New developments in perturbative quantum field theory

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QCD factorisation

- The 'master formula' for LHC observables:

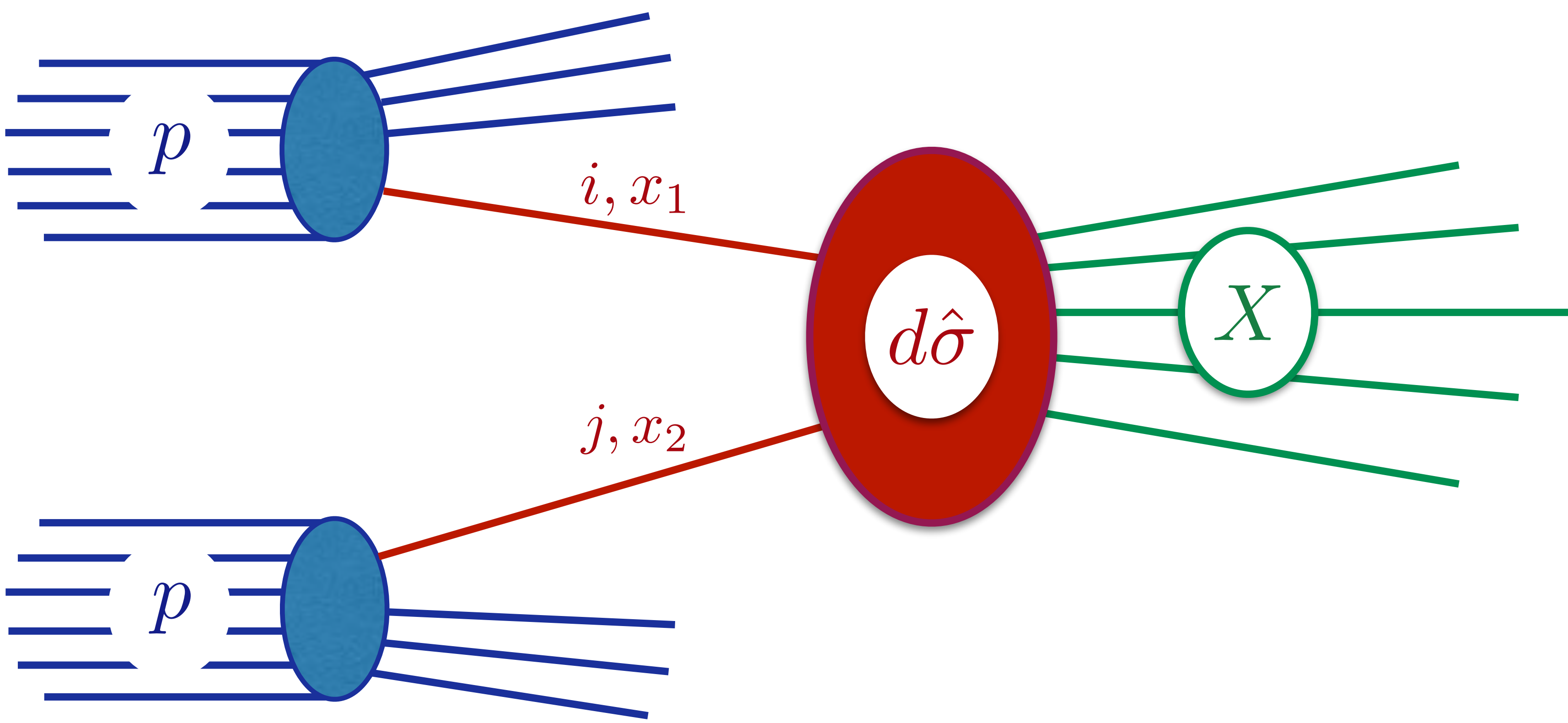
$$d\sigma(p\,p \to X) = \sum_{i,j} \int_0^1 dx_1\, dx_2\, f_i(x_1)\, f_j(x_2)\, d\hat{\sigma}(i\,j \to X)$$

<u>Parton Distribution Functions</u>

non-perturbative;
describe structure of the proton

<u>Partonic cross section</u>

computable in perturbation theory
as collisions between quarks and gluons

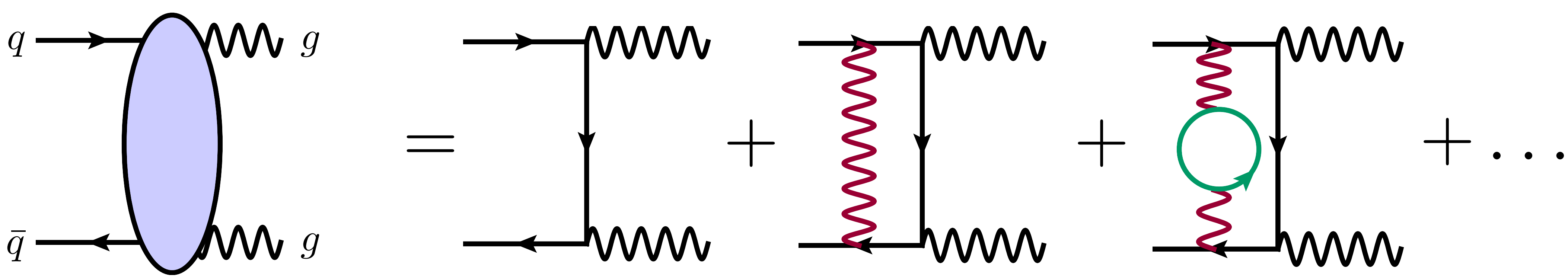


$$d\hat{\sigma} \sim \int dPS\, |\mathcal{A}|^2$$

$\mathcal{A}$ = scattering amplitude

Scattering amplitudes

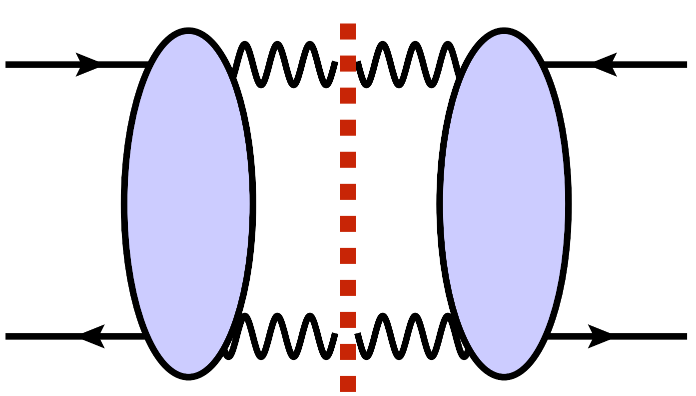
- $\mathcal{A}$ computed from Feynman diagrams:



  - ➡ Each diagram translates into an analytic formula.
  - ➡ Perturbative expansion ~ expansion in number of loops.

