of real, soft gluons, whose transverse momenta sum up to a fixed value q_T

$$\boxed{\beta_t = \sqrt{1 - \frac{4\, m_t^2}{M^2}}}$$

$S_{\mathrm{bare}}(q_T, \beta_t, \theta) \propto \sum$



$\delta\left(q_T - |\sum_i k_{i\perp}|\right) \prod_i \delta^+(k_i^2)$

External momenta $\rightarrow$ Wilson Lines along n, $\bar{n}$, v_3, v_4 (Born kinematics)

$$\boldsymbol{S}_{i\bar{i}} = \sum_{n=0}^{\infty} \boldsymbol{S}_{i\bar{i}}^{(n)} \left(\frac{\alpha_s}{4\pi}\right)^n \qquad \boldsymbol{S}_{i\bar{i}}^{(n)} = \sum_{\{j\}} \boldsymbol{w}_{\{j\}}^{i\bar{i}} I_{\{j\}}$$

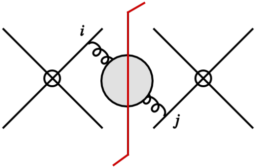
colour matrices

phase space integrals

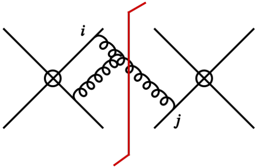
Soft function at NNLO

Three distinct groups of diagrams:

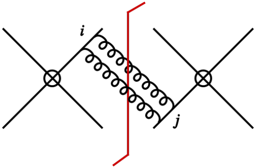
Bubble

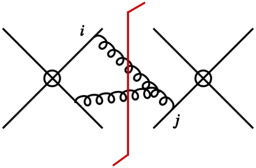


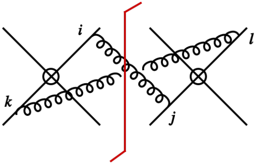
Single-cut



Double-cut







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Soft function at NNLO

Three distinct groups of diagrams:

Bubble

Single-cut

DIFFERENTIAL EQUATIONS

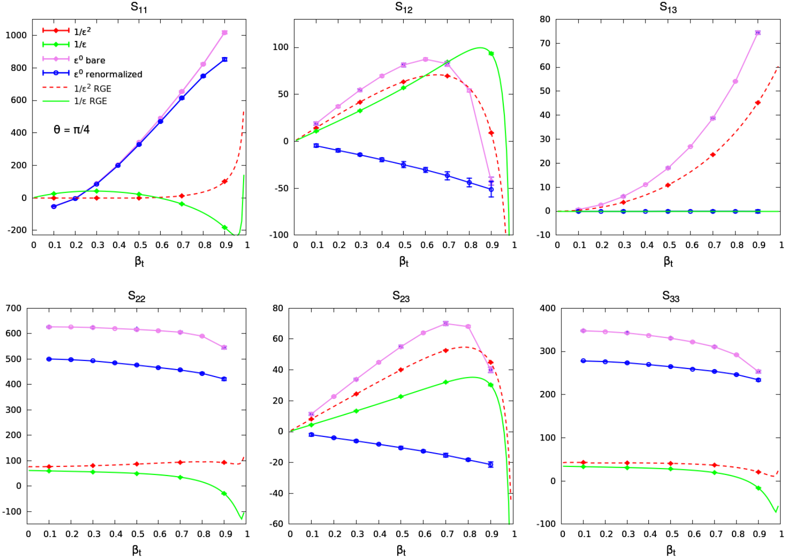
DIRECT INTEGRATION

Double-cut

SECTOR DECOMPOSITION

Numerical check (gg channel)

Coefficient of the soft function matrix as a function of velocity and at scattering angle $\frac{\pi}{4}$

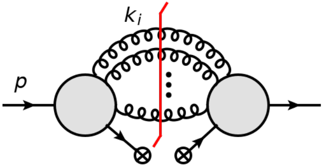


N^3LO beam function

(work in progress)

The beam function

Represents corrections coming from emissions of real, collinear gluons, whose transverse momenta sum up to a fixed value q_T and whose longitudinal component along p sums up to $1-z$

