The quark beam function for zero-jettiness at next-to-next-to-next-to-leading order

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Motivation

- N^{3}LO colour singlet production is interesting
  - $pp \to$ leptons: PDFs, W mass, mixing angle
  - $pp \to H$: Higgs couplings
  - $pp \to VH$: Higgs couplings, Higgs decays to invisibles
  - $pp \to VV'$: Higgs width, anomalous couplings, searches for new resonances
- Differential calculations are very challenging
- Use slicing to separate calculation into NNLO $pp \to V + j$ and genuine N^{3}LO $pp \to V$ in soft or collinear approximations
- Zero-jettiness is a useful slicing variable

$$\mathcal{T} = \sum_j \min_{i \in \{1,2\}} \left[\frac{2p_i \cdot k_j}{Q_i} \right]$$

- Factorisation theorem for $\mathcal{T} \to 0$ [Stewart, Tackmann, Waalewijn '09]

$$\lim_{\mathcal{T} \to 0} \mathrm{d}\sigma^{\mathrm{N^3LO}}_{pp \to V+X} \sim B \otimes B \otimes S \otimes \mathrm{d}\sigma^{\mathrm{N^3LO}}_{pp \to V}$$

Beam functions

- Beam functions are a non-perturbative quantities ($f_k(z, \mu^2)$: PDFs)

$$B_i(t, z, \mu) = \sum_k \mathcal{I}_{ik}(t, z, \mu) \underset{z}{\otimes} f_k(z, \mu^2)$$

- Matching coefficient $\mathcal{I}_{ik}(t, z, \mu)$ can be calculated perturbatively
- Define partonic beam function with external parton states $|j\rangle$ (instead of proton states $|P\rangle$)

$$B_{ij}(t, z, \mu) = \sum_k \mathcal{I}_{ik}(t, z, \mu) \underset{z}{\otimes} f_{kj}(z, \mu^2)$$

$\rightarrow$ parton i: enters hard process, j: external parton

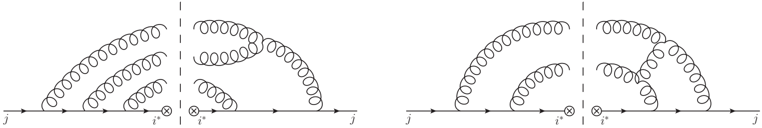
What to calculate?

- Observation from [Ritzmann, Waalewijn '14]: Partonic beam functions can be calculated from collinear limits of QCD amplitudes
- Integrate over constrained phase space

$$B_{qq}^{\text{bare}} \sim \int \prod_{i=1}^{n_R} \frac{\mathrm{d}^d k_i}{(2\pi)^{d-1}\, 2k_i^0} \delta_+(k_i^2) \delta\left(2p \cdot k_{n_R} - \frac{t}{z}\right) \times \delta\left(\frac{2\bar{p} \cdot k_{n_R}}{s} - (1-z)\right) \frac{\hat{C}_p |\mathcal{M}(p, \bar{p}, \{k_i\})|^2}{|\mathcal{M}_0(zp, \bar{p})|^2}$$

- Here, we consider the matching coefficient $\mathcal{I}_{qq}$ at $\mathrm{N}^3\mathrm{LO}$
  - Requires B_{qq} at $\mathrm{N}^3\mathrm{LO}$
  - Requires B_{qq} and B_{qg} at NNLO to $\mathcal{O}(\varepsilon^2)$ and at NLO to $\mathcal{O}(\varepsilon^4)$ $\rightarrow$ see Daniel Baranowski's talk

RRV and RRR

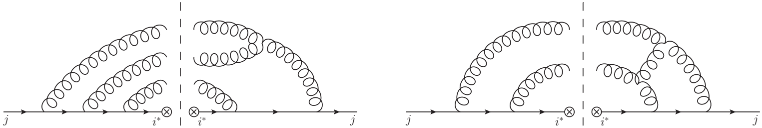


- Generate integrands à la [Catani, Grazzini '99]:
  - Draw diagrams for $j \to i^* +$ emissions (i^* off-shell, would enter the hard process)
  - Replace hard process by suitable projector
  - Use axial gauge so that only emissions from one leg are collinearly enhanced

$$\sum_{\text{pol}} \epsilon_i^\mu(k_i)(\epsilon_i^\nu(k_i))^* = -g^{\mu\nu} + \frac{k_i^\mu \bar{p}^\nu + k_i^\nu \bar{p}^\mu}{k_i \cdot \bar{p}}$$