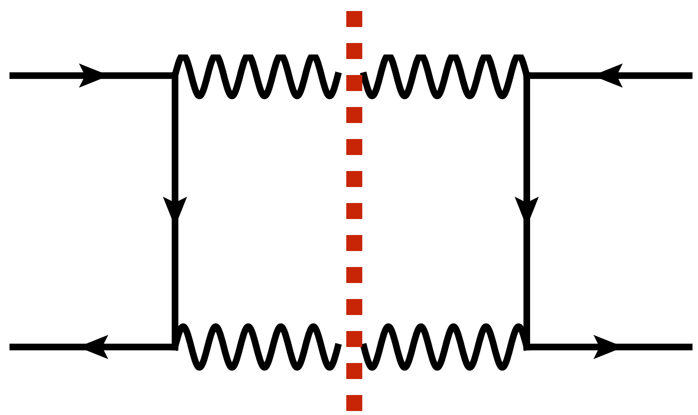
- Probabilities are related to the square of the amplitude:

$$\text{Proba} \sim |\mathcal{A}|^2 = \mathcal{A}\,\mathcal{A}^* =$$

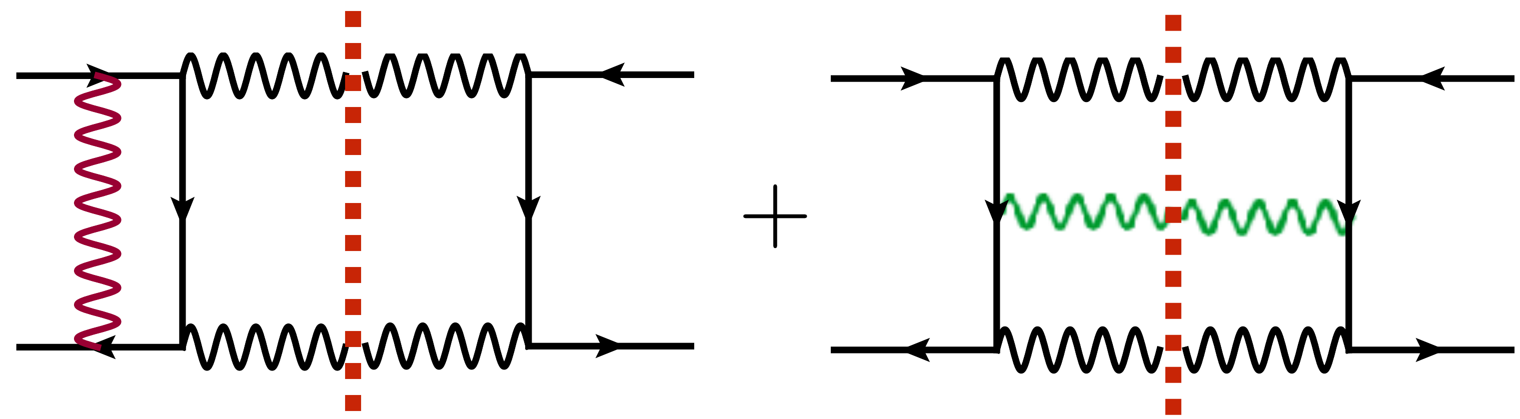


Anatomy of higher orders

- Leading order (LO):



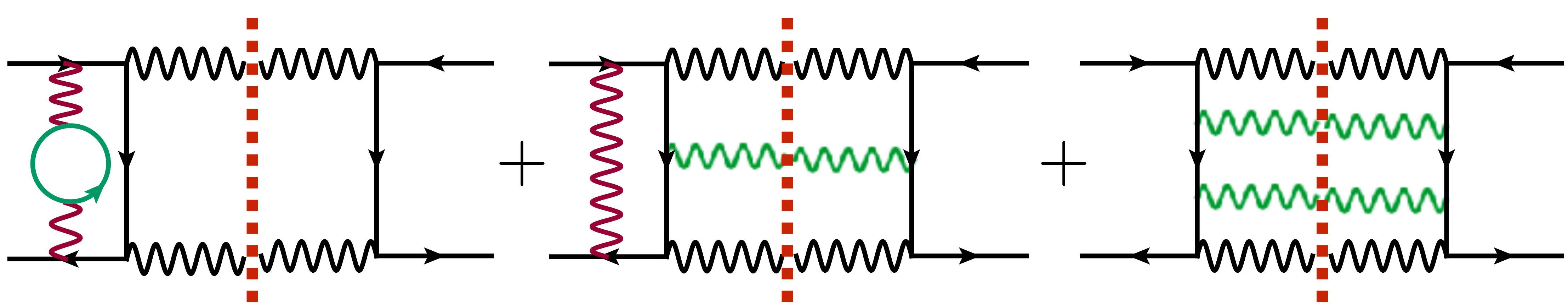
- Next-to-LO (NLO):



Virtual Real

Individually divergent, but sum is finite.

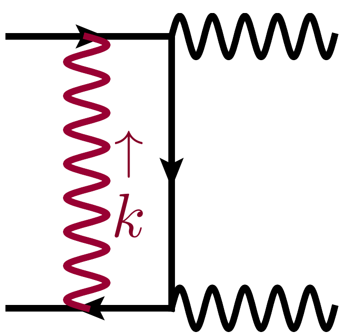
- Next-to-next-to-LO (NNLO):



Anatomy of higher orders

- In the rest of the talk: I focus (mostly) on virtual contributions.
- Virtual corrections require the integration over momentum of unresolved particle.

$$\int d^4k$$



- State of the art:
  - ➡ 1 loop: usually doable.
  - ➡ 2 loops: $g\,g \to g\,g$, $e^+e^- \to q\,\bar{q}\,g$ ~ 2000-05

    Since ~2015: $q\,\bar{q} \to V\,V'$ $\quad g\,g \to H\,H$ (num.) $\quad g\,g \to t\bar{t}$ (num.)

    $g\,g \to g\,g\,g \qquad g\,g \to g\,g\,V$

  - ➡ 3 loop / N3LO:

    Since ~2015: $g\,g \to H \quad g\,g \to H\,H \quad b\,\bar{b} \to H$ $\quad$ VBF

    $q\,\bar{q} \to \gamma^* \qquad q\,\bar{q} \to W^\pm$

Anatomy of higher orders

- Step 1: Sort the Feynman diagrams into (scalar) integral families.
- Step 2: Find a basis of master integrals for each family.
  - ➡ Integration-by-parts (IBP) relations: [Tkachov; Chetyrkin, Tkachov; Laporta; …]

$$\int d^D k \frac{\partial}{\partial k^\mu} \left(\frac{1}{D_1^{n_1} \dots D_p^{n_p}} \right) = 0$$

- Step 3: Evaluate the master integral (e.g., diff. eqs., Feynman parameters, etc.).

Anatomy of higher orders

- Step 1: Sort the Feynman diagrams into (scalar) integral families.
- Step 2: Find a basis of master integrals for each family.

Algebraic complexity ('bookkeeping of algebraic expressions'):

➡ Many scales, huge algebraic expressions.

➡ Huge linear systems to solve (1.000.000's of equations).

- Step 3: Evaluate the master integral (e.g., diff. eqs., Feynman parameters, etc.).

Analytic complexity ('doing the integrals'):

➡ What kind of functions?

➡ How to analytically continue or evaluate them?

Anatomy of higher orders

Language of loop integrals

=

Language of algebraic geometry

Algebraic geometry

- Algebraic geometry ~ Study of polynomial equations.

$$ax + by + c = 0$$ ➡ Straight line

$$x^2 + y^2 = R^2$$ ➡ Circle

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ ➡ Ellipsoid

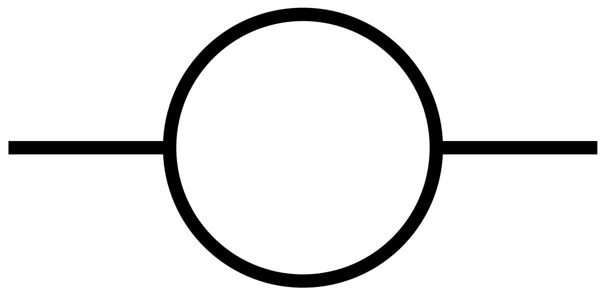
Algebraic varieties

- Period ~ Integral of a rational function over domain specified by polynomials.

Examples:

$$\int_{x^2+y^2\leq 1} dx\, dy = \pi \qquad \int_{1\leq x\leq z} \frac{dx}{x} = \log z$$

Algebraic geometry



$$\sim \int d^D k \, \frac{1}{k^2 - m_1^2} \, \frac{1}{(k+p)^2 - m_2^2}$$

Rational function

➡ Feynman integrals are periods! [Bogner, Weinzierl]

- New developments over the last 10 years:

➡ We can use insights from algebraic geometry to compute Feynman integrals.

➡ Many of these ideas were originally discovered in the context of scattering amplitudes in N=4 Super Yang-Mills.

Algebraic geometry

- There was (and still is) translation work to be done!
- Examples:

"The de Rham cohomology groups of an algebraic variety are finite."