$B_{\text{bare}}(q_T, z) \propto \sum$ 

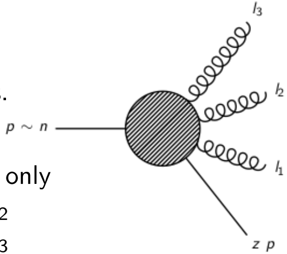
$$\times\, \delta\left(q_T - \left|\sum_i k_{i\perp}\right|\right) \prod_i \delta^+(k_i^2)\, \delta\left(\bar{n}\cdot \sum k_i - (1-z)\,\bar{n}\cdot p\right)$$

$$p = \frac{\bar{n}\cdot p}{2} n = \frac{p_-}{2} n$$

$$n^2 = \bar{n}^2 = 0$$

$$n\cdot\bar{n} = 2$$

N^3LO propagators



Possible denominators that may cause divergencies.

light-cone

$n \cdot l_1$
$n \cdot l_2$
$n \cdot l_3$
$\bar{n} \cdot l_1$
$\bar{n} \cdot l_2$
$\bar{n} \cdot l_3$
$n \cdot l_1 + n \cdot l_2$
$n \cdot l_1 + n \cdot l_3$
$n \cdot l_2 + n \cdot l_3$
$\bar{n} \cdot l_1 + \bar{n} \cdot l_2$
$\bar{n} \cdot l_1 + \bar{n} \cdot l_3$
$\bar{n} \cdot l_2 + \bar{n} \cdot l_3$

internal only

$l_1 \cdot l_2$
$l_1 \cdot l_3$
$l_2 \cdot l_3$
$l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3$

internal+external

$p_-\, n \cdot l_1$
$p_-\, n \cdot l_2$
$p_-\, n \cdot l_3$
$l_1 \cdot l_2 - p_-\, n \cdot l_1 - p_-\, n \cdot l_2$
$l_1 \cdot l_3 - p_-\, n \cdot l_1 - p_-\, n \cdot l_3$
$l_2 \cdot l_3 - p_-\, n \cdot l_2 - p_-\, n \cdot l_3$

The way to go

The beam function

$$B_{\text{bare}}(z, q_T) = \sum_i \mathcal{I}_i \,,$$

can be calculated if each integral is represented as

$$\mathcal{I}_i = \sum_{j \,\in\, \text{sectors}} \int_0^1 \frac{dx_1}{x_1^{1+a_1\epsilon}} \frac{dx_2}{x_2^{1+a_2\epsilon}} \frac{dx_3}{x_3^{1+a_3\epsilon}} \frac{dx_4}{x_4^{1+a_4\epsilon}} \, dx_5 \cdots dx_9 \, \mathcal{W}_j(x_1, x_2, \ldots, x_9) \,.$$

$\mathcal{W}_j(x_1, x_2, \ldots, x_9)$ has to be finite if $x_1, \ldots, x_4 \to 0$.

Then we can use

$$\frac{1}{x_i^{1+a_i\epsilon}} = -\frac{1}{a_i\epsilon}\delta(x_i) + \sum_{n=0}^{\infty} \frac{(-a_i\epsilon)^n}{n!} \left[\frac{\log^n(x_i)}{x_i}\right]_+ \,.$$

N^3LO propagators

The first problem: It is impossible to parameterize the momenta such that all scalar products look simple simultaneously.

Example

$$n = [1,0,0,0,1] \qquad \bar{n} = [1,0,0,0,-1] \qquad l_1 = \left[\frac{l_{1-}^2 + l_{1T}^2}{2\,l_{1-}}, 0, 0, 0, \frac{l_{1-}^2 - l_{1T}^2}{2\,l_{1-}}\right]$$

$$l_3 = \left[\frac{l_{3-}^2 + l_{3T}^2}{2\,l_{3-}},\ 0,\ l_{3T}\sin\chi_1,\ l_{3T}\cos\chi_1,\ \frac{l_{3-}^2 - l_{3T}^2}{2\,l_{3-}}\right]$$

$$l_2 = \left[\frac{l_{2-}^2 + l_{2+}^2}{2\,l_{2-}^2},\ l_{2T}\sin\phi_1\sin\phi_2,\ l_{2T}\cos\phi_2\sin\phi_1,\ l_{2T}\cos\phi_1,\ \frac{l_{2-}^2 - l_{2+}^2}{2\,l_{2-}}\right]$$

$$\bar{n}\cdot l_1 = l_{1-} \quad \bar{n}\cdot l_2 = l_{2-} \quad \bar{n}\cdot l_3 = l_{3-}$$

