Numerical methods and beam functions at NNLO and beyond

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Young Scientists Forum, 17 June 2020

Outline

1. At hadron colliders, there are many interesting final states to observe. Among these, the production of colour-neutral final states has gotten much attention.
2. There are many methods to compute cross sections at NLO and NNLO. However, at N^3LO the implementation of these methods is involved.
3. I shall present our methods to compute beam function which have already been computed at NLO [Becher, Neubert '11], NNLO [Gehrmann, Lübbert, Yang '12, '14] and recently N^3LO [Luo, Wang, Xu, Yang, Yang, Zhu '19, Ebert, Mistlberger, Vita, '20]
4. The beam function was the last ingredient needed to implement the q_T slicing method at N^3LO.

The $\mathrm{q_T}$ slicing method

[Catani, Grazzini ‘07, ‘15]

$$\mathrm{p} + \mathrm{p} \to \mathrm{F}\,(\mathrm{q_T}) + \mathrm{X}$$

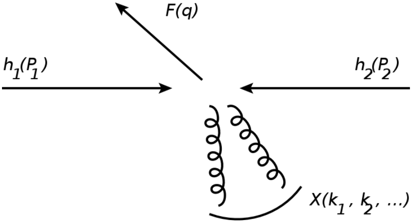
$$\sigma^{\mathrm{F}}_{\mathrm{N^m LO}} = \int_0^{\mathrm{q_{T,cut}}} \mathrm{d}\mathrm{q_T}\, \frac{\mathrm{d}\sigma^{\mathrm{F}}_{\mathrm{N^m LO}}}{\mathrm{d}\mathrm{q_T}} + \int_{\mathrm{q_{T,cut}}}^{\infty} \mathrm{d}\mathrm{q_T}\, \frac{\mathrm{d}\sigma^{\mathrm{F}}_{\mathrm{N^m LO}}}{\mathrm{d}\mathrm{q_T}}$$

$$= \int_0^{\mathrm{q_{T,cut}}} \mathrm{d}\mathrm{q_T}\, \frac{\mathrm{d}\sigma^{\mathrm{F}}_{\mathrm{N^m LO}}}{\mathrm{d}\mathrm{q_T}} + \int_{\mathrm{q_{T,cut}}}^{\infty} \mathrm{d}\mathrm{q_T}\, \frac{\mathrm{d}\sigma^{\mathrm{F+jet}}_{\mathrm{N^{m-1} LO}}}{\mathrm{d}\mathrm{q_T}}$$

enough to know in
small-$\mathrm{q_T}$ approximation

known

Factorization



where $\mathrm{F} = \mathrm{H}, \mathrm{Z}, \mathrm{W}, \mathrm{ZZ}, \mathrm{WW}, \mathrm{t\bar{t}}, \dots$
If not colourless the final state must be at least massive

$\mathrm{q}^2 \sim \mathrm{q}_\mathrm{T}^2 \gg \Lambda_\mathrm{QCD}$ collinear factorization

$$\frac{\mathrm{d}\sigma_\mathrm{F}}{\mathrm{d}\Phi} = \phi_1 \otimes \phi_2 \otimes \mathrm{C} + \mathcal{O}\left(\frac{1}{\mathrm{q}^2}\right)$$

$\mathrm{q}^2 \gg \mathrm{q}_\mathrm{T}^2 > \Lambda_\mathrm{QCD}$ small-q_T factorization

$$\frac{\mathrm{d}\sigma_\mathrm{F}}{\mathrm{d}\Phi} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes \mathcal{H} \otimes \mathcal{S} + \mathcal{O}\left(\frac{\mathrm{q}_\mathrm{T}^2}{\mathrm{q}^2}\right)$$

