The full angle-dependence of the four-loop cusp anomalous dimension in QED

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after the Higgs Discovery

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Wilson lines

$$W\left[C(z,y)\right] = \mathrm{P}\exp\left[ig_{\mathrm{YM}}\int_C \mathrm{d}x_\mu A^\mu(x)\right]$$

$A_\mu^a(x)T_F^a$

Gauge transformations:

$$\psi(x) \to U(x)\psi(x)$$

$$W\left[C(z,y)\right] \to U(z)W\left[C(z,y)\right]U^\dagger(y)$$

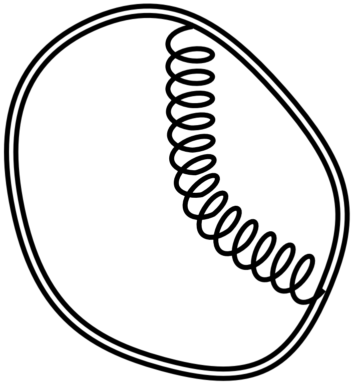
Divergences of Wilson loops

Consider Wilson line with closed time-like contour

$$\mathrm{Tr}\langle 0 | \mathrm{T\,P} \exp \left[i g_{\mathrm{YM}} \oint_C \mathrm{d}x_\mu A^\mu(x) \right] | 0 \rangle$$

gauge invariance

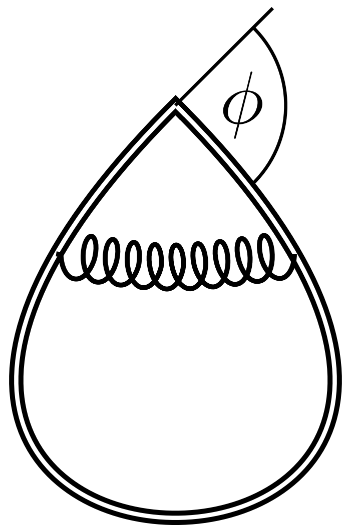
time-like



finite in dimensional regularization
$d = 4 - 2\epsilon$

Divergences of Wilson loops

$$W_{\text{cusp}}^{\text{bare}} =$$



$$\propto \frac{1}{\epsilon}(\phi \cot \phi - 1)$$

[Polyakov '80]

renormalizes multiplicatively: $Z^{-1} W_{\text{cusp}}^{\text{bare}} = W_{\text{cusp}}^{\text{ren}} = \mathcal{O}(\epsilon)$

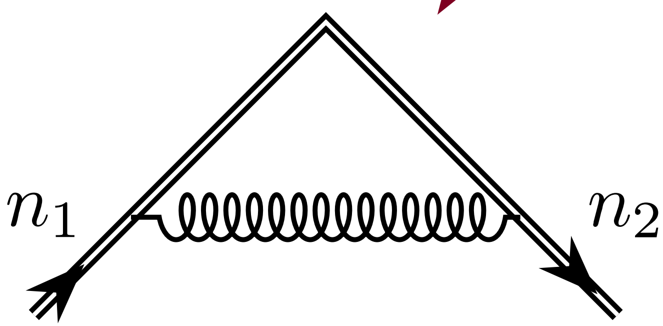
[Brandt, Neri, Sato '81 | Korchemsky, Radyushkin, '87]

cusp anomalous dimension: $\Gamma_{\text{cusp}}(\phi, \alpha_s) = \dfrac{\mathrm{d}\log Z}{\mathrm{d}\log \mu}$

Light-like cusp

Light like cusp anomalous dimension encoded in $\Gamma_{\mathrm{cusp}}(\phi, \alpha_s)$ as limiting case: [Korchemsky, Radyushkin, '86 and '92]

$$\Gamma_{\mathrm{cusp}}(\phi, \alpha_s) = -i\phi K(\alpha_s) + \mathcal{O}(\phi^0) \qquad (\phi \to i\infty)$$



$$n_1^2 = n_2^2 = 0$$

no angle dependence

IR divergences of amplitudes

Scattering amplitudes (in general) have infrared divergences

$$\mathcal{A}_f = \mathbf{Z}_{\mathrm{IR}}^{-1} \mathcal{A}$$

finite

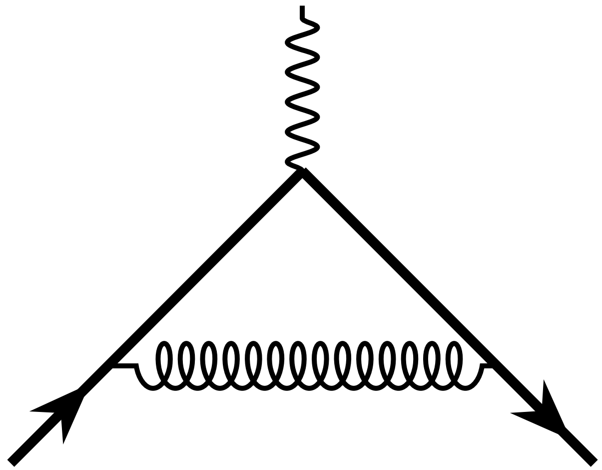
IR divergent

[Korchemsky, Radyushkin, '86 and '92 | Dixon, '09 | Dixon, Gardi, Magnea '09 | Becher, Neubert, '09, ...]