$$l_1\cdot l_2 = \frac{l_{1T}^2 l_{2-}}{2\,l_{1-}} + \frac{l_{2T}^2 l_{1-}}{2\,l_{2-}} - l_{1T}l_{2T}\cos\phi_1 \quad \Rightarrow \quad \phi_1 = 0 \ \& \ \frac{l_{1T}}{l_{1-}} = \frac{l_{2T}}{l_{2-}}$$

$$l_2\cdot l_3 = \frac{l_{2T}^2 l_{3-}}{2\,l_{2-}} + \frac{l_{3T}^2 l_{2-}}{2\,l_{3-}} - l_{2T}l_{3T}\cos\chi_1\cos\phi_1 - l_{2T}l_{3T}\cos\phi_2\sin\chi_1\sin\phi_1$$

Step 1: selector functions

7 triple collinear
$(l_1 \cdot l_2)\,(n \cdot l_1)\,(n \cdot l_2)$
$(l_1 \cdot l_3)\,(n \cdot l_1)\,(n \cdot l_3)$
$(l_2 \cdot l_3)\,(n \cdot l_2)\,(n \cdot l_3)$
$(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)$
$(l_1 \cdot l_3)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_3)$
$(l_2 \cdot l_3)\,(\bar{n} \cdot l_2)\,(\bar{n} \cdot l_3)$
$(l_1 \cdot l_2)\,(l_1 \cdot l_3)\,(l_2 \cdot l_3)$

12 double collinear	
$(n \cdot l_1)\,(\bar{n} \cdot l_2)$	$(l_1 \cdot l_3)\,(n \cdot l_2)$
$(n \cdot l_1)\,(\bar{n} \cdot l_3)$	$(l_2 \cdot l_3)\,(n \cdot l_1)$
$(n \cdot l_2)\,(\bar{n} \cdot l_3)$	$(l_1 \cdot l_2)\,(n \cdot l_3)$
$(\bar{n} \cdot l_1)\,(n \cdot l_2)$	$(l_1 \cdot l_3)\,(\bar{n} \cdot l_2)$
$(\bar{n} \cdot l_1)\,(n \cdot l_3)$	$(l_2 \cdot l_3)\,(\bar{n} \cdot l_1)$
$(\bar{n} \cdot l_2)\,(n \cdot l_3)$	$(l_1 \cdot l_2)\,(\bar{n} \cdot l_3)$

$$d_{1,2;1} = (l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)\,,$$

$$S_{1,2;2} = \frac{1}{d_{1,2;1}\mathcal{D}}\,,$$

$$\mathcal{D} = \sum_{i,j,k} \frac{1}{d_{i,j;k}} + \sum_{i,j,k,l} \frac{1}{d_{i,j;k,l}}\,,$$

$$S_{1,2;2} = \frac{1}{1 + \frac{(l_1 \cdot l_2)\,(\bar{n} \cdot l_2)}{(l_1 \cdot l_3)\,(\bar{n} \cdot l_3)} + \frac{(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)}{(l_1 \cdot l_3)} + \cdots}\,,$$

Step 2: sector decomposition

Let's focus on the sector $(l_1 \cdot l_2)(\bar{n} \cdot l_1)(\bar{n} \cdot l_2)$. All other singularities are suppressed by the corresponding selector functions.

In this sector, divergencies can be generated by the following propagators:

$$
\begin{aligned}
\bar{n} \cdot l_1 &\longrightarrow l_{1-} \\
\bar{n} \cdot l_2 &\longrightarrow l_{2-} \\
n \cdot l_1 & \\
n \cdot l_2 & \\
l_1 \cdot l_2 &\longrightarrow \boxed{\frac{l_{1T}^2 l_{2-}}{2 l_{1-}} + \frac{l_{2T}^2 l_{1-}}{2 l_{2-}} - l_{1T} l_{2T} \cos\phi_1} \\
n \cdot l_1 + n \cdot l_2 & \\
\bar{n} \cdot l_1 + \bar{n} \cdot l_2 &\longrightarrow l_{1-} + l_{2-} \\
l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3 &
\end{aligned}
$$