All those functions

To get the cross section at $\mathrm{N^mLO}$, we need to know all those functions at $\mathrm{N^mLO}$

$$\frac{\mathrm{d}\sigma_{\mathrm{F}}^{\mathrm{N^mLO}}}{\mathrm{d}\Phi} = \mathcal{B}_1^{\mathrm{N^mLO}} \otimes \mathcal{B}_2^{\mathrm{N^mLO}} \otimes \mathcal{H}^{\mathrm{N^mLO}} \otimes \mathcal{S}^{\mathrm{N^mLO}}$$

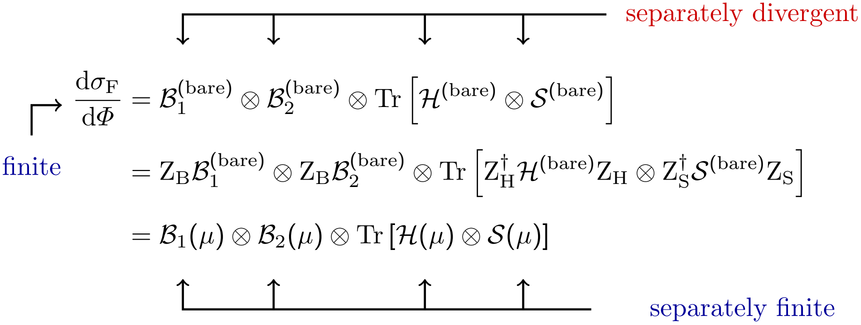
$\mathcal{B}$ - beam function - radiation collinear to the beam, process-independent, known up to $\mathrm{N^3LO}$

$\mathcal{H}$ - hard function - virtual corrections, process-dependent

$\mathcal{S}$ - soft function - soft, real radiation, process-dependent

Today, I will focus on $\mathcal{B}$.

Renormalization



separately divergent

finite

$$\frac{\mathrm{d}\sigma_{\mathrm{F}}}{\mathrm{d}\Phi} = \mathcal{B}_1^{(\mathrm{bare})} \otimes \mathcal{B}_2^{(\mathrm{bare})} \otimes \mathrm{Tr}\left[\mathcal{H}^{(\mathrm{bare})} \otimes \mathcal{S}^{(\mathrm{bare})}\right]$$

$$= \mathrm{Z_B}\mathcal{B}_1^{(\mathrm{bare})} \otimes \mathrm{Z_B}\mathcal{B}_2^{(\mathrm{bare})} \otimes \mathrm{Tr}\left[\mathrm{Z}_{\mathrm{H}}^{\dagger}\mathcal{H}^{(\mathrm{bare})}\mathrm{Z_H} \otimes \mathrm{Z}_{\mathrm{S}}^{\dagger}\mathcal{S}^{(\mathrm{bare})}\mathrm{Z_S}\right]$$

$$= \mathcal{B}_1(\mu) \otimes \mathcal{B}_2(\mu) \otimes \mathrm{Tr}\left[\mathcal{H}(\mu) \otimes \mathcal{S}(\mu)\right]$$

separately finite

$\frac{\mathrm{d}}{\mathrm{d}\mu}\frac{\mathrm{d}\sigma_{\mathrm{F}}}{\mathrm{d}\Phi} = 0 \quad \rightarrow \quad$ Renormalization Group Equations for $\mathcal{B}, \mathcal{H}$ and $\mathcal{S}$

Small-q_T factorization in SCET

Gluons' momenta in light-cone coordinates

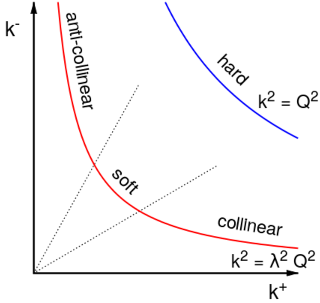
$$k_i^\mu = \left(k_i^+, k_i^-, \boldsymbol{k}_i^\perp\right) \qquad \text{where} \qquad k^\pm = k^0 \pm k^3$$