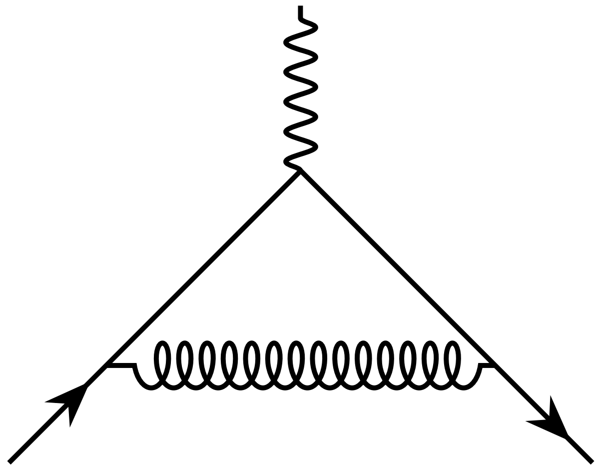
IR divergences of amplitudes

Scattering amplitudes (in general) have infrared divergences

$$\boldsymbol{\mathcal{A}}_f = \mathbf{Z}_{\mathrm{IR}}^{-1} \boldsymbol{\mathcal{A}}$$



$$\propto \frac{1}{\epsilon_{\mathrm{IR}}} \Gamma_{\mathrm{cusp}}^{(1)}(\phi)$$



$$\propto \frac{1}{\epsilon_{\mathrm{IR}}^2} K^{(1)}$$

IR divergences of amplitudes

Scattering amplitudes (in general) have infrared divergences

$$\boldsymbol{\mathcal{A}}_f = \mathbf{Z}_{\mathrm{IR}}^{-1} \boldsymbol{\mathcal{A}}$$

$\mathbf{Z}_{\mathrm{IR}}$ built from K and $\Gamma_{\mathrm{cusp}}(\phi)$ (+ other anom. dim.)

What is known?

All order: QED without light fermions $(n_f = 0)$

$$\Gamma_{\text{cusp}}(x,\alpha) = \frac{\alpha}{\pi}\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right] + \cancel{\mathcal{O}(\alpha^2)}$$

$$x = e^{i\phi}$$

What is known?

All order: QED without light fermions $(n_f = 0)$

$$\Gamma_{\rm cusp}(x,\alpha) = \frac{\alpha}{\pi}\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right] + \cancel{\mathcal{O}(\alpha^2)}$$

Three loops: QED with light fermions $(n_f > 0)$

$$\Gamma_{\rm cusp}(x,\alpha) = K(\alpha)\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right] + \mathcal{O}(\alpha^4)$$

$$\frac{\alpha}{\pi} - \frac{5n_f}{9}\left(\frac{\alpha}{\pi}\right)^2 + \mathcal{O}(\alpha^3, n_f)$$

What is known in QED?

Four loops: QED with light fermions $(n_f > 0)$

Conjecture: [Grozin, Henn, Korchemsky, Marquard '16]

$$\Gamma_{\text{cusp}}(x,\alpha) = K(\alpha)\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right] + \mathcal{O}(\alpha^5)$$

??

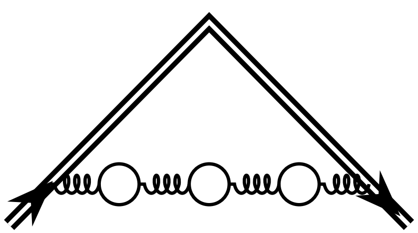
What is known in QED?

Four loops: QED with light fermions $(n_f > 0)$

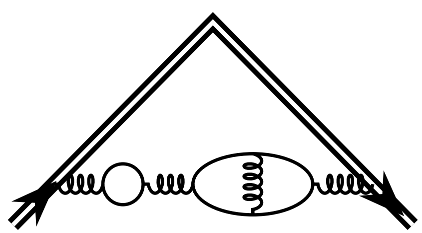
Conjecture: [Grozin, Henn, Korchemsky, Marquard '16]

$$\Gamma_{\text{cusp}}(x,\alpha) = K(\alpha)\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right] + \mathcal{O}(\alpha^5)$$

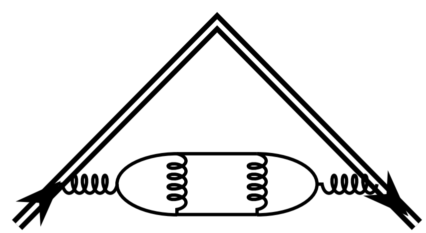
conjecture works



$n_f^3 C_F$

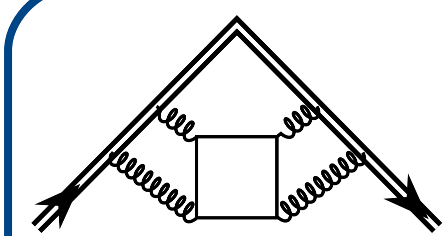


$n_f^2 C_F^2$



$n_f C_F^3$

disagreement with conjecture



$n_f d_F d_F$

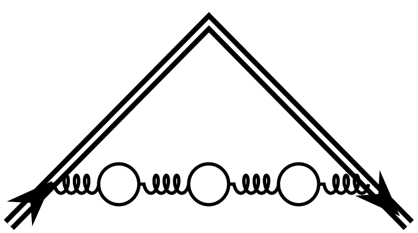
[Grozin, '04 | Grozin, Henn, Korchemsky, Marquard '16 | Grozin, '18]

[Grozin, Henn, Stahlhofen, '17 | RB, Grozin, Henn, Stahlhofen, '19]

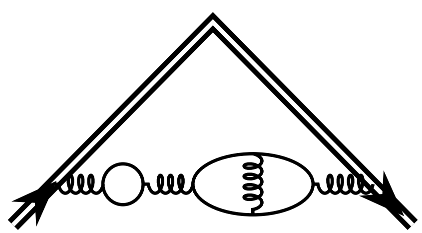
What is known in QED?