"The number of master integrals is finite."

"Feynman integrals define families of periods, and are naturally equipped with a Gauss-Manin connection."

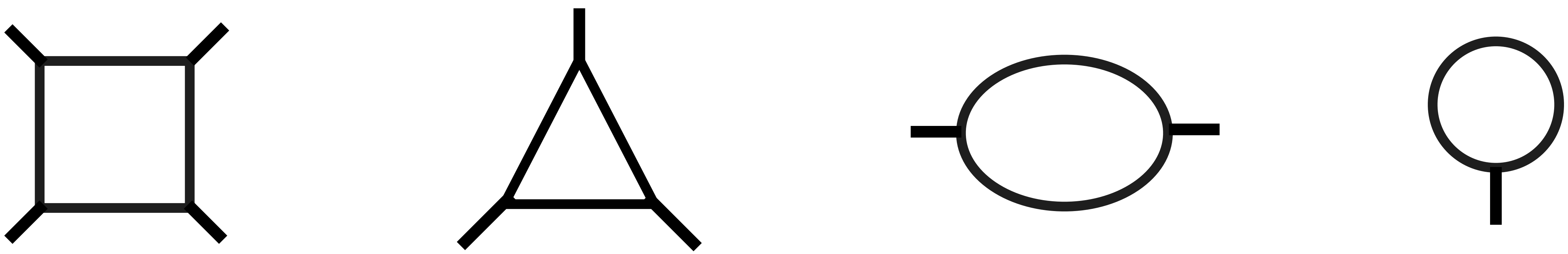
"Master integrals satisfy a system of first-order linear differential equations."

Algebraic geometry

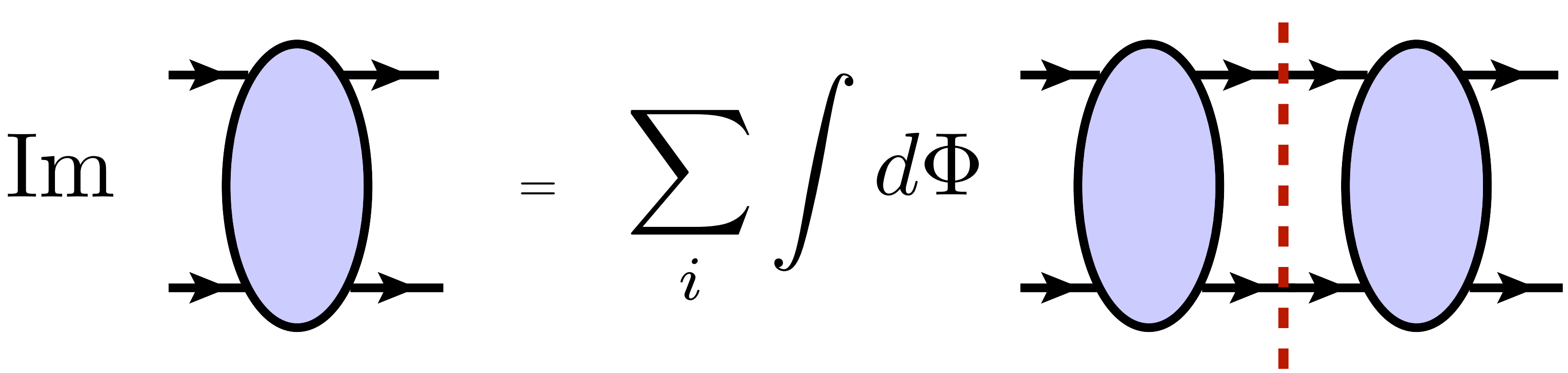
- Algebraic complexity:
  - ➡ Structure of integrand (=rational functions)?
  - ➡ Bookkeeping of algebraic expressions?
  - ➡ Decomposition into a basis (master integrals)?
- Analytic complexity:
  - ➡ What kind of functions do appear?
  - ➡ Algebraic and analytic properties of these functions?
  - ➡ Numerical evaluation?

Algebraic complexity

- One-loop computations are considered a solved problem (at least conceptually).
- Important ingredient: Every one-loop integral in 4D can be decomposed into integrals with only a few propagators:

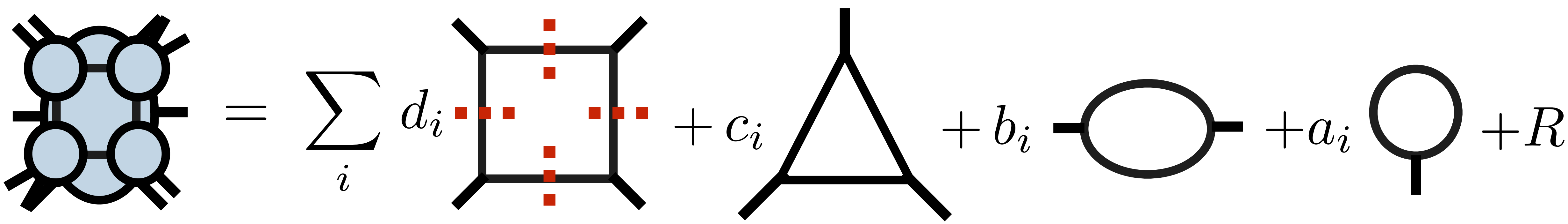


- Coefficients can be determined from unitarity.
  - ➡ Unitarity/Optical theorem:



Algebraic complexity

- Key idea: Use unitarity to reduce loop computation to tree computation.



box coefficient ~ Product of four tree amplitudes

- Computation of integral coefficients reduced to a tree-level computation! [Bern, Dixon, Dunbar, Kosower]

- Ossola-Papadopoulos-Pittau (OPP) & Giele-Kunszt-Melnikov (GKM): Parametrise loop integrand and fix coefficients with unitary cuts.

- There was no immediate extension beyond one loop.

Algebraic complexity

- Cuts/discontinuities ~ multi-variate residue calculus.

$$\frac{1}{p^2 - m^2 + i0} \to \delta(p^2 - m^2) \qquad \oint \frac{dz}{2\pi i} \frac{f(z)}{z} = f(0) = \int dz\, f(z)\, \delta(z)$$

- Breakthrough: use idea from calculations in polynomial rings to obtain parametrisation of loop integrand.

[Kosower, Gluza; Papadopoulos; Larsen, Yang; Ita; …]

➡ Similar in spirit to OPP / GKM at one-loop.

- Recently there was a first public code (CARAVEL) for numerical unitarity at two loops.

[Abreu, Dormans, Febres-Cordero, Ita, Kraus, Page, Pascual, Ruf, Sotnikov]

Algebraic complexity

- We know all integrals needed for 5-parton scattering!

[Gehrmann, Henn, Lo Presti; Papadopoulos, Tommasini, Wever; Gehrmann, Henn, Wasser, Zhang, Zoia; Chicherin, Sotnikov]

- Two-loop results for 2-to-3 scattering are within reach!
  - ➡ All planar two-loop amplitudes for 3-jet production.

    [Abreu, Dormans, Frebres Cordero, Ita, Page, Sotnikov]
  - ➡ Special helicity configuration beyond planar limit.

    [Badger, Chicherin, Gehrmann, Heinrich, Henn, Peraro, Wasser, Zhang, Zoia]
  - ➡ First steps towards W+2j at two loops.

    [Bayu Hartanto, Badger, Brønnum-Hansen, Peraro; see also Canko, Papadopoulos, Syrrakos]

$\mathcal{A}^{(2)[N_f^0]}/\mathcal{A}^{(\text{norm})}$	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}
$(1_g^+, 2_g^+, 3_g^+, 4_g^+, 5_g^+)$	0	0	−5.000000000	−29.38541207	−62.68413553
$(1_g^-, 2_g^+, 3_g^+, 4_g^+, 5_g^+)$	0	0	−5.000000000	−42.33840431	−159.9778589
$(1_g^-, 2_g^-, 3_g^+, 4_g^+, 5_g^+)$	12.50000000	84.83123596	243.4660216	301.9565843	−152.0528809
$(1_g^-, 2_g^+, 3_g^-, 4_g^+, 5_g^+)$	12.50000000	84.83123596	269.4635002	551.6251881	984.0882231

[Abreu, Frebres Cordero, Ita, Page, Sotnikov]

Algebraic complexity

- Mathematical interpretation of IBPs

$$\int d^D k \frac{\partial}{\partial k^\mu} \left(\frac{1}{D_1^{n_1} \dots D_p^{n_p}} \right) = 0$$

➡ IBPs ~ find relations among integrand up to total derivatives.