Expansion parameter

$$\lambda = \sqrt{\frac{q_T^2}{q^2}} \ll 1$$

Phase space regions

collinear	$k_i^\mu \sim (1, \lambda^2, \lambda)\, Q^2$	$\mathcal{B}_1$
anti-collinear	$k_i^\mu \sim (\lambda^2, 1, \lambda)\, Q^2$	$\mathcal{B}_2$
hard	$k_i^\mu \sim (1, 1, 1)\, Q^2$	$\mathcal{H}$
soft	$k_i^\mu \sim (\lambda, \lambda, \lambda)\, Q^2$	$\mathcal{S}$



Soft Collinear Effective Theory (SCET)

$$\mathrm{SCET} \simeq \mathrm{QCD}\Big|_{\mathrm{IR\ limit}}$$

Hard degrees of freedom are integrated out into Wilson coefficients, which are then used to adjust new couplings of the (effective) theory.

QCD fields written as sums of collinear, anti-collinear and soft components:

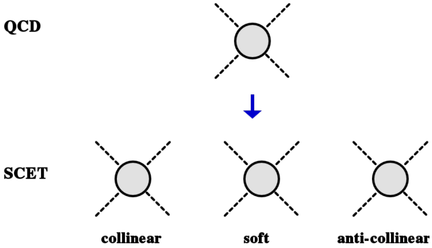
$$\phi(\mathrm{x}) = \phi_{\mathrm{c}}(\mathrm{x}) + \phi_{\bar{\mathrm{c}}}(\mathrm{x}) + \phi_{\mathrm{s}}(\mathrm{x})$$

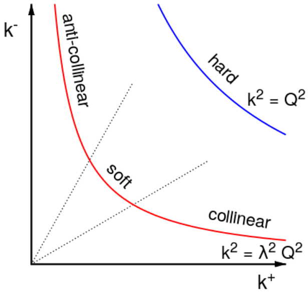
The new fields decouple in the Lagrangian

$$\mathcal{L}_{\mathrm{SCET}} = \mathcal{L}_{\mathrm{c}} + \mathcal{L}_{\bar{\mathrm{c}}} + \mathcal{L}_{\mathrm{s}}$$

At leading power, the decoupled Lagrangians are copies of the QCD lagrangian.
The separation of fields in the Lagrangian into collinear, anti-collinear and soft sectors, facilitates proofs of factorization theorems.

Rapidity divergences and analytic regulator





Modification of the measure [Becher, Bell '12]

$$\int \mathrm{d^d k}\, \delta^+(\mathrm{k}^2) \to \int \mathrm{d^d k} \left(\frac{\nu}{\mathrm{k}_+}\right)^\alpha \delta^+(\mathrm{k}^2)$$

The regulator is necessary at intermediate steps of the calculation.

Rapidity divergences and analytic regulator

Rapidity divergences do not appear in QCD, hence, the complete SCET result has to stay finite in the limit $\alpha \to 0$.

They appear in real emission diagrams due to the expansion in different momentum regions (collinear, soft, ..).

$$\mathrm{s}_{\mathrm{i}_1 \mathrm{i}_2 .. \mathrm{i}_\mathrm{k}} = (\mathrm{p}_{\mathrm{i}_1} + \mathrm{p}_{\mathrm{i}_2} + .. + \mathrm{p}_{\mathrm{i}_\mathrm{k}})^2$$

If we look at a phase space region in which $\mathrm{p}_{\mathrm{i}_1}$ and $\mathrm{p}_{\mathrm{i}_2}$ are collinear while all other momenta are anti-collinear, one expands

$$\mathrm{s}_{\mathrm{i}_1 \mathrm{i}_2 .. \mathrm{i}_\mathrm{k}} = (\mathrm{p}^-_{\mathrm{i}_1} + \mathrm{p}^-_{\mathrm{i}_2})(\mathrm{p}^+_{\mathrm{i}_3} + .. + \mathrm{p}^+_{\mathrm{i}_\mathrm{k}}) + \mathcal{O}(\lambda^2)$$

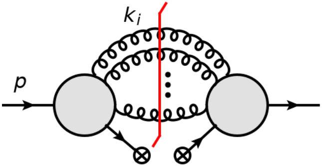
It can be shown that divergences arising from this expansion, can be regulated by analytic regularization [Becher, Bell ‘11].