Four loops: QED with light fermions $(n_f > 0)$

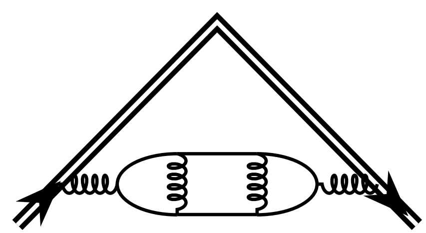
$$\Gamma_{\text{cusp}}(x,\alpha) = \gamma(\alpha)\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right] + \left(\frac{\alpha}{\pi}\right)^4 n_f B(x) + \mathcal{O}(\alpha^5)$$



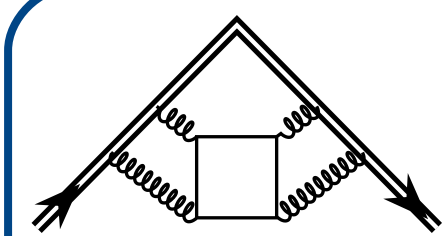
$n_f^3 C_F$



$n_f^2 C_F^2$



$n_f C_F^3$

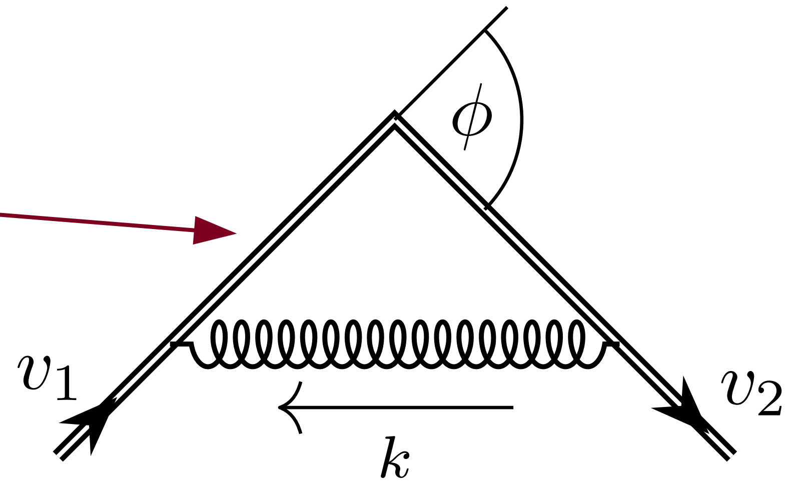


$n_f d_F d_F$

Calculation

Use heavy quark effective theory setup and dimensional regularisation $(d = 4 - 2\epsilon)$ for computation

$$\frac{1}{-2v_1 \cdot k + 1}$$

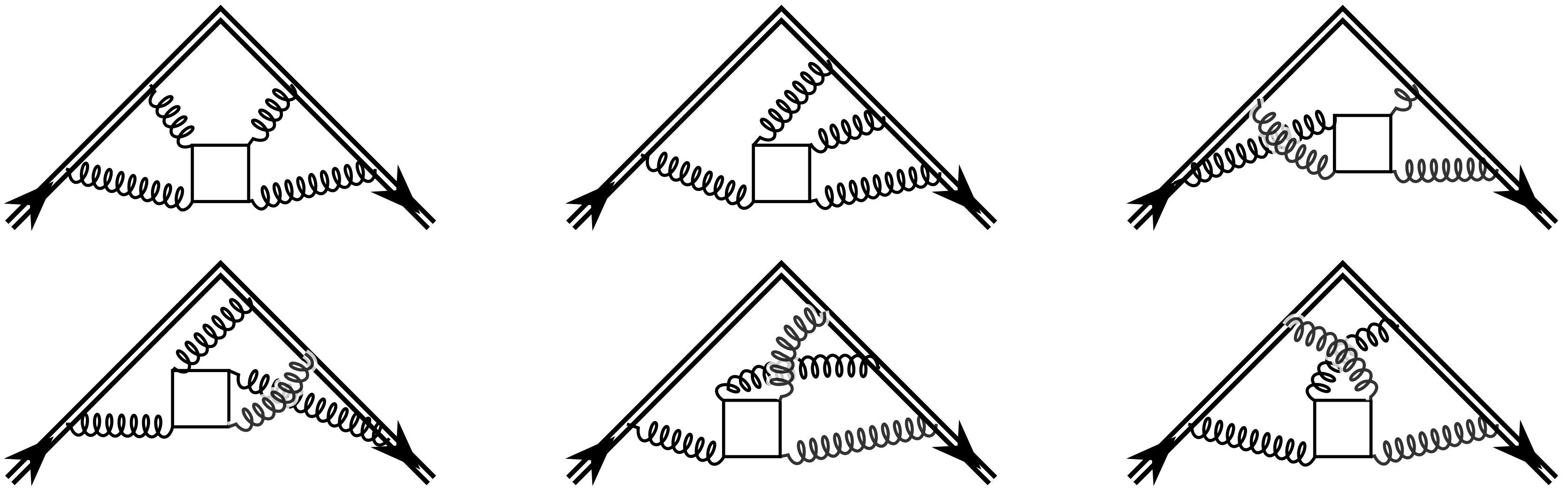


$$v_1^2 = v_2^2 = 1$$

$$v_1 \cdot v_2 = \cos\phi = \frac{1}{2}\left(x + \frac{1}{x}\right)$$

[Grozin, Henn, Korchemsky, Marquard '16]