➡ de Rham cohomology: $\omega_1 \sim \omega_2 \Leftrightarrow \omega_1 - \omega_2 = d\eta$

- Novel approach: Use (twisted) de Rham cohomology to perform decomposition into master integrals.

➡ Cohomology groups = **vector space** of Feynman integrals.

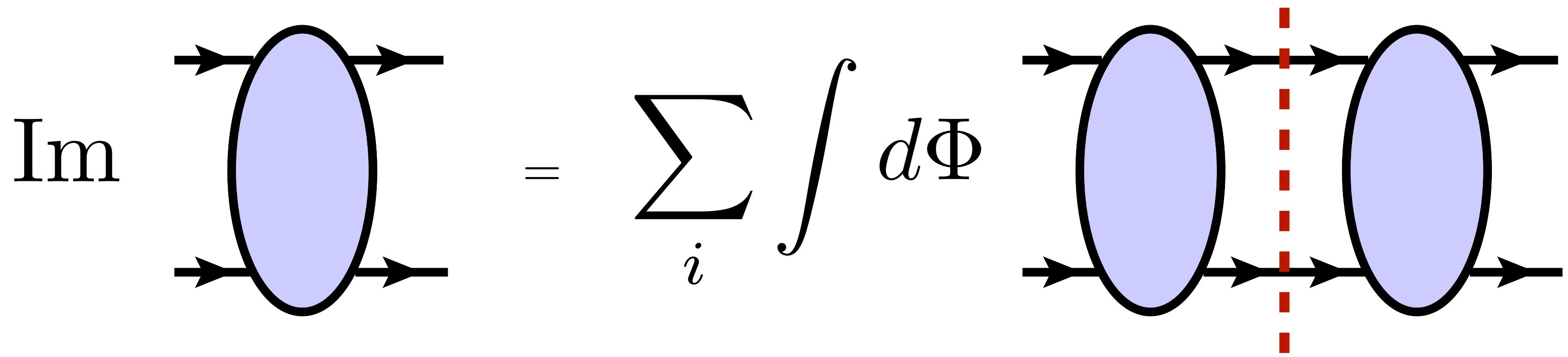
➡ Master integrals = **basis** of this vector space.

➡ Intersection pairing = '**scalar product**' on this vector space.

[Mastrolia, Mizera; Frellesvig, Gasparotto, Laporta, Mandal, Mastrolia, Matiazzi, Mizera]

Analytic complexity

- Unitarity implies that amplitudes are multi-valued functions:



- One-loop integrals:

$$\log z = \int_1^z \frac{dt}{t} \qquad \mathrm{Li}_2(z) = -\int_0^z \frac{dt}{t} \log(1-t)$$

➡ Multiple polylogarithms: extension to two-loop integrals with massless partons.

➡ Well understood, thanks to algebraic geometry!

[Goncharov; Brown; Goncharov, Spradling, Vergu, Volovich; CD; Panzer; …]

Analytic complexity

- Large classes of loop integrals can be expressed in terms of polylogarithms.

$$G(0;z) = \log z$$

$$G(a_1;z) = \log\left(1 - \frac{z}{a_1}\right)$$

$$G(0,1;z) = -\mathrm{Li}_2(z)$$

$$G(a_1,\ldots,a_n;z) = \int_0^z \frac{dt}{t-a_1}\, G(a_2,\ldots,a_n;t)$$

Weight = n = # integrations

[Poincaré; Kummer; Lappo-Danilevsky; Goncharov; …]

- Related to active research in pure mathematics! [cf. Brown; Goncharov; …]

- Polylogarithms satisfy many identities:

➡ Example: $\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) - \log(1-z)\,\log z + \dfrac{\pi^2}{6}$ [Euler]

Why should I care…?

Analytic complexity

1. To compute integrals:

$$\int_0^z \frac{dt}{t} \log\frac{1+t}{1-t}$$

$$= \int_0^z \frac{dt}{t}\log(1+t) - \int_0^z \frac{dt}{t}\log(1-t)$$

$$= -\mathrm{Li}_2(-z) + \mathrm{Li}_2(z)$$

$$\log\frac{1+t}{1-t} = \log(1+t) - \log(1-t)$$

$$\mathrm{Li}_2(z) = -\int_0^z \frac{dt}{t}\log(1-t)$$

➡ Identities between special functions are important when computing integrals.

Analytic complexity

1. To compute integrals:
2. To simplify expressions / evaluate amplitudes numerically:
   - ➡ Mathematica does not know $G(0,a,b;1)$ …
   - ➡ … but it does know $\log x$ and $\text{Li}_n(x)$!

$$G(0,a,b;1) =$$

$$-\text{Li}_3\left(\frac{a(1-b)}{a-b}\right) + \text{Li}_3\left(\frac{b-1}{b-a}\right) - \text{Li}_3\left(\frac{b}{b-a}\right) + \text{Li}_3\left(\frac{1}{a}\right) + \text{Li}_3(1-b)$$

$$+\log\left(1-\frac{1}{b}\right)\left[\text{Li}_2\left(\frac{a(1-b)}{a-b}\right) - \text{Li}_2\left(\frac{b-1}{b-a}\right) - \text{Li}_2(1-b)\right]$$

$$-\frac{1}{6}\log^3\left(\frac{ab}{a-b}\right) + \frac{1}{2}\log^2\left(1-\frac{1}{b}\right)\left[-\log\left(\frac{a-1}{a-b}\right) + \log\left(\frac{(a-1)b}{a-b}\right) - \log b\right]$$

$$-\frac{\pi^2}{6}\log\left(\frac{ab}{a-b}\right) + \frac{1}{6}\log^3 b + \frac{\pi^2}{6}\log b$$

$$(a,b) = (1.2, 1.1) \longrightarrow \text{[Mathematica]} \longrightarrow 0.490485\ldots$$

Analytic complexity

1. To compute integrals:
2. To simplify expressions / evaluate amplitudes numerically:
3. To discover new structures in QFT:

   Example: 'Principle maximal transcendentality'

   ➡ An L loop amplitude in N=4 Super Yang only contains polylogarithms of 'transcendentality'/weight $2L$.

$$\mathcal{A}_4^{(1)} \sim \frac{1}{2}\log^2\frac{s}{t} + \frac{2\pi^2}{3} \qquad \log(-1) = \pm i\pi \qquad \text{Weight: } 2L = 2$$

[Kotikov, Lipatov]

   ➡ In other theories: weight bounded by $2L$.

   ➡ Sometimes (but not always) the maximal weight term is identical between N=4 SYM and QCD!

Analytic complexity

- Polylogarithms form a Hopf algebra. [Goncharov; Brown]
  - ➡ Algebra: Vector space with an operation that allows one to 'fuse' two elements into one (multiplication).
  - ➡ Coalgebra: Vector space with an operation that allows one to break one element apart (coproduct Δ).
- A Hopf algebra is
  - ➡ at the same time an algebra & a coalgebra.
  - ➡ such that the product and coproduct are compatible

$$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b)$$

  - ➡ plus some other properties.