NNLO and N^{3}LO beam function

The beam function

Represents corrections coming from emissions of real, collinear gluons, whose transverse momenta sum up to a fixed value $\mathrm{q_T}$ and whose longitudinal component along p sums up to $1 - \mathrm{z}$

$\mathrm{B_{bare}}(\mathrm{q_T}, \mathrm{z}) \propto \sum$



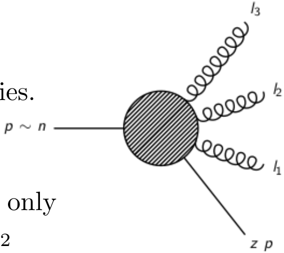
$$\times\, \delta\left(\mathrm{q_T} - \left|\textstyle\sum_i \mathrm{k}_{i\perp}\right|\right) \textstyle\prod_i \delta^+(\mathrm{k}_i^2)\, \delta\left(\bar{\mathrm{n}} \cdot \sum \mathrm{k}_i - (1 - \mathrm{z})\, \bar{\mathrm{n}} \cdot \mathrm{p}\right)$$

$$\mathrm{p} = \frac{\bar{\mathrm{n}} \cdot \mathrm{p}}{2}\mathrm{n} = \frac{\mathrm{p}_-}{2}\mathrm{n}$$

$$\mathrm{n}^2 = \bar{\mathrm{n}}^2 = 0$$

$$\mathrm{n} \cdot \bar{\mathrm{n}} = 2$$

N^{3}LO propagators



Possible denominators that may cause divergencies.

light-cone

$n \cdot l_1$

$n \cdot l_2$

$n \cdot l_3$

$\bar{n} \cdot l_1$

$\bar{n} \cdot l_2$

$\bar{n} \cdot l_3$

$n \cdot l_1 + n \cdot l_2$

$n \cdot l_1 + n \cdot l_3$

$n \cdot l_2 + n \cdot l_3$

$\bar{n} \cdot l_1 + \bar{n} \cdot l_2$

$\bar{n} \cdot l_1 + \bar{n} \cdot l_3$

$\bar{n} \cdot l_2 + \bar{n} \cdot l_3$

internal only

$l_1 \cdot l_2$

$l_1 \cdot l_3$

$l_2 \cdot l_3$

$l_1 \cdot l_2 + l_1 \cdot l_3 + l_2 \cdot l_3$

internal+external

$p_- \, n \cdot l_1$

$p_- \, n \cdot l_2$

$p_- \, n \cdot l_3$

$l_1 \cdot l_2 - p_- \, n \cdot l_1 - p_- \, n \cdot l_2$

$l_1 \cdot l_3 - p_- \, n \cdot l_1 - p_- \, n \cdot l_3$

$l_2 \cdot l_3 - p_- \, n \cdot l_2 - p_- \, n \cdot l_3$

The way to go

The beam function

$$B_{\mathrm{bare}}(z, q_T) = \sum_i \mathcal{I}_i \,,$$

can be calculated if each integral is represented as

$$\mathcal{I}_i = \sum_{j \in \mathrm{sectors}} \int_0^1 \frac{dx_1}{x_1^{1+a_1\epsilon}} \frac{dx_2}{x_2^{1+a_2\epsilon}} \frac{dx_3}{x_3^{1+a_3\epsilon}} \frac{dx_4}{x_4^{1+a_4\epsilon}} \, dx_5 \cdots dx_9 \, \mathcal{W}_j(x_1, x_2, \ldots, x_9) \,.$$