Calculation



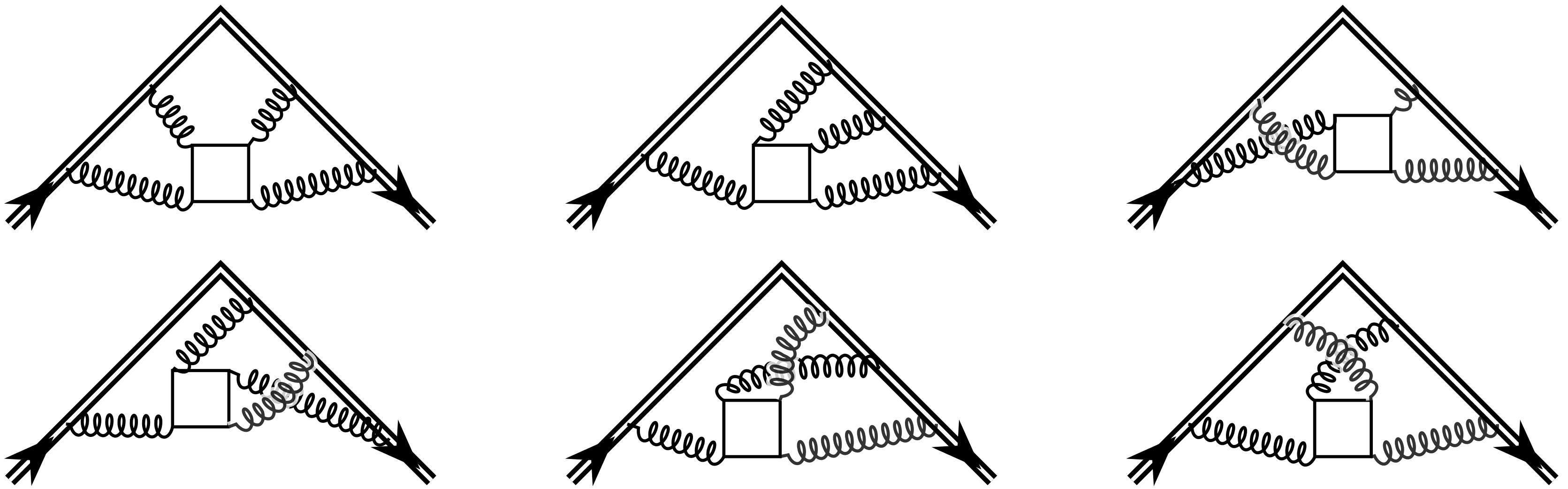
Integration by parts reduction (FIRE6, LiteRed)

[Smirnov, Chuharev, '19] [Lee, '13]

$\rightarrow$ each family $\mathcal{O}(500)$ master integrals

total master integrals ~ 900

Calculation

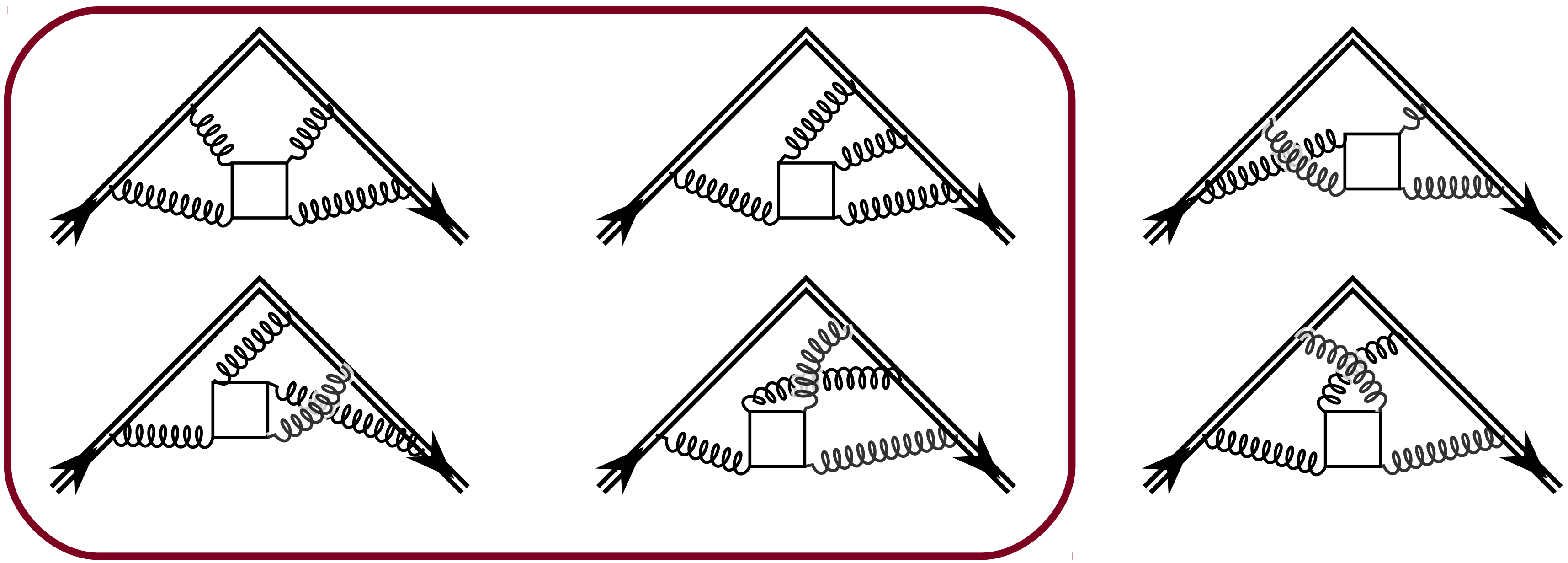


Use differential equations to compute master integrals $\vec{f}(x, \epsilon)$

⟶ choice of master integrals important [Kotikov, '91 | Remiddi '97 | Gehrmann, Remiddi, '00 | Argeri, Mastrolia '07 | Henn '13]

- INITIAL [Dlapa, Henn, Kai, '20, https://github.com/UT-team/INITIAL]
- Lee's algorithm [Lee, '20]

