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Solving Differential Equations with Imaginary Mass

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Based on *Liu, Ma, and Wang, arXiv:1711.09572*, and work in preperation *Brønnum-Hansen, Melnikov, and Wang*.

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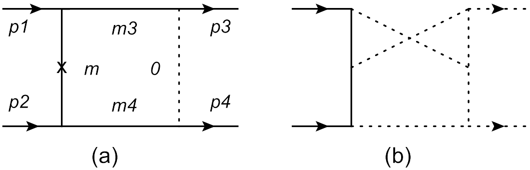
Outline

1. Solving differential equations with imaginary mass
2. NLO on-shell $gg \to WW$ with full m_t dependence
3. Conclusion
4. References

Imaginary mass method

Motivation



$$I = \lim_{\eta \to 0^+} \int \prod_{i=1}^{L} \frac{d^d l_i}{(2\pi)^d} \prod_a \frac{1}{(q_a^2 - m_a^2 + i\eta)^{\nu_a}}$$

- Higher experimental precision at the LHC requires more accurate theoretical predictions for processes where jets, tops, Higgses, and vector bosons are produced.
- Many important processes at colliders involve Feynman diagrams with massive internal lines ($t, H, W/Z$). Computing such diagrams is challenging.

Imaginary mass method

Existing Methods

$$I = \lim_{\eta \to 0^+} \int \prod_{i=1}^{L} \frac{d^d l_i}{(2\pi)^d} \prod_a \frac{1}{(q_a^2 - m_a^2 + i\eta)^{\nu_a}}$$

- Sector decomposition *Binoth and Heinrich 2000; Heinrich 2008; A. V. Smirnov 2016; Borowka et al. 2018*
  - Time consuming.
  - Hard to achieve high precision.
- Mellin–Barnes representation *V. A. Smirnov 1999; Tausk 1999; Czakon 2006; A. V. Smirnov and V. A. Smirnov 2009; Ochman and Riemann 2015*
  - Hard to deal with non-planar diagrams.
- Differential equations method *Kotikov 1991; Remiddi 1997; Gehrmann and Remiddi 2000; Henn 2013; Lee 2015*
  - Boundary conditions are process-dependent and can be difficult to compute.

Differential equations method is promising, but **process-dependent**.

Imaginary mass method

Differential equations

$$I = \lim_{\eta \to 0^+} \int \prod_{i=1}^{L} \frac{d^d l_i}{(2\pi)^d} \prod_a \frac{1}{(q_a^2 - m_a^2 + i\eta)^{\nu_a}}$$

- Treat parameter η as a free variable and define

$$I(\eta) = \int \prod_{i=1}^{L} \frac{d^d l_i}{(2\pi)^d} \prod_a \frac{1}{(q_a^2 - m_a^2 + i\eta)^{\nu_a}}.$$

  Require $I = \lim_{\eta \to 0^+} I(\eta)$.
- Consider the DEs with respect to η

$$\frac{\partial I_i(\eta)}{\partial \eta} = \sum_n \frac{M_{ij}^n(\varepsilon)}{(\eta - \eta_n)^{a_n}} I_j(\eta)$$

  The boundary conditions at $\eta = \infty$ are **process-independent**.

Imaginary mass method

Boundary conditions

$$I(\eta) = \int \prod_{i=1}^{L} \frac{d^d l_i}{(2\pi)^d} \prod_a \frac{1}{(q_a^2 - m_a^2 + i\eta)^{\nu_a}}$$

- At $\eta = \infty$, there is only one region $l_\mu \sim \sqrt{|\eta|} k_\mu$.
- Indeed, for each propagator

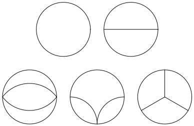
$$\frac{1}{((\sqrt{|\eta|}k + p)^2 - m^2 + i\eta)^\nu} = \frac{1}{\eta^\nu} \frac{1}{(k^2 + i)^\nu} + \cdots$$

  i.e. remove all legs and set all masses to $m^2 = -i$.
- The boundary conditions of DEs with respect to η are simple at $\eta = \infty$!

The vacuum bubbles are well-studied objects.