- Integrate over constrained phase space

RRV and RRR



- Map phase space integrals to loop integrals via reverse unitarity, e.g., [Anastasiou, Melnikov '02]

$$2\pi i\delta(k_i^2) \to \frac{1}{k_i^2 + i0} - \frac{1}{k_i^2 - i0}$$

- Linearly dependent propagators require partial fractioning to map to integral families, e.g. for RRV due to $2k_{12} \cdot \bar{p} = s(1-z)$,

$$\frac{1}{(k_1 \cdot \bar{p})(k_2 \cdot \bar{p})} = \frac{2}{s(1-z)} \left[\frac{1}{k_1 \cdot \bar{p}} + \frac{1}{k_2 \cdot \bar{p}} \right]$$

- Use IBP reductions (`Reduze 2` [Manteuffel, Studerus '12]) to express amplitudes in terms of master integrals
  - RRR: 91 MI from 19 families
  - RRV: 128 MI from 19 families

Calculating the master integrals

- Integrals depend on 3 variables: $s = 2p \cdot \bar{p}$, t and z
- Two variables can be scaled out trivially
  $k_i = \tilde{k}_i \sqrt{t/z}$, $p = \tilde{p}\sqrt{t/z}$ and $\bar{p} = s\sqrt{z/t}$

$$I^{\text{top}}_{n_1,\dots,n_8}(s,t,z) = \left(\frac{1}{s}\right)^{1+n_7+n_8} \left(\frac{t}{z}\right)^{3-(n_1+\dots+n_6)-3\varepsilon} I^{\text{top}}_{n_1,\dots,n_8}(1,z,z)$$

- Derive differential equation system for z dependence
- Bring to canonical form (we use `Fuchsia` [Gituliar, Magerya '17])
- For RRR: Square roots like $\sqrt{z(4-z)}$ appear
  $\rightarrow$ rationalise by variable transformation: $z = \frac{(1+x)^2}{x}$
- Denominators appearing in the differential equations:

$$\text{RRV}: \quad \frac{1}{z+1}, \ \frac{1}{z}, \ \frac{1}{z-1}, \ \frac{1}{z-2}$$

$$\text{RRR}: \quad \frac{1}{x+1}, \ \frac{1}{x}, \ \frac{1}{x-1}, \ \frac{1}{x^2+x+1}, \ \frac{1}{x^2+3x+1}, \ \frac{1}{x^2+1}$$

- Solve in terms of GPLs

Calculating boundary conditions

- Fix integration constants in the limit $z \to 1$
- Determine singularity structure around the limit $z \to 1$ from the DEQs
- Work out expected singularity structure from integral representation $\to$ leads to linear relations between integration constants
- Calculate remaining boundary conditions by evaluating the integrals in the limit $z \to 1$
  - Simpler cases: Solve integrals directly $\to$ See Daniel Baranowski's talk for similar examples
  - More involved cases: Derive auxiliary DEQs in the $z \to 1$ limit

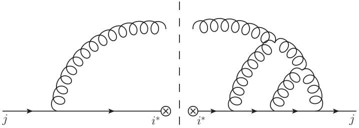
Peculiarities of the calculation

- Partial fractioning relations also exist between master integrals
- Even more relations exist (besides IBPs and partial fractioning), e.g.,

$$\int \frac{\mathrm{d}\Phi_{t,z}}{k_{12}^2(k_1-p)^2(k_{123}-p)^2(k_2\cdot\bar{p})(k_3-p)^2} + \frac{(1-z)z}{2}\int \frac{\mathrm{d}\Phi_{t,z}}{k_{12}^2(k_1-p)^2(k_{123}-p)^2(k_2\cdot\bar{p})(k_3\cdot\bar{p})} = 0$$

  - Can be proven from the differential equations
  - Systematisation?
- "Bad denominators" (mixing d and z in non-factorisable ways) occur in DEQs
  - Make canonicalisation difficult
  - Solution so far: Replace master integrals by others from same sector

RVV



- Challenging due to linear propagators from axial gauge
- Circumvent calculation by using available 2-loop splitting functions [Duhr, Gehrmann, Jaquier '14]
- Perform remaining phase space integrals

$$B^{\text{bare}}_{qq,\text{RVV}} = \int \mathrm{d}\Phi_{t,z} \left(\frac{\alpha_s}{2\pi}\right)^2 g_s^2 \frac{2}{s_{12}} \left(-\frac{s_{12}}{\mu^2}\right)^{-2\varepsilon} \frac{P^{(2)}_{qq}(z)}{z}$$

- Integrals are trivial for a single emission
- Important: Unexpanded ε dependence for singular behaviour ($t^{n\varepsilon}$, $(1-z)^{m\varepsilon}$) $\rightarrow$ luckily available in the literature
- One-loop squared calculated from RV^2 master integrals