Calculation



$$\mathrm{d}\vec{f}(x,\epsilon) = \epsilon \sum_k \mathbf{M}_k [\mathrm{d}\log \alpha_k(x)] \vec{f}(x,\epsilon)$$

$$\left\{ x, 1 \pm x, i \pm x, 1 - x + x^2, \frac{1-\sqrt{-x}}{1+\sqrt{-x}}, \frac{1-\sqrt{-x}+x}{1+\sqrt{-x}+x} \right\}$$

Calculation

$$\mathrm{d}\vec{f}(x,\epsilon) = \epsilon \sum_k \mathbf{M}_k [\mathrm{d}\log \alpha_k(x)] \vec{f}(x,\epsilon)$$

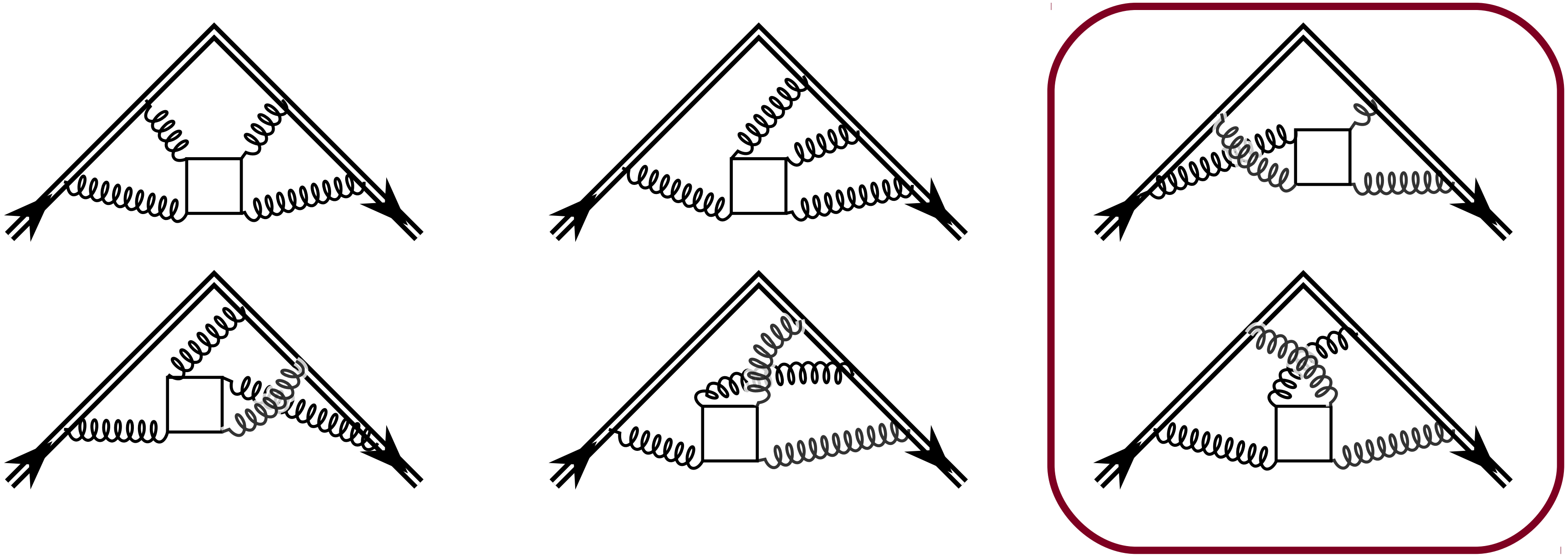
Solve order by order in ϵ: $\vec{f}(x,\epsilon) = \sum_k \epsilon^k \vec{f}^{(k)}(x)$

$\mathbb{Q}$-linear combination of multiple polylogarithms

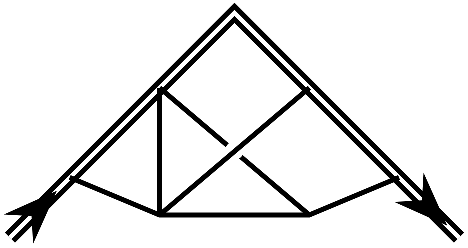
$$G(a_1, a_2, \ldots, a_n | x) = \int_0^x \frac{\mathrm{d}t}{t - a_1} G(a_2, \ldots, a_n | t)$$

$$G(a_1 | x) = \log(x - a_1)$$

Calculation



Possibly non-polylogarithmic integral sector:



We could not find an algebraic transformation that cast differential equation in desired form

QED results

$$B(x) = \frac{1+x^2}{1-x^2} B_1(x) + \frac{x}{1-x^2} B_2(x) + \frac{1-x^2}{x} B_3(x) + B_4(x)$$

$B_i(x) \;\rightarrow\;$ $\mathbb{Q}$-linear combination of multiple polylogarithms

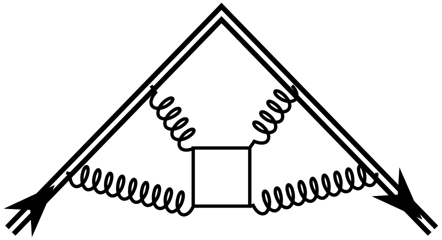
(weight three to seven)

Only subset of integration kernels appear in final result!

$$\left\{ \underbrace{x, 1 \pm x}_{\text{appear already at two loops}}, \underbrace{i \pm x}_{\text{new at 4 loops}} \right\}$$