Analytic complexity

- Examples:

$$\Delta(\log z) = \log z \otimes 1 + 1 \otimes \log z$$

$$\Delta(\mathrm{Li}_2(z)) = \mathrm{Li}_2(z) \otimes 1 + 1 \otimes \mathrm{Li}_2(z) - \log(1-z) \otimes \log z$$

- Example: $T = -\mathrm{Li}_2(z) - \log(1-z)\log z$

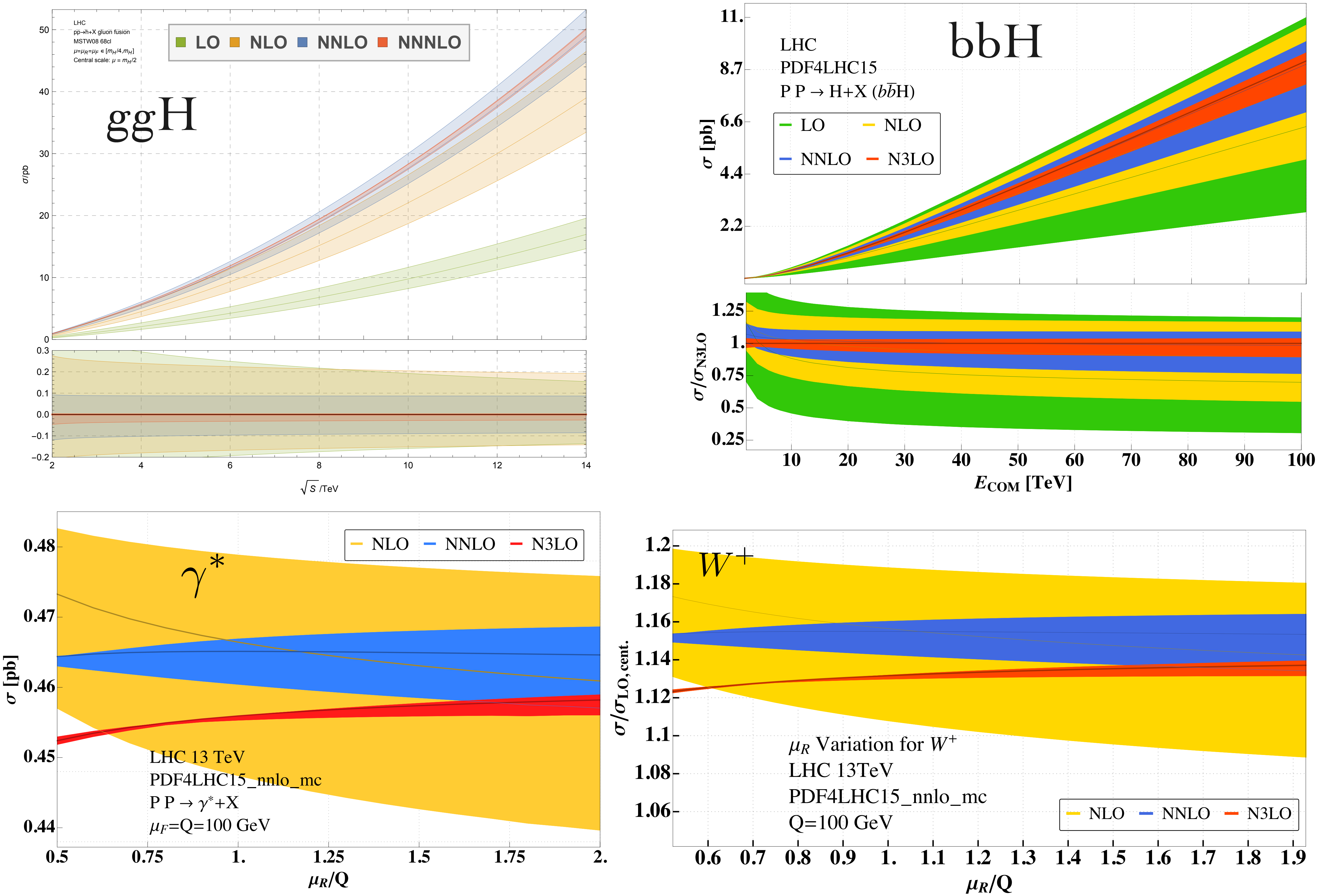
Can this simplified?

$$\Delta(T) = T \otimes 1 + 1 \otimes T + \Delta'(T)$$

$$\begin{aligned}\Delta'(T) &= \log(1-z) \otimes \log z - [\log(1-z) \otimes \log z + \log z \otimes \log(1-z)] \\ &= -\log z \otimes \log(1-z) \\ &= \Delta'(\mathrm{Li}_2(1-z))\end{aligned}$$

➡ At $z = 0$: $T = -\mathrm{Li}_2(z) - \log(1-z)\log z = \mathrm{Li}_2(1-z) - \frac{\pi^2}{6}$

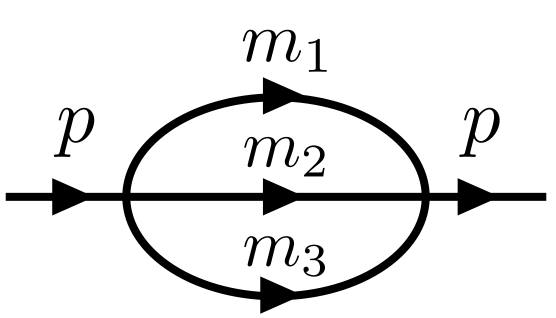
N3LO cross sections



Beyond polylogarithms

- Starting from two loops: new functions arise!

- Prototype example: the massive sunrise graph.



➡ Closely related to elliptic integral:

$$\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}}$$

[Sabry; Broadhurst; Bauberger, Berends, Bohm, Buza; Caffo, Czyz, Laporta, Remiddi; Laporta Remiddi]

➡ No closed analytic result since 60's.

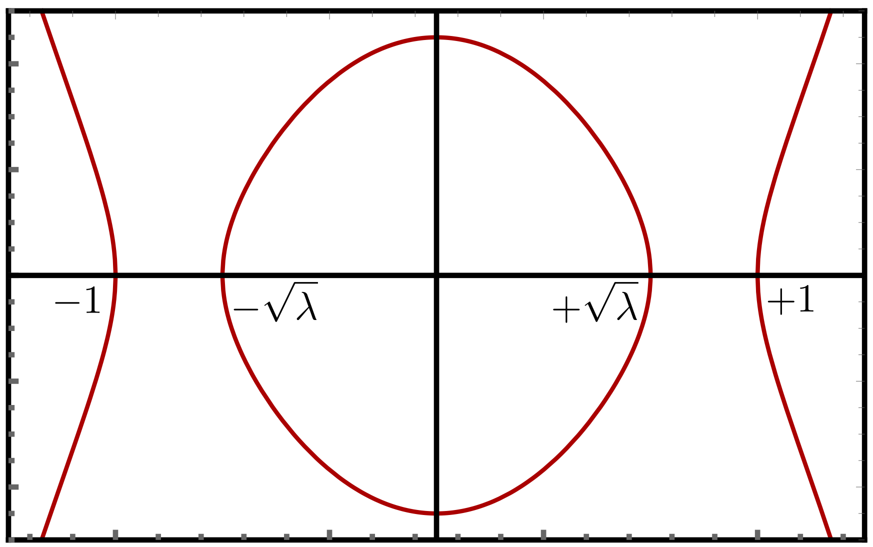
- Breakthrough in 2013: The sunrise graph evaluates to a dilogarithm on an elliptic curve!