Davydychev, V. A. Smirnov, and Tausk 1993; Broadhurst 1999;

Schroder and Vuorinen 2005; Luthe 2015; Kalmykov and Kniehl 2017

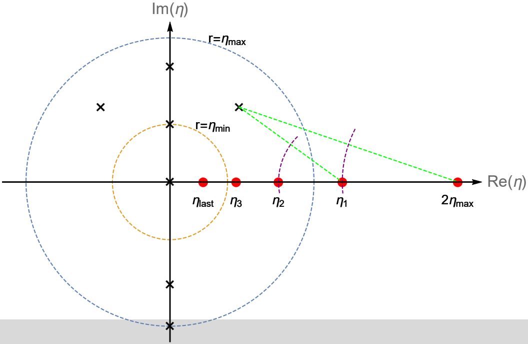


Imaginary mass method

Solving the DEs

$$\frac{\partial I_i(\eta)}{\partial \eta} = \sum_n \frac{M^n_{ij}(\varepsilon)}{(\eta-\eta_n)^{a_n}} I_j(\eta)$$

- Determine boundary conditions near $\eta = \infty$.
- Taylor expand at each finite point $\eta = \eta_k$.
- Match with the general solution $I_i(\eta) = \left[P(\eta)\eta^{M_{-1}}\right]_{ij} I_j$ near $\eta = 0$.



Imaginary mass method

Conclusion

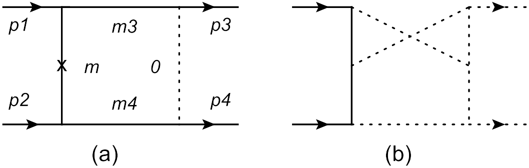


Figure: Test cases.

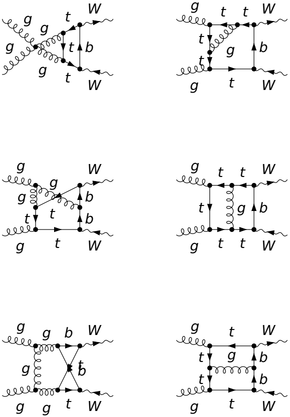
- Efficient and precise.
  Diagram (b) up to 6 digits in the threshold region:
  - Our new method: 168 master integrals in $\mathcal{O}(10^2)$ CPU seconds.
  - `FIESTA4`: single master integral in $\mathcal{O}(10^7)$ CPU seconds to reach comparable precision.

  $\mathcal{O}(10^5)$ times faster than sector decomposition!
- Boundary conditions are processes-independent. Systematic method to deal with multi-scale problems.

NLO $gg \to WW$ with finite m_t

Motivation

- Background to Higgs decay and anomalous gauge coupling.
- Suppressed by α_s, but relevant at LHC due to high gluon luminosity.
- NLO massless results are available for off-shell $gg \to VV$. *Caola, Henn, et al. 2015; Caola, Melnikov, et al. 2015; Manteuffel and Tancredi 2015; Caola, Melnikov, et al. 2016*
- Full m_t dependency is important at high energy region where $1/m_t$ & threshold expansion breaks down. *Melnikov and Dowling 2015; Davies et al. 2020*



Analytic calculation is hard.

$\downarrow$

Evaluate numerically with the imaginary mass method.

NLO $gg \to WW$ with finite m_t

Differential equations

- Only add an imaginary part to the top propagator

$$\frac{i}{p^2 - m_t^2} \to \frac{i}{p^2 - m_t^2 + i\eta}$$

- Solve the differential equations at each phase space point

$$\frac{\partial I_i(x,s,t)}{\partial x} = \sum_n \frac{M_{ij}^n(\varepsilon,s,t)}{(x-x_n)^{a_n}} I_j(x,s,t)$$

  where $x = -i\eta/m_W^2$.

- Determine boundary conditions near $x = \infty$.
- Taylor expand at each finite point $x = x_k$, moving along positive imaginary axis direction.
- Finally we reach $x = 0$.

NLO $gg \to WW$ with finite m_t

Numerical evaluation

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Figure: Topologies of the boundary integrals.

- 334 master integrals in 26 topologies.
- More kinds of boundary topologies appear.
- All 6 boundary integrals are known analytically. *Gehrmann, Huber, and Maitre 2005*
- Typical run time: 10 digits, 1–5 CPU seconds per integral.
- Checked with `pySecDec` at an unphysical point.