Renormalisation

$$\mathcal{A}_{ij}(t,z)\mathcal{A}^*_{ij}(t,z)$$

$$\alpha_s = Z_{\alpha_s}\alpha_{s,0}$$

$$B^{\text{bare}}_{ij}(t,z) = Z_i(t,\mu) \underset{t}{\otimes} B_{ij}(t,z,\mu)$$

$$B_{ij}(t,z,\mu) = \sum_{k\in\{q,\bar{q},g\}} \mathcal{I}_{ik}(t,z,\mu) \underset{z}{\otimes} f_{kj}(z,\mu)$$

$$\mathcal{I}_{ij}(t,z,\mu)$$

Renormalisation

$$\mathcal{A}_{ij}(t,z)\mathcal{A}^*_{ij}(t,z)$$

$$\alpha_s = Z_{\alpha_s}\alpha_{s,0}$$

$$\gamma_i(t,\mu) = \gamma^i_B\delta(t) - 2\Gamma^i_{\text{cusp}}L_0\left(\frac{t}{\mu}\right)$$

$$\frac{\mathrm{d}Z_i(t,\mu)}{\mathrm{d}\ln\mu} = -Z_i(t,\mu)\underset{t}{\otimes}\gamma_i(t,\mu)$$

$$B^{\text{bare}}_{ij}(t,z) = Z_i(t,\mu)\underset{t}{\otimes}B_{ij}(t,z,\mu)$$

$$\frac{\mathrm{d}f_{ij}(z)}{\mathrm{d}\ln\mu^2} = \frac{\alpha_s}{2\pi}\sum_k P_{ik}(z)\underset{z}{\otimes}f_{kj}(z)$$

$$B_{ij}(t,z,\mu) = \sum_{k\in\{q,\bar{q},g\}}\mathcal{I}_{ik}(t,z,\mu)\underset{z}{\otimes}f_{kj}(z,\mu)$$

$$\mathcal{I}_{ij}(t,z,\mu)$$

Results: Matching coefficient $I_{qq}^{(3)}$

- Organise results according to their structure in $t \to 0$:

$$\mathcal{I}_{qq}^{(3)} = \sum_{k=0}^{5} L_k\left(\frac{t}{\mu^2}\right) F_{+}^{(3,k)}(z) + \delta(t) F_{\delta}^{(3)}(z)$$

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plus distributions: $L_k\left(\frac{t}{\mu^2}\right) = \frac{1}{\mu^2}\left[\frac{\mu^2}{t}\ln^k\left(\frac{t}{\mu^2}\right)\right]_+$

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$$F_\delta^{(3)}(z) = C_{-1}^{(3)}\delta(1-z) + \sum_{k=0}^{5} C_k^{(3)} D_k(z) + F_{\delta,h}^{(3)}(z)$$

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- Coefficients $F_+^{(3,k)}(z)$ and $C_k^{(3)}$ available in literature
  [Billis, Ebert, Michel, Tackmann '19]
  $\to$ allows for checks

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- New result: "Hard" contributions $F_{\delta,h}^{(3)}(z)$ in $N_c \sim N_f \gg 1$ limit
  [AB, Melnikov, Rietkerk, Tancredi, Wever '19]

$$F_{\delta,h}^{(3)}(z) = N_f^2 T_F^2 N_c F_1(z) + N_f T_F N_c^2 F_2(z) + N_c^3 F_3(z)$$

Results: Matching coefficient $I_{qq}^{(3)}$

- Orga…

$\mathcal{I}_{qq}^{(3)}$ …

- Isola…

$F_{\delta}^{(3)}$ …

- Coef…

[Billis, E…

$\rightarrow$ allows for checks

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$$F_{\delta,h}^{(3)}(z) = N_f^2 T_F^2 N_c F_1(z) + N_f T_F N_c^2 F_2(z) + N_c^3 F_3(z)$$