[Bloch, Vanhove]

Elliptic Curves

- Elliptic curve ~ set of points (x, y) such that

$$y^2 = (1-x^2)(1-\lambda x^2) \quad \leftrightarrow \quad y = \pm\sqrt{(1-x^2)(1-\lambda x^2)}$$

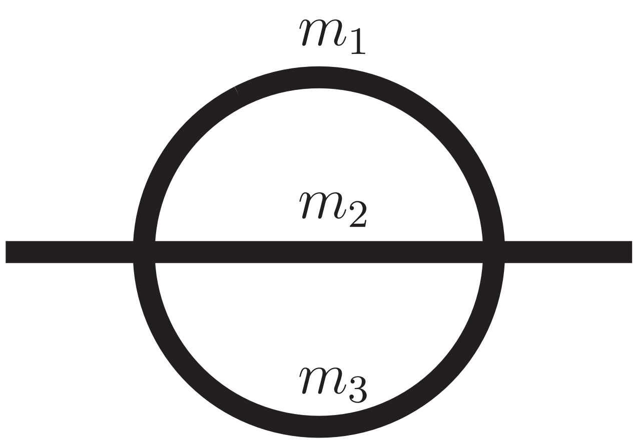


$$\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}}$$

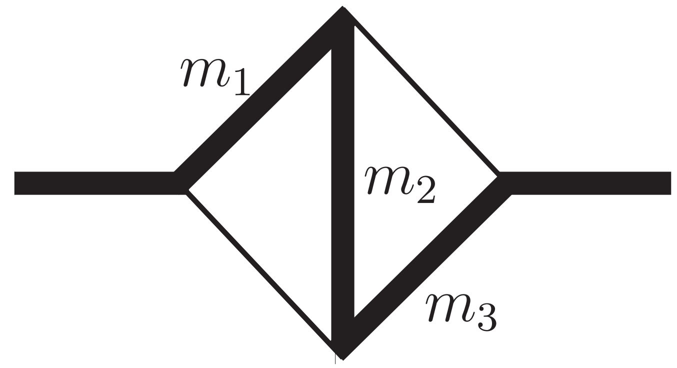
➡ Elliptic curves are important in algebraic geometry, number theory, cryptography, string theory, …

- Elliptic polylogarithms are very new mathematics: original math literature from 2010! [Brown, Levin]

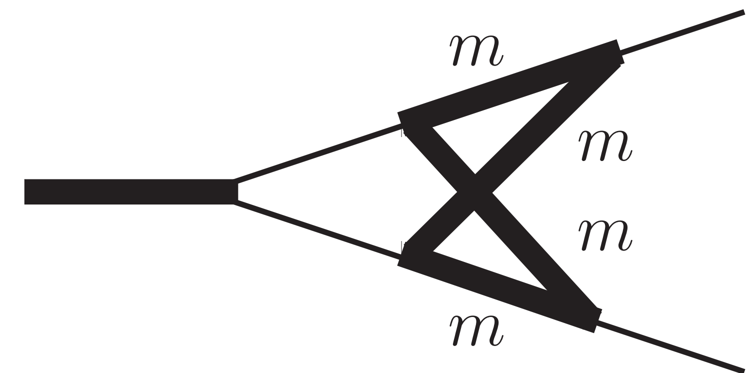
Elliptic Feynman integrals



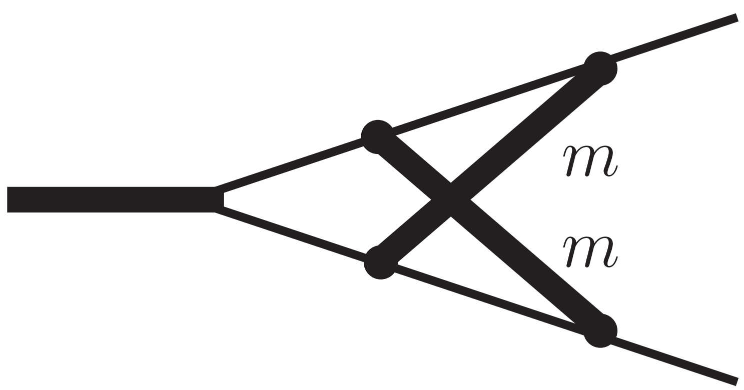
$t\,\bar{t}$



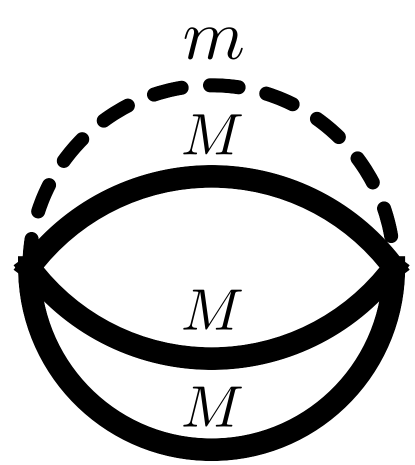
e self-energy



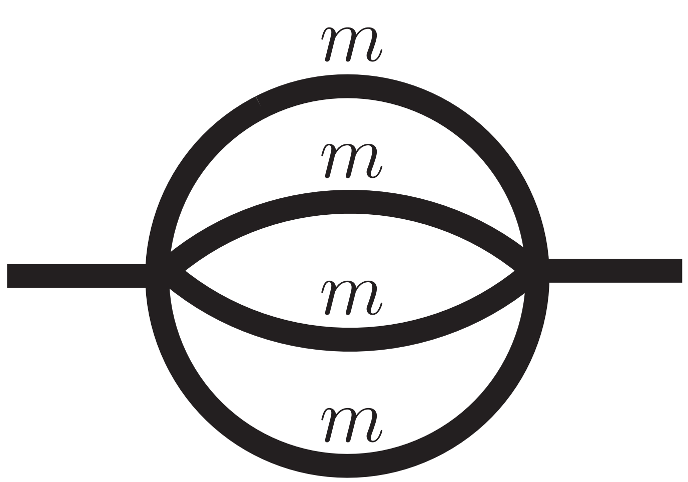
$t\,\bar{t}$, $\gamma\,\gamma$, etc.