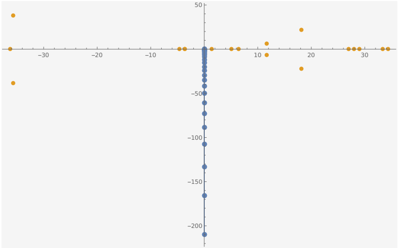

Figure: A typical contour on x plane.

NLO $gg \to WW$ with finite m_t

Preliminary results

$A^{(2)}/A^{(1)}$		ϵ^{-2}	ϵ^{-1}	ϵ^{0}
LLLL				
N_c	amp	1.0000	$18.7405 - 4.4380i$	$162.6122 - 85.6331i$
	pole	1.0000	$18.7405 - 4.4380i$	-
$1/N_c$	amp	0.0000	$1.3762 + 0.0298i$	$-27.2042 + 1.0664i$
	pole	0	$1.3762 + 0.0298i$	-
LRLL				
N_c	amp	1.0000	$19.4950 - 5.5603i$	$-176.5145 + 106.9308i$
	pole	1.0000	$19.4950 - 5.5603i$	-
$1/N_c$	amp	0.0000	$1.0276 + 0.3533i$	$-21.6031 - 3.8137i$
	pole	0	$1.0276 + 0.3533i$	-

Table: K factor of two helicity amplitudes at $\sqrt{s} = 200$ GeV and $\sqrt{|t|} \approx 71.52$ GeV.

- Divergence cancels with UV counter terms and matches Catani's IR operator.
- Building a grid in phase space, around 1 CPU hour per phase space point to reach above precision on average.

Conclusion

- Imaginary mass method provides an efficient and precise way to evaluate multi-loop integrals.
- Boundary conditions contain only vacuum bubble diagrams. Systematic method to deal with multi-scale problems.
- Capable to evaluate interesting process like $gg \to WW$ and beyond.
- Thank you!

Backup

Boundary at $\eta = \infty$

- For a Feynman integral

$$I = \int \prod_{i=1}^{L} \frac{d^d l_i}{(2\pi)^d} \prod_a \frac{1}{(q_a^2 - m_a^2 + i\eta)^{\nu_a}}$$

$$\propto \int_0^\infty \prod_a \frac{dx_a}{x_a} x_a^{\nu_a} \delta(1-x) U^{-\frac{d}{2}} \left(\frac{F}{U} - i\eta\right)^{L\frac{d}{2}-\nu}$$

where $x = \sum_a x_a$ and $\nu = \sum_a \nu_a$.

- Since $\frac{F}{U}$ is bounded according to graph theory, at $\eta = \infty$

$$\left(\frac{F}{U} - i\eta\right)^{L\frac{d}{2}-\nu} = (-i\eta)^{L\frac{d}{2}-\nu} \left[1 + \left(\nu - L\frac{d}{2}\right)\frac{F}{Ui\eta} + \cdots\right]$$

$$= (-i\eta)^{L\frac{d}{2}-\nu} \sum_{n=0}^{\infty} \frac{\left(\nu - L\frac{d}{2}\right)_n}{n!} \left(\frac{F}{Ui\eta}\right)^n$$

Backup

Boundary at $\eta = \infty$

- For the leading term

$$I_{n=0} \propto \eta^{L\frac{d}{2}-\nu} \int_0^{\infty} \prod_a \frac{dx_a}{x_a} x_a^{\nu_a} \delta(1-x) U^{-\frac{d}{2}} (-i)^{L\frac{d}{2}-\nu}$$

  which is just a vacuum bubble diagram, with all masses $m^2 = -i$.
- Actually every terms correspond to vacuum bubble diagrams

$$I(d,\eta) = I_0^{\text{vac}}(d,\eta) + \eta^{-1} \sum_k c_{1k} I_{1k}^{\text{vac}}(d+2,\eta) + \cdots$$

- The BC of DE with respect to η is simple at $\eta = \infty$!

The vacuum bubbles are well-studied objects.

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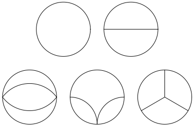

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