Checks

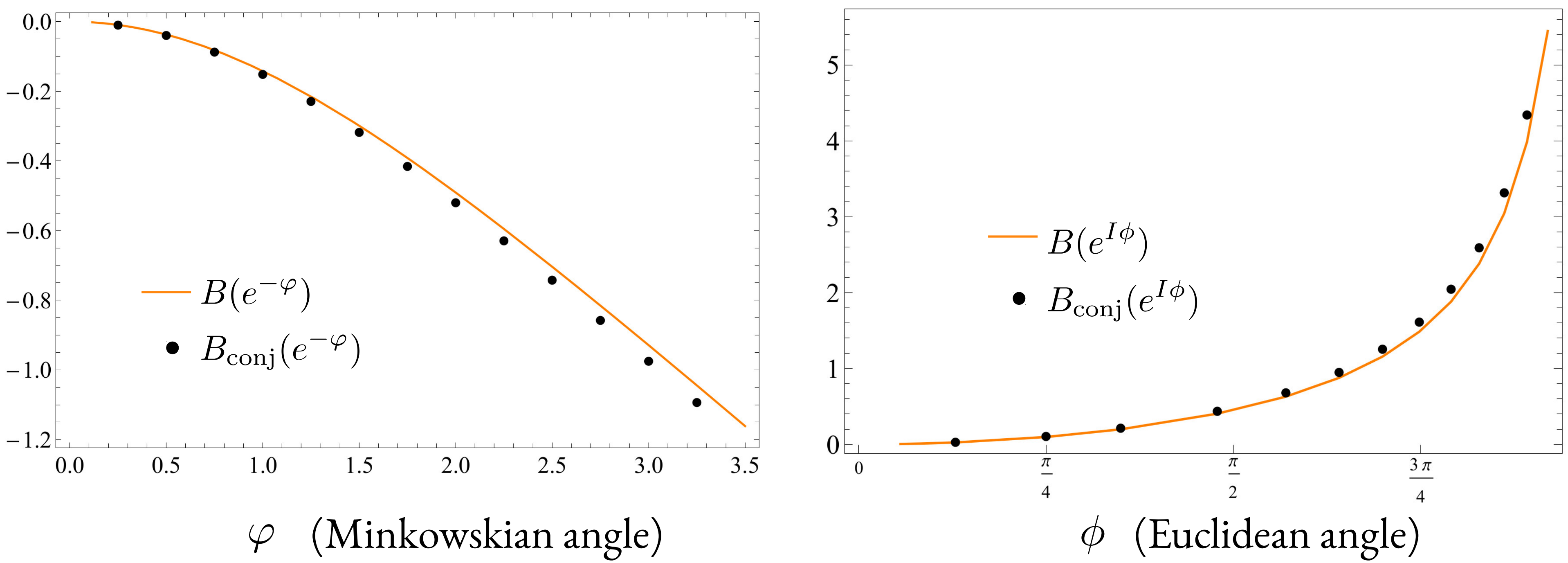
Result passes several non-trivial tests:



- gauge invariants $\rightarrow$ compute in covariant gauge
- small angle limit $\rightarrow$ known results
  $(x \rightarrow 1)$
  [Grozin, Henn, Stahlhofen, '17 | RB, Grozin, Henn, Stahlhofen, '19]
- light-like limit $\rightarrow$ recover light-like cusp
  $(x \rightarrow 0)$
  [Lee, Smirnov, Smirnov, Steinhauser, '19 | Henn, Peraro, Stahlhofen, Wasser, '19]
- anti-parallel lines limit $\rightarrow$ static potential
  $(x \rightarrow -1)$
  [Lee, Smirnov, Smirnov, Steinhauser, '16]

Compare to conjecture

$$B_{\text{conj}} = \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}\right)\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right]$$



Conclusion

- Cusp anomalous dimension Γ_{cusp} is ubiquitous in gauge theories

- Considered QED and computed last missing piece $B(x)$ at four loops

  $\longrightarrow$ important step to compute full QCD result

- Conjecture expression for Γ_{cusp} not correct but good approximation (for large part of parameter space)

Backup slides

Intriguing observation

We have expansion in the strong coupling constant

$$\Gamma_{\text{cusp}}(\phi, \alpha_s) = \sum_{k\geq 1} \left(\frac{\alpha_s}{\pi}\right)^k \Gamma^{(k)}_{\text{cusp}}(\phi)$$

Expand in new coupling $\lambda = \dfrac{\pi K}{C_R}$ $\qquad (T^a_R T^a_R = C_R \mathbb{1}_R)$

$$\Gamma_{\text{cusp}}(\phi, \alpha_s(\lambda)) = \sum_{k\geq 1} \left(\frac{\lambda}{\pi}\right)^k \Omega^{(k)}(\phi)$$

Intriguing observation

Observation up to three loops $(k = 1, 2, 3)$

$$\boxed{\Omega^{(k)}_{\mathrm{YM}} = \Omega^{(k)}_{\mathrm{QCD}} = \Omega^{(k)}_{\mathcal{N}=4}} \qquad (*)$$

$\longrightarrow$ any dependence of the number of fermions and scalars in the theory enters only through the light-like cusp

Conjecture: $(*)$ holds to all orders in perturbation theory

[Grozin, Henn, Korchemsky, Marquard '16]

Implications of the conjecture

Conjecture predicts fermionic color structures in QCD

- Re-expand $\Gamma_{\rm cusp}(\phi, \alpha_s(\lambda))$ in α_s

$$\Gamma_{\rm cusp} = \frac{\alpha_s}{\pi}\Omega^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{K^{(2)}\Omega^{(1)}}{C_R} + \Omega^{(2)}\right] + \mathcal{O}(\alpha_s^3)$$