EW form factor



ρ-parameter

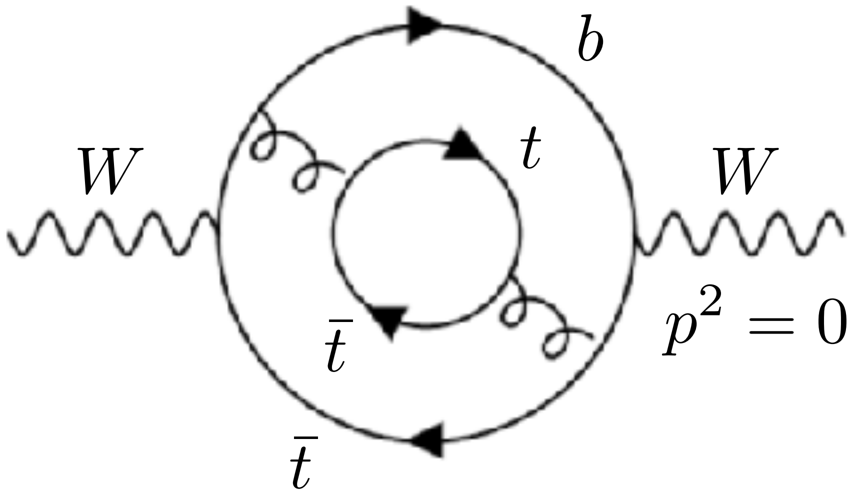


$g\,g \to H$

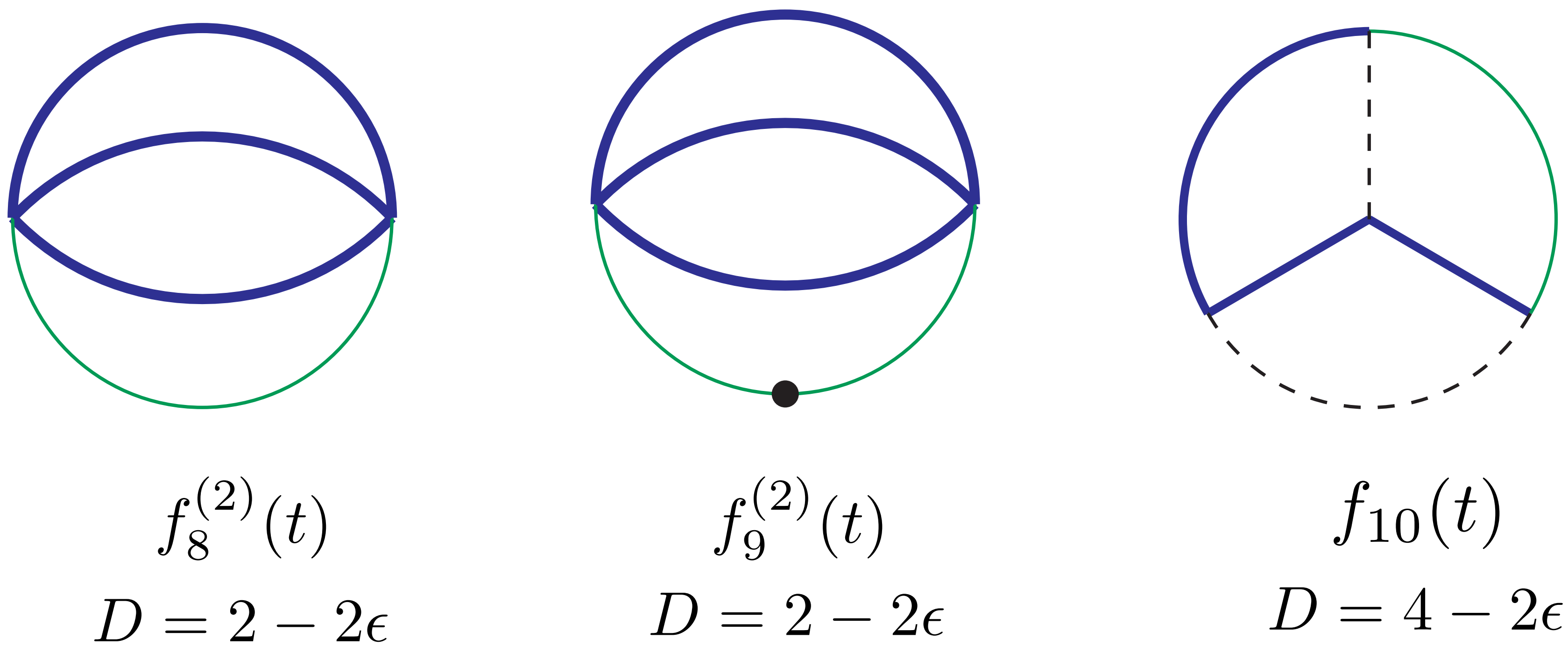
[Bloch, Vanhove; Adams, Bogner, Chaubey, Schweitzer, Weinzierl; Brödel, CD, Dulat, Marzucca, Tancredi, Penante; Hiddings, Moriello; …]

The rho parameter

- Example: the rho parameter at 3 loops with quark-mass dependence.
  - ➡ Known numerically [Grigo, Hoff, Marquard, Steinhauser; see also Blümlein, de Freitas, van Hoeij, Imamoglu, Marquard].



- Integrals that were unknown analytically:



$$t = \frac{m^2}{M^2}$$

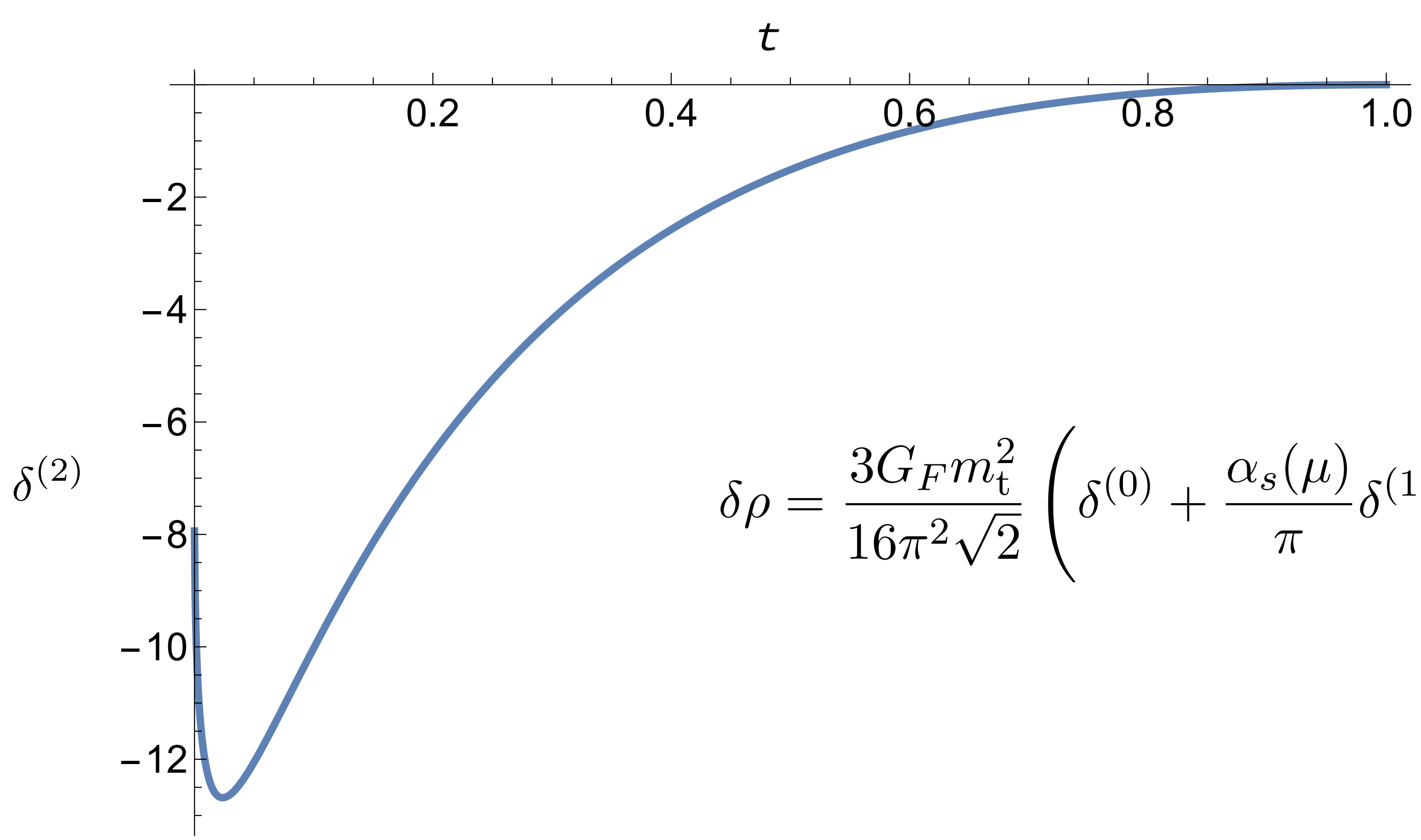
The rho parameter

- These integrals can be evaluated in terms of the same class of functions are the sunrise and banana graphs.

  [Abreu, Becchetti, CD, Marzucca; see also Blümlein, de Freitas, van Hoeij, Imamoglu, Marquard]

  - ➡ Iterated integrals of Eisenstein series.
  - ➡ Analytic continuation and numerical evaluation of these functions well understood.

  [CD, Tancredi]



$$\delta\rho = \frac{3G_F m_t^2}{16\pi^2\sqrt{2}}\left(\delta^{(0)} + \frac{\alpha_s(\mu)}{\pi}\delta^{(1)} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2\delta^{(2)} + \mathcal{O}(\alpha_s(\mu)^3)\right)$$

[Abreu, Becchetti, CD, Marzucca]

Pure Mathematics

- Can physics inspire new results in mathematics?
- Consider the identity:

$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(z) - \log(1-z)\,\log z + \frac{\pi^2}{6}$$

- Version of the dilogarithm ('Bloch-Wigner dilogarithm')

$$D(z) = \mathrm{Im}\left[\mathrm{Li}_2(z) + \frac{1}{2}\log|z|^2\,\log(1-z)\right]$$

with the properties

- $D(z)$ is single-valued (i.e., no branch cuts).
- $D(z)$ satisfies 'clean' identity, e.g., $D(1-z) = -D(z)$.