$$\begin{aligned}
F_3(z) = {} & \frac{1}{2916}(715565z - 197242) + \frac{35}{108}(698z - 69)H_1 + \frac{181}{27}(31z+1)H_{1,1} + \frac{1}{9}(1403z + 662)H_{1,1,1} \\
& + \frac{8}{3}(32z+23)H_{1,1,1,1} + 60(z+1)H_{1,1,1,1,1} + \frac{1}{1-z}\left[\frac{1}{648}\left(-217440z^2 + 191022z - 186085\right)H_0\right] \\
& + \frac{1}{1-z}\left[\frac{1}{162}\left(-52174z^2 + 38784z - 38101\right)H_2 + \frac{1}{162}\left(-32914z^2 + 29415z - 33625\right)H_{1,0}\right. \\
& \left. + \frac{1}{972}\left(50848z^2 - 34734z - 1747\right)\pi^2\right] + \frac{1}{1-z}\left[\frac{1}{18}\left(-4800z^2 + 1759z - 3599\right)H_3\right. \\
& + \frac{1}{18}\left(-3843z^2 + 2024z - 3645\right)H_{2,1} + \frac{1}{54}\left(-8357z^2 + 3903z - 8099\right)H_{2,0} \\
& + \frac{1}{9}\left(-1704z^2 + 795z - 1793\right)H_{1,2} - \frac{2}{9}\left(554z^2 - 277z + 541\right)H_{1,1,0} \\
& + \frac{1}{54}\left(-7033z^2 + 2574z - 7429\right)H_{1,0,0} - \frac{13}{27}\left(407z^2 - 96z + 185\right)H_{0,0,0} \\
& + \frac{1}{108}\left(4442z^2 - 2067z - 243\right)\pi^2 H_1 + \frac{1}{108}\left(6139z^2 - 2356z + 4431\right)\pi^2 H_0 \\
& \left. + \frac{1}{54}\left(15898z^2 - 5313z - 10099\right)\zeta_3\right] + \frac{1}{1-z}\left[\frac{1}{18}\left(-3653z^2 + 726z - 1559\right)H_4\right. \\
& + \frac{1}{3}\left(-572z^2 + 186z - 327\right)H_{3,1} + \frac{1}{9}\left(-1388z^2 + 477z - 656\right)H_{3,0}
\end{aligned}$$

Remarks about the results

Checks

- All poles cancel after renormalisation procedure
- Coefficients $F_{+}^{(3,k)}(z)$ and $C_k^{(3)}$ agree with available literature

Functions appearing in result

- Recall: Intermediate results contain GPLs with more general letters (containing also x)

$$\frac{1}{z-2}, \frac{1}{x^2+x+1}, \frac{1}{x^2+3x+1}, \frac{1}{x^2+1}$$

- Final results for $N_c \sim N_f \gg 1$ limit contain only HPLs of z
- We expect a similar cancellation also for whole matching coefficient

Outlook: Next steps

- Complete the subleading pieces of the matching coefficient $\mathcal{I}_{qq}^{(3)}$
  - Everything related to pure gluon emissions is complete
  - RVV is complete
  - Missing master integrals for $q \to q^* + gq\bar{q}$ and $q \to q^* + q\bar{q}$
    $\to$ surprisingly large number of new integrals
    (RRR: ~ 140, RRV: ~ 120)
    $\to$ in principle "just more of the same"
- Calculate the soft function
  $\to$ next slide
- Long term
  - Calculate remaining matching coefficients
  - Combine with NNLO $pp \to V + j$ calculation

Outlook: Soft function for zero-jettiness

- Calculation of soft function follows similar principle:
  Integrate soft limit of QCD amplitudes over constrained phase space

$$S_{\mathrm{RRR}} = \int \prod_{i=1}^{3} [\mathrm{d}^d k_i] \delta(\mathcal{T} - \tau) \mathrm{Eik}(\{k_i\}, p_1, p_2)$$

- However: More complicated phase space constraint
  $\rightarrow$ not only one direction is relevant

$$\mathcal{T} = \sum_j \min_{i \in \{1,2\}} \left[\frac{2p_i \cdot k_j}{Q_i} \right]$$

- Minimum introduces step function to distinguish cases
  for emissions closer to p_1 or p_2; e.g. at NLO (with $Q_i \rightarrow 1$):

$$\delta(\mathcal{T} - \tau) = \theta(2\bar{p} \cdot k - 2p \cdot k)\delta(2p \cdot k - \tau) + \theta(2p \cdot k - 2\bar{p} \cdot k)\delta(2\bar{p} \cdot k - \tau)$$

- Interesting challenge: Devise methods to deal with step functions
  as in reverse unitarity $\rightarrow$ translate to propagator-like structures

Conclusions

- Differential $\mathrm{N^3LO}$ distributions for colour singlet production achievable via slicing methods
- Motivates calculation of zero-jettiness beam functions at $\mathrm{N^3LO}$
- Matching coefficient $\mathcal{I}_{qq}^{(3)}$ now available in large $N_c \sim N_f$ limit
- Next steps:
  - Completion of subleading $\mathcal{I}_{qq}^{(3)}$ pieces
  - Calculation of soft function at $\mathrm{N^3LO}$
  - Complete remaining matching coefficients
  - Combine with NNLO $pp \to V + j$ calculation