$\mathcal{W}_j(x_1, x_2, \ldots, x_9)$ has to be finite if $x_1, \ldots, x_4 \to 0$.
Then we can use

$$\frac{1}{x_i^{1+a_i\epsilon}} = -\frac{1}{a_i\epsilon}\delta(x_i) + \sum_{n=0}^{\infty} \frac{(-a_i\epsilon)^n}{n!} \left[\frac{\log^n(x_i)}{x_i}\right]_+ \,.$$

In order to further simplify the computation, we substitute the delta $\delta(k_\perp)$ for $e^{-k_\perp^2}$ and rescale the integral correspondingly.

N^{3}LO propagators

The first problem: It is impossible to parameterize the momenta such that all scalar products look simple simultaneously.

Example

$$\mathrm{n} = [1,0,0,0,1] \qquad \bar{\mathrm{n}} = [1,0,0,0,-1] \quad \mathrm{l}_1 = \left[\frac{\mathrm{l}_{1-}^2 + \mathrm{l}_{1\mathrm{T}}^2}{2\,\mathrm{l}_{1-}}, 0, 0, 0, \frac{\mathrm{l}_{1-}^2 - \mathrm{l}_{1\mathrm{T}}^2}{2\,\mathrm{l}_{1-}}\right]$$

$$\mathrm{l}_3 = \left[\frac{\mathrm{l}_{3-}^2 + \mathrm{l}_{3\mathrm{T}}^2}{2\,\mathrm{l}_{3-}},\ 0,\ \mathrm{l}_{3\mathrm{T}} \sin\chi_1,\ \mathrm{l}_{3\mathrm{T}} \cos\chi_1,\ \frac{\mathrm{l}_{3-}^2 - \mathrm{l}_{3\mathrm{T}}^2}{2\,\mathrm{l}_{3-}}\right]$$

$$\mathrm{l}_2 = \left[\frac{\mathrm{l}_{2-}^2 + \mathrm{l}_{2+}^2}{2\,\mathrm{l}_{2-}^2},\ \mathrm{l}_{2\mathrm{T}} \sin\phi_1 \sin\phi_2,\ \mathrm{l}_{2\mathrm{T}} \cos\phi_2 \sin\phi_1,\ \mathrm{l}_{2\mathrm{T}} \cos\phi_1,\ \frac{\mathrm{l}_{2-}^2 - \mathrm{l}_{2+}^2}{2\,\mathrm{l}_{2-}}\right]$$

$$\bar{\mathrm{n}} \cdot \mathrm{l}_1 = \mathrm{l}_{1-} \quad \bar{\mathrm{n}} \cdot \mathrm{l}_2 = \mathrm{l}_{2-} \quad \bar{\mathrm{n}} \cdot \mathrm{l}_3 = \mathrm{l}_{3-}$$

$$\mathrm{l}_1 \cdot \mathrm{l}_2 = \frac{\mathrm{l}_{1\mathrm{T}}^2 \mathrm{l}_{2-}}{2\,\mathrm{l}_{1-}} + \frac{\mathrm{l}_{2\mathrm{T}}^2 \mathrm{l}_{1-}}{2\,\mathrm{l}_{2-}} - \mathrm{l}_{1\mathrm{T}} \mathrm{l}_{2\mathrm{T}} \cos\phi_1 \quad \Rightarrow \quad \phi_1 = 0 \ \& \ \frac{\mathrm{l}_{1\mathrm{T}}}{\mathrm{l}_{1-}} = \frac{\mathrm{l}_{2\mathrm{T}}}{\mathrm{l}_{2-}}$$

$$\mathrm{l}_2 \cdot \mathrm{l}_3 = \frac{\mathrm{l}_{2\mathrm{T}}^2 \mathrm{l}_{3-}}{2\,\mathrm{l}_{2-}} + \frac{\mathrm{l}_{3\mathrm{T}}^2 \mathrm{l}_{2-}}{2\,\mathrm{l}_{3-}} - \mathrm{l}_{2\mathrm{T}} \mathrm{l}_{3\mathrm{T}} \cos\chi_1 \cos\phi_1 - \mathrm{l}_{2\mathrm{T}} \mathrm{l}_{3\mathrm{T}} \cos\phi_2 \sin\chi_1 \sin\phi_1$$