Step 2: sector decomposition

The nonlinear transformation

$$\zeta = \frac{1}{2} \frac{(l_{1T} l_{2-} - l_{1-} l_{2T})^2 (1 + \cos\phi_1)}{l_{1T}^2 l_{2-}^2 + l_{1-}^2 l_{2T}^2 - 2 l_{1-} l_{2-} l_{1T} l_{2T} \cos\phi_1}$$

turns

$$l_1 \cdot l_2 = \frac{l_{1T}^2 l_{2-}}{2 l_{1-}} + \frac{l_{2T}^2 l_{1-}}{2 l_{2-}} - l_{1T} l_{2T} \cos\phi_1$$

into

$$l_1 \cdot l_2 = \frac{(l_{1T}^2 l_{2-}^2 - l_{1-}^2 l_{2T}^2)^2}{2\, l_{1-} l_{2-} (l_{1T}^2 l_{2-}^2 + l_{1-}^2 l_{2T}^2 - 2\, l_{1-} l_{2-} l_{1T} l_{2T} (1 - 2\zeta))}$$

Step 2: sector decomposition

$$l_{1-} = l_{1T}\, l_{1-}$$
$$l_{2-} = l_{2T}\, l_{2-}$$

$\boxed{l_{2-} \to l_{2-} l_{1-}}$ $l_{1-} > l_{2-}$

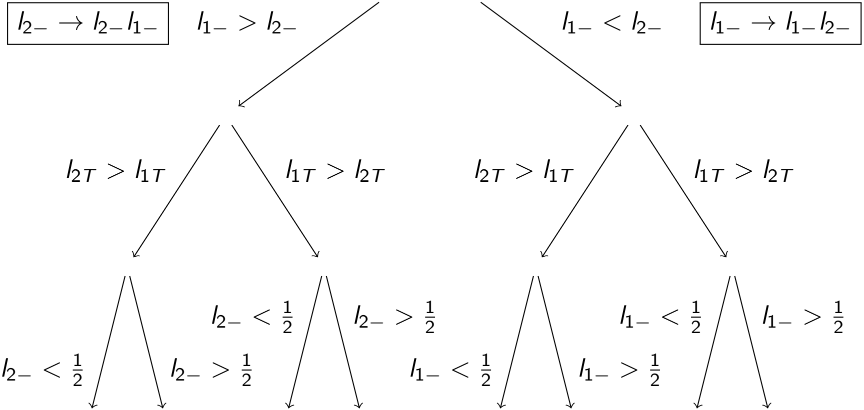
$l_{1-} < l_{2-}$ $\boxed{l_{1-} \to l_{1-} l_{2-}}$

$$l_{1-} + l_{2-} \to l_{1-}(1 + l_{2-})$$

$$l_{1-} + l_{2-} \to l_{2-}(1 + l_{1-})$$

Step 2: sector decomposition

$$
\begin{aligned}
l_{1-} &= l_{1T}\, l_{1-} \\
l_{2-} &= l_{2T}\, l_{2-}
\end{aligned}
$$

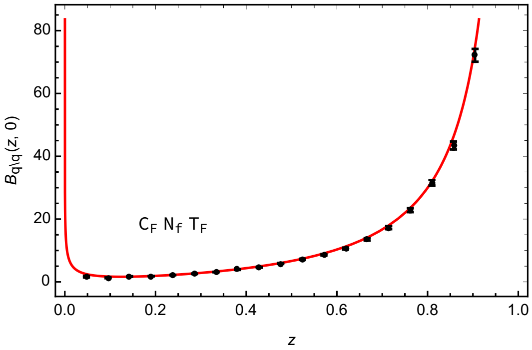


- This algorithm factorizes all overlapping singularities

NNLO beam function

Known analytically [Gehrmann, Lübbert, Yang '12, '14].

We checked that our method reproduces that result



Status

The integrals take now the desired form

$$\mathcal{I}_i = \sum_{j \in \text{sectors}} \int_0^1 \frac{dx_1}{x_1^{1+a_1\epsilon}} \frac{dx_2}{x_2^{1+a_2\epsilon}} \frac{dx_3}{x_3^{1+a_3\epsilon}} \frac{dx_4}{x_4^{1+a_4\epsilon}} \, dx_5 \cdots dx_9 \, \mathcal{W}_j(x_1, x_2, \ldots, x_9)$$

We checked that, for the case of the $q \to q\bar{q}qg$ contribution to the beam function, the weights $\mathcal{W}_j$ are finite in the limit of $x_i \to 0$, as required

We are now ready to evaluate the integrals