$$K^{(2)} = -C_R n_f \frac{5}{18} + \ldots$$

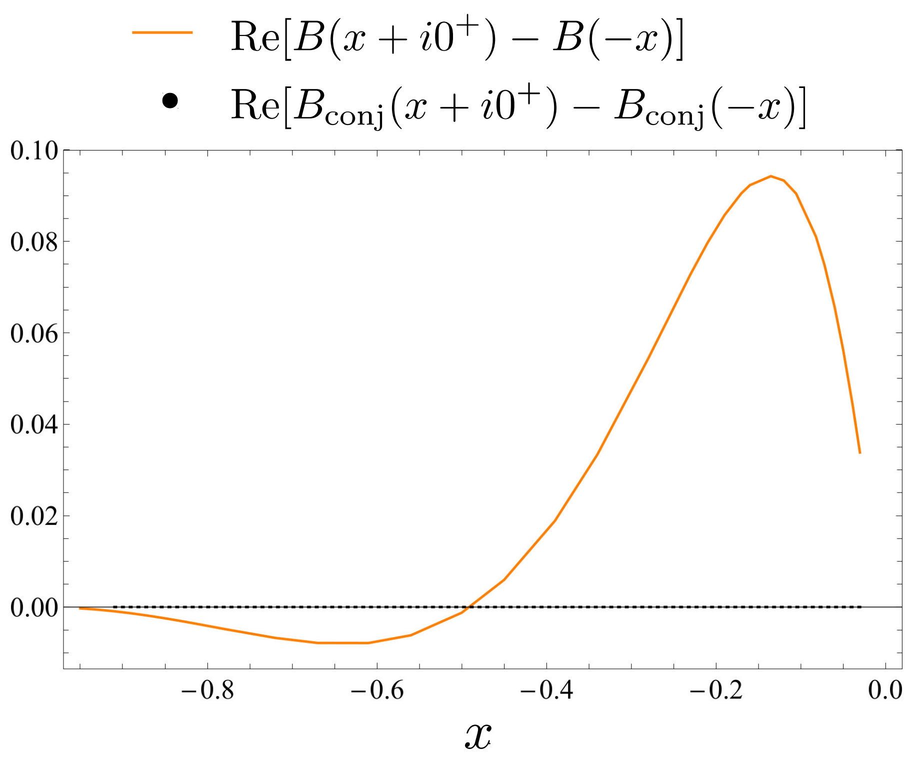
predicts n_f term
at two-loops

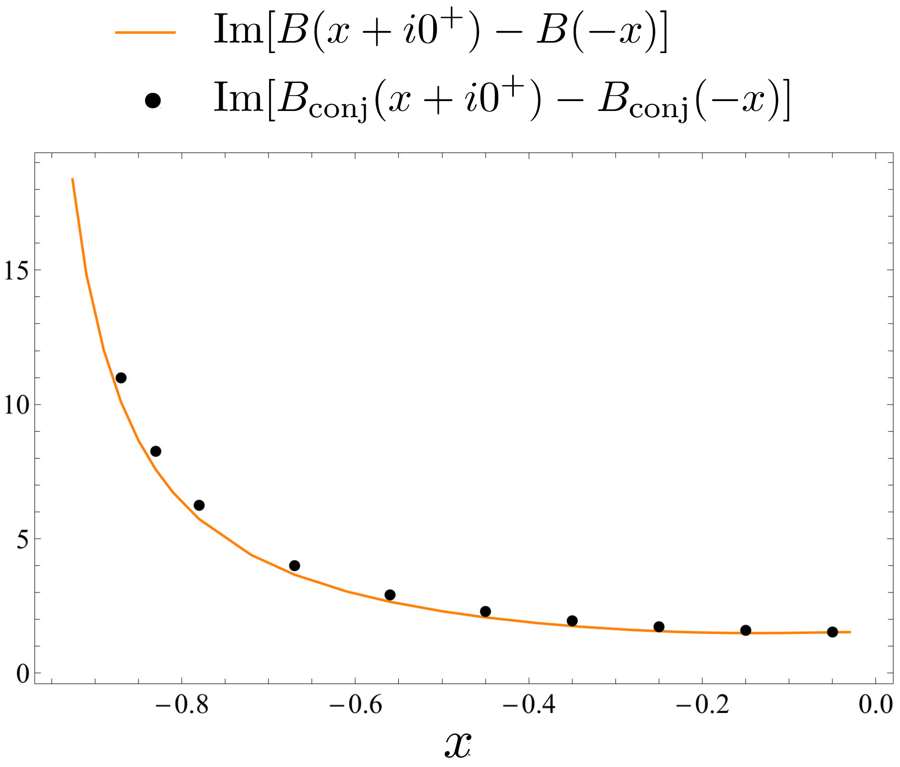
no prediction
for gluonic part

- Same pattern at higher loop order

Compare to conjecture

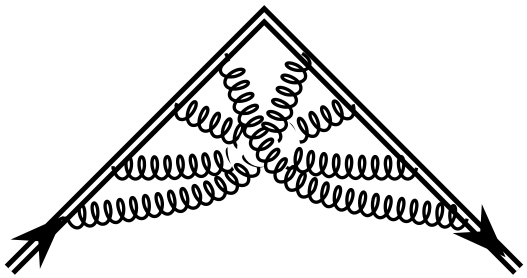
$$B_{\text{conj}} = \left(\frac{\pi^2}{6} - \frac{\zeta_3}{3} - \frac{5\zeta_5}{3}\right)\left[-\frac{1+x^2}{1-x^2}\log(x) - 1\right]$$



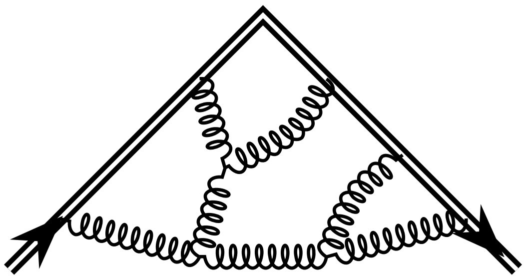


Towards full QCD result

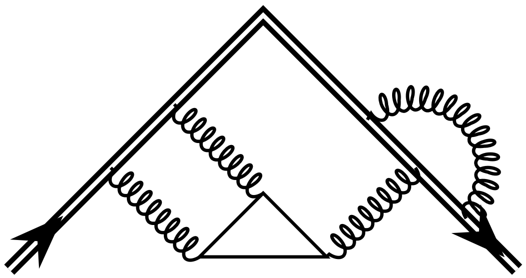
Missing color structures:



$d_R d_A / N_R$



$C_R C_A^3$



$n_f C_R C_A^2$

- Compute $\mathcal{N} = 4$ sYM expression
- Perform supersymmetric decomposition using our result

- Only planar calculation needed
- Can recycle (fraction) of computed master integrals

$$\Gamma_{\text{cusp}}(x, \alpha_s)\Big|_{\alpha_s^4} = \left(\frac{\alpha_s}{\pi}\right)^4 \left(\frac{d_R d_F}{N_R}\left[n_f B(x) + n_s C(x)\right] + \ldots\right)$$

Impression of B(x)

$$
\begin{aligned}
B_4(x) = {} & + \frac{5\pi^2}{8} - \frac{2149\pi^4}{8640} + \frac{847\pi^4 G_{-1}}{8640} + \frac{\pi^2 G_0}{72} - \frac{1267\pi^4 G_0}{8640} + \frac{847\pi^4 G_1}{8640} - \frac{1}{72}\pi^2 G_{-1,0} \\
& + \frac{17}{18}\pi^2 G_{0,0} - \frac{1}{72}\pi^2 G_{1,0} - \frac{77}{72}\pi^2 G_{-1,-1,0} + \frac{17}{3} G_{-1,0,0} + \frac{11}{9}\pi^2 G_{-1,0,0} - \frac{77}{72}\pi^2 G_{-1,1,0} \\
& - \frac{1}{6} G_{0,-1,0} + \frac{83}{72}\pi^2 G_{0,-1,0} - \frac{11}{2} G_{0,0,0} - \frac{14}{9}\pi^2 G_{0,0,0} + \frac{1}{6}\pi^2 G_{0,-i,0} + \frac{1}{6}\pi^2 G_{0,i,0} - \frac{1}{6} G_{0,1,0} \\
& + \frac{83}{72}\pi^2 G_{0,1,0} - \frac{77}{72}\pi^2 G_{1,-1,0} + \frac{17}{3} G_{1,0,0} + \frac{11}{9}\pi^2 G_{1,0,0} - \frac{77}{72}\pi^2 G_{1,1,0} + \frac{1}{6} G_{-1,0,-1,0} \\
& - \frac{1}{6} G_{-1,0,0,0} + \frac{1}{6} G_{-1,0,1,0} - \frac{1}{6} G_{0,0,-1,0} + \frac{19}{2} G_{0,0,0,0} - \frac{1}{6} G_{0,0,1,0} + \frac{1}{6} G_{1,0,-1,0} - \frac{1}{6} G_{1,0,0,0} \\
& + \frac{1}{6} G_{1,0,1,0} + \frac{77}{6} G_{-1,-1,0,-1,0} - \frac{77}{6} G_{-1,-1,0,0,0} + \frac{77}{6} G_{-1,-1,0,1,0} - \frac{77}{6} G_{-1,0,0,-1,0} \\
& + \frac{71}{2} G_{-1,0,0,0,0} - \frac{77}{6} G_{-1,0,0,1,0} + \frac{77}{6} G_{-1,1,0,-1,0} - \frac{77}{6} G_{-1,1,0,0,0} + \frac{77}{6} G_{-1,1,0,1,0} \\
& - \frac{77}{6} G_{0,-1,0,-1,0} + \frac{40}{3} G_{0,-1,0,0,0} - \frac{77}{6} G_{0,-1,0,1,0} - \frac{35}{3} G_{0,0,-1,0,0} + \frac{83}{6} G_{0,0,0,-1,0} \\
& - \frac{88}{3} G_{0,0,0,0,0} + \frac{83}{6} G_{0,0,0,1,0} - \frac{35}{3} G_{0,0,1,0,0} + 4G_{0,-i,0,0,0} + 4G_{0,i,0,0,0} - \frac{77}{6} G_{0,1,0,-1,0} \\
& + \frac{40}{3} G_{0,1,0,0,0} - \frac{77}{6} G_{0,1,0,1,0} + \frac{77}{6} G_{1,-1,0,-1,0} - \frac{77}{6} G_{1,-1,0,0,0} + \frac{77}{6} G_{1,-1,0,1,0} \\
& - \frac{77}{6} G_{1,0,0,-1,0} + \frac{71}{2} G_{1,0,0,0,0} - \frac{77}{6} G_{1,0,0,1,0} + \frac{77}{6} G_{1,1,0,-1,0} - \frac{77}{6} G_{1,1,0,0,0} + \frac{77}{6} G_{1,1,0,1,0} \\
& - \frac{61\zeta_3}{24} - \frac{107}{144}\pi^2\zeta_3 - \frac{1}{12} G_{-1}\zeta_3 + \frac{G_0\zeta_3}{12} - \frac{1}{12} G_1\zeta_3 - \frac{77}{12}\zeta_3 G_{-1,-1} + \frac{77}{12}\zeta_3 G_{-1,0} \\
& - \frac{77}{12}\zeta_3 G_{-1,1} + \frac{77}{12}\zeta_3 G_{0,-1} - \frac{23}{24}\zeta_3 G_{0,0} + \frac{77}{12}\zeta_3 G_{0,1} - \frac{77}{12}\zeta_3 G_{1,-1} + \frac{77}{12}\zeta_3 G_{1,0} - \frac{77}{12}\zeta_3 G_{1,1} \\
& + \frac{457\zeta_5}{32}
\end{aligned}
$$

80 terms (simplest of the B_i)