Step 1: selector functions

7 triple collinear
$(l_1 \cdot l_2)\,(n \cdot l_1)\,(n \cdot l_2)$
$(l_1 \cdot l_3)\,(n \cdot l_1)\,(n \cdot l_3)$
$(l_2 \cdot l_3)\,(n \cdot l_2)\,(n \cdot l_3)$
$(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)$
$(l_1 \cdot l_3)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_3)$
$(l_2 \cdot l_3)\,(\bar{n} \cdot l_2)\,(\bar{n} \cdot l_3)$
$(l_1 \cdot l_2)\,(l_1 \cdot l_3)\,(l_2 \cdot l_3)$

12 double collinear	
$(n \cdot l_1)\,(\bar{n} \cdot l_2)$	$(l_1 \cdot l_3)\,(n \cdot l_2)$
$(n \cdot l_1)\,(\bar{n} \cdot l_3)$	$(l_2 \cdot l_3)\,(n \cdot l_1)$
$(n \cdot l_2)\,(\bar{n} \cdot l_3)$	$(l_1 \cdot l_2)\,(n \cdot l_3)$
$(\bar{n} \cdot l_1)\,(n \cdot l_2)$	$(l_1 \cdot l_3)\,(\bar{n} \cdot l_2)$
$(\bar{n} \cdot l_1)\,(n \cdot l_3)$	$(l_2 \cdot l_3)\,(\bar{n} \cdot l_1)$
$(\bar{n} \cdot l_2)\,(n \cdot l_3)$	$(l_1 \cdot l_2)\,(\bar{n} \cdot l_3)$

$$d_{1,2;1} = (l_1 \cdot l_2)\,(\bar{n} \cdot l_1)\,(\bar{n} \cdot l_2)\,,$$

$$S_{1,2;2} = \frac{1}{d_{1,2;1}\mathcal{D}}\,,$$

$$\mathcal{D} = \sum_{i,j,k} \frac{1}{d_{i,j;k}} + \sum_{i,j,k,l} \frac{1}{d_{i,j;k,l}}\,,$$

$$S_{1,2;2} = \frac{1}{1 + \frac{(l_1 \cdot l_2)\,(\bar{n} \cdot l_2)}{(l_1 \cdot l_3)\,(\bar{n} \cdot l_3)} + \frac{(l_1 \cdot l_2)\,(\bar{n} \cdot l_1)}{(l_1 \cdot l_3)} + \cdots}\,,$$

Step 2: nonlinear transformations

Let’s focus on the sector $(\mathrm{l}_1 \cdot \mathrm{l}_2)(\bar{\mathrm{n}} \cdot \mathrm{l}_1)(\bar{\mathrm{n}} \cdot \mathrm{l}_2)$. All other singularities are suppressed by the corresponding selector functions.

In this sector, divergencies can be generated by the following propagators:

$$
\begin{aligned}
\bar{\mathrm{n}} \cdot \mathrm{l}_1 &\longrightarrow \mathrm{l}_{1-} \\
\bar{\mathrm{n}} \cdot \mathrm{l}_2 &\longrightarrow \mathrm{l}_{2-} \\
\mathrm{n} \cdot \mathrm{l}_1 & \\
\mathrm{n} \cdot \mathrm{l}_2 & \\
\mathrm{l}_1 \cdot \mathrm{l}_2 &\longrightarrow \frac{\mathrm{l}_{1\mathrm{T}}^2 \mathrm{l}_{2-}}{2\mathrm{l}_{1-}} + \frac{\mathrm{l}_{2\mathrm{T}}^2 \mathrm{l}_{1-}}{2\mathrm{l}_{2-}} - \mathrm{l}_{1\mathrm{T}} \mathrm{l}_{2\mathrm{T}} \cos\phi_1 \\
\mathrm{n} \cdot \mathrm{l}_1 + \mathrm{n} \cdot \mathrm{l}_2 & \\
\bar{\mathrm{n}} \cdot \mathrm{l}_1 + \bar{\mathrm{n}} \cdot \mathrm{l}_2 &\longrightarrow \mathrm{l}_{1-} + \mathrm{l}_{2-} \\
\mathrm{l}_1 \cdot \mathrm{l}_2 + \mathrm{l}_1 \cdot \mathrm{l}_3 + \mathrm{l}_2 \cdot \mathrm{l}_3 &
\end{aligned}
$$

Step 2: nonlinear transformations

The nonlinear transformation

$$\zeta = \frac{1}{2}\frac{(\mathrm{l}_{1\mathrm{T}}\mathrm{l}_{2-} - \mathrm{l}_{1-}\mathrm{l}_{2\mathrm{T}})^2\,(1+\cos\phi_1)}{\mathrm{l}_{1\mathrm{T}}^2\mathrm{l}_{2-}^2 + \mathrm{l}_{1-}^2\mathrm{l}_{2\mathrm{T}}^2 - 2\mathrm{l}_{1-}\mathrm{l}_{2-}\mathrm{l}_{1\mathrm{T}}\mathrm{l}_{2\mathrm{T}}\cos\phi_1}$$

turns

$$\mathrm{l}_1\cdot\mathrm{l}_2 = \frac{\mathrm{l}_{1\mathrm{T}}^2\mathrm{l}_{2-}}{2\mathrm{l}_{1-}} + \frac{\mathrm{l}_{2\mathrm{T}}^2\mathrm{l}_{1-}}{2\mathrm{l}_{2-}} - \mathrm{l}_{1\mathrm{T}}\mathrm{l}_{2\mathrm{T}}\cos\phi_1$$

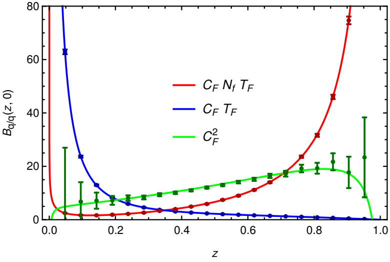
into

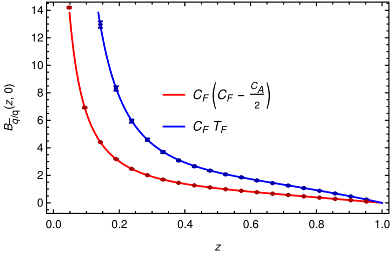
$$\mathrm{l}_1\cdot\mathrm{l}_2 = \frac{(\mathrm{l}_{1\mathrm{T}}^2\mathrm{l}_{2-}^2 - \mathrm{l}_{1-}^2\mathrm{l}_{2\mathrm{T}}^2)^2}{2\,\mathrm{l}_{1-}\mathrm{l}_{2-}\big(\mathrm{l}_{1\mathrm{T}}^2\mathrm{l}_{2-}^2 + \mathrm{l}_{1-}^2\mathrm{l}_{2\mathrm{T}}^2 - 2\,\mathrm{l}_{1-}\mathrm{l}_{2-}\mathrm{l}_{1\mathrm{T}}\mathrm{l}_{2\mathrm{T}}(1-2\zeta)\big)}$$

NNLO beam function

Known analytically [Gehrmann, Lübbert, Yang '12, '14].

We checked that our method reproduces that result





Furthermore, we are able to get higher orders in the expansion parameters at the cost of more computing time