Clean identities

- Does this generalise to arbitrary multiple polylogarithms?
- Example:

$$G(0,a,b;1) = -\mathrm{Li}_3\left(\frac{a-ab}{a-b}\right) - \mathrm{Li}_3\left(-\frac{b}{a-b}\right) + \mathrm{Li}_3\left(\frac{b-1}{b-a}\right) + \mathrm{Li}_3\left(\frac{1}{a}\right) + \mathrm{Li}_3(1-b)$$

$$+\log\left(\frac{b-1}{b}\right)\left(\mathrm{Li}_2\left(\frac{a-ab}{a-b}\right) - \mathrm{Li}_2\left(\frac{b-1}{b-a}\right) - \mathrm{Li}_2(1-b)\right) - \frac{1}{6}\log^3\left(\frac{ab}{a-b}\right)$$

$$+\frac{1}{2}\log^2\left(\frac{b-1}{b}\right)\left(\log\left(\frac{(a-1)b}{a-b}\right) - \log(b) - \log\left(\frac{a-1}{a-b}\right)\right) - \frac{1}{6}\pi^2\log\left(\frac{ab}{a-b}\right)$$

$$+\frac{\log^3(b)}{6} + \frac{1}{6}\pi^2\log(b) + i\pi\log^2\left(\frac{b-a}{ab}\right)\mathrm{sgn}(\mathrm{Im}(b))\,\mathcal{H}_1(a,b)$$

$$+i\pi\log^2\left(\frac{b-a}{b}\right)\mathrm{T}\left(1, 1-\frac{1}{b}, 1-\frac{a}{b}\right)\mathrm{sgn}\left(\mathrm{Im}\left(\frac{a}{b}\right)\right).$$

➡ Goal: single-valued functions $C(0,a,b;1)$ and $C(0,0,1;a)$ such that $(\mathrm{Li}_3(a) = -G(0,0,1;a))$

$$C(0,a,b;1) =$$

$$= C\left(0,0,1;\frac{a-ab}{a-b}\right) + C\left(0,0,1;\frac{b}{a-b}\right) - C\left(0,0,1;\frac{1-b}{a-b}\right) - C\left(0,0,1;\frac{1}{a}\right) - C(0,0,1;1-b)$$

Clean identities

$$
\begin{aligned}
C(0,a,b;1) = {} & \frac{1}{3}G(b,1)G(0,\bar{a})G(a,b) - \frac{1}{3}G(b,1)G(0,\bar{b})G(a,b) - \frac{1}{3}G(0,a)G(b,1)G(\bar{a},\bar{b}) + \frac{1}{3}G(0,b)G(b,1)G(\bar{a},\bar{b}) \\
& - \frac{1}{3}G(b,1)G(0,\bar{a})G(\bar{a},\bar{b}) - \frac{1}{3}G(b,1)G(0,\bar{a},\bar{b}) + \frac{1}{3}G(b,1)G(\bar{a},0,\bar{b}) - \frac{1}{3}G(0,\bar{a})G(\bar{a},1)G(b,a) \\
& + \frac{1}{3}G(0,\bar{b})G(\bar{b},1)G(a,b) + \frac{1}{3}G(0,a)G(\bar{b},1)G(\bar{a},\bar{b}) + \frac{1}{3}G(0,\bar{a})G(\bar{b},1)G(\bar{a},\bar{b}) \\
& - \frac{1}{3}G(0,a)G(\bar{a},1)G(\bar{b},\bar{a}) - \frac{1}{3}G(0,a,1)G(\bar{b},\bar{a}) + \frac{1}{3}G(0,b,1)G(\bar{a},\bar{b}) - \frac{1}{3}G(\bar{a},1)G(0,b,a) \\
& - \frac{2}{3}G(0,\bar{a},1)G(b,a) - \frac{2}{3}G(0,\bar{a},1)G(\bar{b},\bar{a}) + \frac{1}{3}G(\bar{b},1)G(0,\bar{a},\bar{b}) + \frac{2}{3}G(0,\bar{b},1)G(a,b) \\
& + \frac{2}{3}G(0,\bar{b},1)G(\bar{a},\bar{b}) - \frac{1}{3}G(\bar{a},1)G(0,\bar{b},\bar{a}) + \frac{1}{3}G(\bar{b},1)G(a,0,b) - \frac{1}{3}G(\bar{a},1)G(b,0,a) \\
& - \frac{1}{3}G(0,\bar{a})G(b,a,1) + \frac{1}{3}G(0,a)G(b,\bar{a},1) + \frac{1}{3}G(0,\bar{a})G(b,\bar{a},1) + \frac{1}{3}G(0,a)G(\bar{a},b,1) \\
& + \frac{1}{3}G(0,\bar{a})G(\bar{a},b,1) + \frac{1}{3}G(0,a)G(\bar{a},\bar{b},1) + \frac{1}{3}G(0,\bar{a})G(\bar{a},\bar{b},1) - \frac{1}{3}G(\bar{a},1)G(\bar{b},0,\bar{a}) \\
& - \frac{1}{3}G(0,a)G(\bar{b},\bar{a},1) - \frac{1}{3}G(0,\bar{a})G(\bar{b},\bar{a},1) + \frac{1}{3}G(0,a,\bar{b},1) + \frac{4}{3}G(0,\bar{a},\bar{b},1) + \frac{1}{3}G(0,\bar{b},a,1) \\
& + \frac{1}{3}G(0,\bar{b},\bar{a},1) - \frac{2}{3}G(b,\bar{a},0,1) - \frac{2}{3}G(\bar{a},0,b,1) + \frac{2}{3}G(\bar{a},0,\bar{b},1) - \frac{2}{3}G(\bar{a},b,0,1) + \frac{2}{3}G(\bar{a},\bar{b},0,1) \\
& + \frac{1}{3}G(\bar{b},0,a,1) + \frac{1}{3}G(\bar{b},0,\bar{a},1) + \frac{2}{3}G(\bar{b},\bar{a},0,1) + \frac{1}{3}G(0,a)G(b,1)G(a,b) + \frac{1}{3}G(b,1)G(0,a,b) \\
& - \frac{1}{3}G(b,1)G(a,0,b) - \frac{1}{3}G(0,a,1)G(b,a) + \frac{1}{3}G(0,b,1)G(a,b) - \frac{1}{3}G(0,a)G(b,a,1) + \frac{1}{3}G(0,a,b,1) \\
& - \frac{2}{3}G(0,b,a,1) - \frac{2}{3}G(b,0,a,1)
\end{aligned}
$$

$$
\begin{aligned}
C(0,0,1;a) = {} & \frac{1}{3}G(0,0,a)G(1,\bar{a}) - \frac{1}{3}G(0,1,a)G(0,\bar{a}) - \frac{1}{3}G(0,a)G(0,1,\bar{a}) + \frac{1}{3}G(0,a)G(1,0,\bar{a}) \\
& - \frac{1}{3}G(0,1,0,\bar{a}) + \frac{1}{3}G(1,0,0,\bar{a}) + \frac{1}{3}G(0,0,1,a) - \frac{1}{3}G(0,1,0,a)
\end{aligned}
$$

Clean identities

- Construction is totally algorithmic! [Charlton, CD, Gangl]
  - ➡ Based on the Hopf algebra and the coproduct of polylogarithms.
- Can proof that the resulting functions will satisfy all relations of polylogarithms, but in a clean version!
- Joint-venture between math and physics!
  - ➡ Insight & experience from physics played a crucial role!
- Applications to physics..?

Conclusion

- Language of loop integrals = Language of algebraic geometry
- Taming the algebraic complexity:
  - ➡ Ideas from algebraic geometry lead to new ways to develop a "unitarity program" beyond one loop.
  - ➡ Recent application: 2-to-3 scattering at 2 loops.
- Taming the analytic complexity:
  - ➡ The simplest class of functions (polylogarithms) are now under very good control.
  - ➡ New insight into elliptic Feynman integrals.